

UDC 511.11

MINIMAL INVOLUTIVE GENERATING SETS OF  
EXTENDED SPECIAL LINEAR GROUP  $ESL_5(\mathbb{Z})$ ,  
ANALYTICAL FORMULAS OF ROOTS IN  $ESL_2(\mathbb{F})$ ,  
 $GL_2(\mathbb{F}_p)$ .

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In this research we continue our previous investigation of wreath product normal structure [1], where we generalize the group of unimodular matrices [2]. We recall a concept of  $ESL_2(\mathbb{F}_p)$  introduced in our previous work [1].

**Definition 1.** The set of matrices

$$\{M_i : \det(M_i) = \pm 1, M_i \in GL_2(\mathbb{F}_p)\} \quad (1)$$

forms the extended special linear group and is denoted by  $ESL_2(\mathbb{F}_p)$ .

Let  $SL_3(\mathbb{F}_p)$  denotes the special linear group of degree 3 over finite field.

**Definition 2.** *The set of matrices over the integer ring or over a field*

$$\{M_i : \det(M_i) = \pm 1, M_i \in GL_3(\mathbb{Z}), \text{ or } M_i \in GL_3(\mathbb{F}_p)\}$$

*forms extended special linear group and is denoted by  $ESL_3(\mathbb{Z})$  or by  $ESL_3(\mathbb{F}_p)$ , respectively.*

We note that  $ESL_3(\mathbb{Z}) \simeq GL_3(\mathbb{Z})$  however, this isomorphism turns into an isomorphic embedding  $ESL_3(\mathbb{F}_p) < GL_3(\mathbb{F}_p)$  over a finite field  $\mathbb{F}_p$ .

To construct split extension of  $SL(3, \mathbb{F}_p)$  we consider the following class of matrix having  $\det(D_1) = -1$ , but do not centralizing the group  $SL_3(\mathbb{F}_p)$ .

$$D_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Based on the above, we conclude that structure of group generated as extension of  $SL_3(\mathbb{F}_p)$  by  $\langle D_1 \rangle$  is a semidirect product

$$\langle D_1 \rangle \rtimes SL_3(\mathbb{F}_p) \simeq ESL_3(\mathbb{F}_p)$$

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with kernel  $K = SL_3(\mathbb{F}_p)$ . But if we extend the kernel by  $D_{1,2,3} = D_1 D_2 D_3$  which belongs to centralizer of  $K$ , then the semidirect product degenerates into a direct product  $\langle D_{1,2,3} \rangle \times SL_3(\mathbb{F}_p) \simeq ESL_3(\mathbb{F}_p)$ .

A diagonal matrix with one diagonal element  $-1$  and the rest of them  $1$  be called **elementary diagonal** matrix.

**Remark 1.** If  $n = 2k + 1$ , then there exist exactly  $2^{n-1}$  extensions by subgroup generated by elementary diagonal matrices of the group  $SL(n, \mathbb{F}_p)$  to the group  $ESL(n, \mathbb{F}_p)$ .

**Proposition 1.** Minimal generating set for  $ESL_3(\mathbb{Z})$  consists of 2 generators:

$$P = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } t_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Definition 3.** By the extended group of (upper) unitriangular matrices  $EUT_n(\mathbb{F})$  (over a field  $\mathbb{F}$ ) we mean the unitriangular group  $UT_n(\mathbb{F})$  [7] that admits not only  $1$  but also  $-1$  on the diagonal.

In terms of generating set  $P_3, t_{12}$  wherein  $P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ , main relations

are the following:  $P_3 t_{12} P_3^{-1} = t_{31}$ ,  $P_3 t_{31} P_3^{-1} = t_{23}$ ,  $P_3 t_{13} P_3^{-1} = t_{32}$ ,  $P_3 t_{12} P_3^{-1} = t_{23}$ ,  $P_3^3 = E$ .

