

## Normal subgroups of iterated wreath products of symmetric groups

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In this research we continue our previous investigation of wreath product normal structure [1]. Normal subgroups and there properties for finite and infinite iterated wreath products  $S_{n_1} \wr \dots \wr S_{n_m}$ ,  $n, m \in \mathbb{N}$  and  $A_n \wr S_n$  are founded. Special classes of normal subgroups are investigated, their generators are found and presented as in the form of Kaloujnine [2] tables as well as in the form of wreath recursion.

**Definition 1.** The permutational *subwreath product*  $G \wr H$  is the semi-direct product  $G \rtimes \tilde{H}^X$ , where  $G$  acts on the subdirect product [3]  $\tilde{H}^X$  by the respective permutations of the subdirect factors. Provided that the specification of  $\tilde{H}^X$  is established separately.

**Definition 2.** The set of elements from  $S_n \wr S_n$ ,  $n \geq 3$  which is presented by the Kaloujnine tableaux [2] of the form:  $[e]_0$ ,  $[a_1, a_2, \dots, a_n]_1$ , satisfying the following condition

$$\sum_{i=1}^n \text{dec}([a_i]_1) = 2k, k \in \mathbb{N}, \quad (1)$$

be denoted by  $\tilde{A}_n^{(1)}$ .

Also we prove that  $\tilde{A}_n^{(1)}$  is the normal subgroup of  $S_n \wr S_n$  so it will be called **alternating level subgroup**  $\tilde{A}_n^{(1)}$  with structure  $E \wr \tilde{A}_n$ , where  $\tilde{A}_n$  is the subdirect product of  $n$  copies of  $S_n$  uniquely identified by (1).

**Proposition 1.** *The set of elements  $\tilde{A}_n^{(1)}$  forms a normal subgroup in  $S_n \wr S_n$ ,  $n \geq 3$ .*

**Definition 3.** The permutational subwreath product  $G \wr H$  is the semi-direct product  $G \rtimes \tilde{H}^X$ , where  $G$  acts on the subdirect product [3]  $\tilde{H}^X$  by the respective permutations of the subdirect factors.

**Definition 4.** *The subgroup  $E \wr \tilde{A}_n^{(2)}$  be denoted by  $\tilde{A}_n^{(3)}$ .*

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Furthermore we prove that  $E \wr \tilde{A}_n^{(2)} \triangleleft S_n \wr S_n \wr S_n$ . The order of  $E \wr \tilde{A}_n^{(2)}$  is  $(n!)^{3n} : 2^3$ .

**Definition 5.** The set of elements from  $S_n \wr S_n \wr S_n, n \geq 3$  presented by the tables [2] form:

$[e]_0, [e, e, \dots, e]_1, [a_1, a_2, \dots, a_n]_2$ , satisfying the following condition

$$\sum_{i=1}^n \text{dec}([a_i]_2) = 2k, k \in \mathbb{N}, \quad (2)$$

be denoted by  $\tilde{A}_{n^2}^{(2)}$ . Note that condition (2) uniquely identifies subdirect product  $\tilde{A}_{n^2}$  in  $\prod_{i=1}^{n^2} S_n$ . Moreover  $\tilde{A}_{n^2} \triangleleft \prod_{i=1}^{n^2} S_n$ .

**Definition 6.** A subgroup in  $S_n \wr S_n$  is called  $\tilde{T}_n$  if it consists of:

1. elements of  $E \wr A_n$ ,
2. elements with the tableau [2] presentation  $[e]_0, [\pi_1, \dots, \pi_n]_1$ , that  $\pi_i \in S_n \setminus A_n$ .

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq \underbrace{(A_n \times A_n \times \dots \times A_n)}_n \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \dots \boxplus S_n}_n,$$

where the operation  $\boxplus$  of a subdirect product is subject of item 1) and 2).