We prove that  $EUT_3[\mathbb{Z}] \simeq \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$  is *sggi* group [5] with the following involutive generating set:

$$\rho_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where  $[\rho_1, \rho_3] = e$ ,  $[\rho_0, \rho_3] = e$ ,  $[\rho_0, \rho_2] = e$ , furthermore this defines the type of involution generating set  $(2 \times 2, 2 \times 2)$ .

**Remark 2.** If we set  $D_0 = \rho_0$ ,  $F_L = \rho_1$ ,  $I_{23,43} = \rho_2$ , then this order of involutions justifies that  $ESL_5[\mathbb{Z}]$  and  $ESL_5(\mathbb{F}_p)$  are string groups generated by involutions (*sggi*) [5].

For convenience, we fix some notation for *diagonal involutive matrices* from  $ESL_3(\mathbb{Z})$ :

$$I_{12} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, I_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \dots, I_{45} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Theorem 1.** *Let*

$$F_L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_{23,43} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

*The set of involutions  $\langle I_{23,43}, D_0, F_L \rangle$  with two commuting involutions  $I_{23,43}, D_0$  generates  $ESL_3[\mathbb{Z}]$ . Moreover, we obtain triple with **two commuting involutions***

$$D_0, I_{23,43}, F_U, [D_0, I_{23,43}] = e, I_{43} = D_0 I_{23} D_0.$$

**Corollary 1.** *The minimal set  $S$  of involutive generators  $ESL_5(\mathbb{Z})$  consists of  $I_{23,43}, F_L, D_0$ .  $S$  is a Mazurov triple  $(2 \times 2, 2)$  [6], moreover, the set  $S$  satisfies the *sggi* [5] property.*

**Theorem 2.** *The minimal involutive generating set of  $ESL(5, \mathbb{Z})$  as well as of  $ESL(5, \mathbb{F}_p)$  consists of the following 3 involutions  $I_{12}, D_1, F_L$ , where  $D_1$  is given by*

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

**Remark 3.** *The set  $-D_0, I_{23,54}, F_L$  is a generating set for  $ESL_n[\mathbb{F}_p]$  with the relation of commuting involutions  $[I_{23,54}, F_L] = E$ .*

**Definition 4.** We define  $PESL(n, p)$  as a quotient of  $ESL(n, p)$  by its center.

**Theorem 3.** For  $n = 2k$  provided  $\left(\frac{-1}{p}\right) = 1$  (that is,  $p = 4k + 1$ ) we have  $[Z(ESL(n, p)) : Z(SL(n, p))] = 2$ , and  $PESL(n, p) = PSL(n, p)$ .

**Theorem 4.** For  $n = 2k$  provided  $\left(\frac{-1}{p}\right) = -1$  (that is,  $p = 4k + 3$ ) an index of a center  $[Z(ESL(2k, p)) : Z(SL(2k, p))] = 2$ . Furthermore,  $PESL(n, p)$  is simple, except special cases of  $(n, p)$ .

As we have studied  $ESL_3(\mathbb{Z})$  and it has a structure of semidirect product  $SL_3(\mathbb{Z}) \rtimes \langle \mathbb{D}_1 \rangle$ .

Let  $\lambda_1, \lambda_2, \lambda_3$  be the e.v. of  $A \in SL_3[\mathbb{Z}]$  provided  $tr A = \lambda_1 + \lambda_2 + \lambda_3 = a$ ,  $b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ . Let  $\chi_A(x) = x^3 - ax^2 + bx - 1$  denotes a characteristic polynomial for  $A$ . According to Lemma 1 [1] if  $B^2 = A$ , then  $\mu_1 = \sqrt{\lambda_1}$ ,  $\mu_2 = \sqrt{\lambda_2}$ ,  $\mu_3 = \sqrt{\lambda_3}$ , where  $\mu_1, \mu_2, \mu_3$  are the e.v. of  $B$ . We introduce the following notations  $q = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$ ,  $tr B = p = \mu_1 + \mu_2 + \mu_3$ . Let  $\chi_B(x) = x^3 - px^2 + qx - 1$  be the characteristic polynomial of  $B$ .

**Theorem 5.** The square root of a simple matrix  $A \in ESL_3[\mathbb{Z}]$  belongs to  $ESL_3[\mathbb{Z}]$  iff

$$(p^4 - 2ap^2 - 8p + a^2 - 4b)(p^4 - 2ap^2 + 8p + a^2 - 4b) = 0 \quad (2)$$

is solvable over  $p \in \mathbb{Z}$ , then square root (up to matrix similarity)

$$\sqrt{A} \in ESL_3(\mathbb{Z}).$$

**Remark 4.** The square root of a simple matrix  $A$  belongs to  $SL_3[\mathbb{F}_p]$  up to similarity iff

$$p^4 - 2ap^2 - 8p + a^2 - 4b = 0 \quad (3)$$

is solvable over  $p \in \mathbb{F}_p$ , then square root

$$\sqrt{A} \in SL_3(\mathbb{F}_p).$$

The equivalent system of conditions

$$\left. \begin{array}{l} q^2 - 2p = b \in \mathbb{F}_p \\ p^2 - 2q = a \in \mathbb{F}_p \end{array} \right\} \quad (\text{equal to the sign})$$

holds. If  $p = 2$  then each matrix  $A \in SL_3(\mathbb{F}_p)$  has square root  $\sqrt{A} \in SL_3(\mathbb{F}_p)$ .

**Proposition 1.** *If  $A, B \in GL_2(\mathbb{F}_p)$  is a root of equation  $X^3 = A$ , then*

$$B = \frac{A + \operatorname{tr}(\sqrt[3]{A})\sqrt[3]{\det(A)}}{\left(\operatorname{tr}\sqrt[3]{A}\right)^2 - \sqrt[3]{\det(A)}}.$$

In the case  $A \in SL_2(\mathbb{F}_p)$  we specify the values of the formula parameters taking into account that  $\det(A) = 1$ . Let us define sequences  $s_n = B s_{n-1} + t_{n-1}$  and  $t_n = -\det B s_{n-1}$  with initial conditions  $s_1 = 1, t_1 = 0, s_2 = \operatorname{tr} B$  and  $t_2 = -\det B$ . Now we prove the following lemma.

**Lemma 1.** *Sequences  $s_n, t_n$  satisfy a recurrent equation with characteristic polynomial  $c(x)$  which is also characteristic polynomial for the matrix  $B$ .*

**Theorem 6.** *Let  $n \geq 3$  and  $A \in M_2(\mathbb{F}_p)$ . If  $A \neq c \cdot E$  for any  $c \in \mathbb{F}_p$  and  $R = \{B \in M_2(\mathbb{F}_p) \mid B^n = A\}$  set of it's  $n$ -th roots, then next inclusion follows:*

$$R \subset \left\{ B \in M_2(\mathbb{F}_p) \mid B = \frac{A + b Q_{n-2}(a, b) \cdot E}{Q_{n-1}(a, b)}, b^n = \det A, P_n(a, b) = A \right\}.$$

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**МІНІМАЛЬНІ ІНВОЛЮТИВНІ СИСТЕМИ ТВІРНИХ ДЛЯ  
РОЗШИРЕНОЇ СПЕЦІАЛЬНОЇ ЛІНІЙНОЇ ГРУПИ  $ESL_5(\mathbb{Z})$ ,  
АНАЛІТИЧНА ФОРМУЛА КОРЕНІВ В  $ESL_2(\mathbb{F}_p)$ ,  $GL_2(\mathbb{F}_p)$ .**

*В цьому дослідженні ми знаходимо мінімальні системи твірних з інволюцій для груп  $ESL_5(\mathbb{Z})$ ,  $ESL_2(\mathbb{F}_p)$ . Також ми знаходимо такі системи твірних з інволюцій в яких дві з них комутують тобто системи що складаються з трійок інволюцій типу  $(2 \times 2, 2)$*