**Theorem 1.** The monolith of  $S_n \wr S_m$  is  $e \wr A_m$ .

**Proposition 2.** The order of  $\tilde{T}_n$  is  $\frac{(n!)^n}{2^{n-1}}$ .

**Theorem 2.** The subgroup  $\tilde{A}_3^{(1)}$  of  $S_3 \wr S_3$  has the structure  $\tilde{A}_3^{(1)} \simeq (C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)$ .

The structure of subgroup  $\tilde{A}_n^{(1)} \triangleleft S_n \wr S_n$  is  $\tilde{A}_n^{(1)} \simeq (\prod_{i=1}^n A_n) \rtimes (\prod_{i=1}^{n-1} C_2)$ .

**Theorem 3.** There are 8 proper normal subgroups in  $S_n \wr S_m$ , where  $n, m \geq 3$  with  $n, m \neq 4$  of the following types:

1. subgroups that act only on the second level are

$$\tilde{A}_m^{(1)}, \tilde{T}_m, E \wr S_m, E \wr A_m,$$

2. subgroups that act on both levels are

$$A_n \wr \tilde{A}_m, S_n \wr \tilde{A}_m, S_n \wr \tilde{B}_m, A_n \wr S_m,$$

wherein the subgroup  $S_n \wr \tilde{A}_m \simeq S_n \ltimes \underbrace{(S_m \boxtimes S_m \boxtimes S_m \dots \boxtimes S_m)}_n$  is endowed

with the subdirect product satisfying to condition (1). Moreover,  $S_n \wr \tilde{A}_m$  has two isomorphic copies, embedded into  $S_n \wr S_m$  in 2 different ways. The second embedding is denoted by  $S_n \wr \tilde{B}_m$ . Also, 2 non-proper normal subgroups  $E$  and  $S_n \wr S_m$  exist.

The subgroup  $E \wr \tilde{A}_n^{(1)}$  be denoted by  $\tilde{A}_n^{(2)}$ .

**Proposition 1.** The subgroup  $\tilde{A}_n^{(2)} \triangleleft S_n \wr S_n \wr S_n$ .

**Theorem 4.** The subgroup  $\tilde{A}_n^{(1)}$  has **normal rank**  $n-1$  [4] in  $S_n \wr S_n$ ,  $n \geq 3$  provided  $n \equiv 1 \pmod{2}$  and **normal rank**  $n$  iff  $n \equiv 0 \pmod{2}$  and  $n \geq 3$ .

**Definition 5.** A subgroup in  $S_n \wr S_n \wr S_n$  is denoted by  $\tilde{T}_n^{(3)}$  if it consists of:

1. elements of the form  $E \wr E \wr A_n$ ,
2. elements with the tableau [2] presentation  $[e]_0, [e \dots, e]_1, [\pi_1 \dots, \pi_n, \pi_{n+1} \dots, \pi_{n^2}]_2$ , wherein  $\forall i = 1, \dots, n: \pi_i \in S_n \setminus A_n$ .

**Definition 7.** The set of elements from  $\bigcup_{i=0}^{k-1} S_{n_i}, n_i \geq 3$  with depth  $m$  satisfying the following condition

$$\sum_{i=1}^{n^j} dec([a_i]_j) = 2t, t \in \mathbb{N}, m \leq j \leq k, [a_i]_j = e, \text{ whenever } j = \overline{1, m-1} \quad (3)$$

be called  $\tilde{A}_{n^j}^{(m,k)}$ , where  $m < k$ .

**Theorem 5.** The order of normal subgroup  $\tilde{A}_{n^j}^{(0,k-1)}$  is  $(\frac{1}{2})^k \cdot (n!)^{\binom{n(k+1)-1}{n-1}}$  and the order of the quotient  $\bigcup_{i=0}^{k-1} S_{n_i} / \tilde{A}_{n^j}^{(0,k-1)}$  is  $2^k$ . The order of generalized alternating group of  $k$ -th level  $\tilde{A}_{n^k}^{(k)}$  is  $2^{n^k-1}$ .

**Theorem 6.** The quotient  $\bigcup_{i=0}^{k-1} S_{n_i}$  by  $\tilde{A}_{n^j}^{(0,k-1)}$  is the following  $\prod_{i=1}^k \mathbb{Z}_2$ .

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The set of normal subgroup of  $S_n \wr S_n$  is denoted by  $N(S_n \wr S_n)$ . Subgroup with number  $i$  from  $N(S_n \wr S_n)$  is denoted by  $N_i(S_n \wr S_n)$ .

**Theorem 7.** The full list of normal subgroups of  $W = S_n \wr S_n \wr S_n$  consists of 50 normal subgroups. These subgroups are the following:

- 1 **Type**  $T_{023}$  contains:  $E \wr \tilde{A}_n \wr H$ ,  $\tilde{T}_n \wr H$ , where  $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$ . There are 6 subgroups.
- 2 **The second type of subgroups is subclass in**  $T_{023}$  with a new base of wreath product subgroup  $\tilde{A}_{n^2}$ :  $E \wr S_n \wr \tilde{A}_{n^2}$ ,  $E \wr N_i(S_n \wr S_n)$ . Therefore, this class has 12 new subgroups. Thus, the total number of normal subgroups in **Type**  $T_{023}$  is 18.
- 3 **Type**  $T_{003}$ :  $A_{00(n^2)}^{(3)}$ ,  $\tilde{T}_{n^2}$ ,  $\tilde{T}_n^{(3)}$ .
- 4 **Type**  $T_{123}$ :  $N_i(S_n \wr S_n) \wr S_n$ ,  $N_i(S_n \wr S_n) \wr \tilde{A}_n$  and  $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$ . Thus, there are 29 new normal subgroups in  $T_{123}$ , taking into account a repetition.

**Remark 1.** Note that  $E \wr \tilde{A}_n^{(1)} \simeq E \wr (E \wr \tilde{A}_n)$  contains in the family  $E \wr N_i(S_n \wr S_n)$ .

The condition of invariance of subgroups of  $H$  with depth  $l$  [?] of a permutation wreath product  $W$  is presented in the following theorem.

**Definition 8.** The set of elements from  $\bigwrangle_{i=0}^k S_{n_i}, n_i \geq 3$  with depth  $k$  satisfying the following condition

$$\sum_{i=(s-1)n^m+1}^{sn^m} \text{dec}([a_i]_k) = 2t, t \in \mathbb{N}, 1 \leq s \leq n^{k-m}, [a_i]_j = e, \text{ for } j = \overline{0, k-1} \quad (4)$$

be called  $\tilde{A}_{n^m}^{(k)}$ , where  $1 \leq m < k$ .

**Theorem 8.** If  $H \triangleleft W$ , where  $d(H) = l$  and  $l : k - l = m$ , then  $A_{n^m}^{(k)} \triangleleft H$ .

**Theorem 9.** Let  $H \triangleleft \text{Aut}_f X^*$  having depth  $k$ , then  $H$  contains  $k$ -th level subgroup  $P$  having all even vertex permutations  $p_{ki} \in A_n$  on  $X^k$  and trivial permutations in vertices of rest of levels.

Let  $W_\infty = \bigwrangle_{i=1}^\infty S_{n_i}, n_i \geq 3$ . Assume an open set of  $k$ -th level subgroup to be such a normal subgroup  $O_{n^k}^{(k)} = S_{n_1} \wr S_{n_2} \wr S_{n_3} \wr \dots \wr S_{n_{k-1}} \wr \tilde{A}_{n^k} \wr S_{n_{k+1}} \wr S_{n_{k+2}} \wr \dots$ , this set, as a subgroup, is denoted by  $\tilde{W}^k$ . The quotient  $W_\infty$  by

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$\widetilde{W}^k$  is isomorphic to  $Z_2$  and has the form  $E \times E \dots \times E \times S_2 \times E \times E \dots$ . We construct homomorphism from  $W_\infty / \cap_{k=1}^\infty \widetilde{W}^k$  to countable direct product of a group  $Z_2$  i.e. in  $\prod_{i=1}^\infty Z_2$ . Then this **topological group** on base of  $W_\infty$  is homeomorphic to **Cantor set**  $\mathbb{C}$ .

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