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PARASTROPHIC ORBIT OF ALTERNATIVE QUASIGROUPS

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An algebra $(Q, \circ, \overset{l}{\circ}, \overset{r}{\circ})$ with identities

$$(x \circ y) \overset{l}{\circ} y = x, \quad (x \overset{l}{\circ} y) \circ y = x, \quad x \overset{r}{\circ} (x \circ y) = y, \quad x \circ (x \overset{r}{\circ} y) = y$$

is called a *quasigroup*; the operation $(\circ)$ is *main*, $(\overset{l}{\circ})$, $(\overset{r}{\circ})$ are called *left* and *right divisions* of $(\circ)$.

Each inverse of an invertible operation is also invertible. All such operations are called *parastrophes* of $(\circ)$ and they are defined by $x_{1\sigma} \overset{\sigma}{\circ} x_{2\sigma} = x_{3\sigma} :\Leftrightarrow x_1 \circ x_2 = x_3$, where $\sigma \in S_3 := \{\iota, s, l, r, sl, sr\}$, $l := (13)$, $r := (23)$, $s := (12)$. In particular, the left and right divisions of $(\circ)$ are its parastrophes. It is easy to verify that ${}^{\sigma}(\overset{\sigma}{\circ}) = (\overset{\sigma\tau}{\circ})$ holds for all $\sigma, \tau \in S_3$.

Let P be an arbitrary proposition in a class of quasigroup $\mathfrak{A}$. A proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if it can be obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Let ${}^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$. A set of all pairwise parastrophic classes is called a *parastrophic orbit* of $\mathfrak{A}$:

$$Po(\mathfrak{A}) = \{{}^{\sigma}\mathfrak{A} \mid \sigma \in S_3\}.$$

A parastrophic orbit of varieties is uniquely defined by one of its varieties. Parastrophy orbits were studied by Alla Lutsenko and Fedir Sokhatsky [1], [2].

A quasigroup $(Q, \circ)$ is called *left (right) alternative* if it satisfies the identity $x \circ (x \circ y) = (x \circ x) \circ y$ (respectively, $(x \circ y) \circ y = x \circ (y \circ y)$) for all $x, y \in Q$ [3].

If a quasigroup is a left alternative and a right alternative, we will call it an *alternative.*

Theorem 1. *The parastrophic orbit of an alternative quasigroup comprises three varieties:*

$$Po(\mathfrak{A}) = \{\mathfrak{A},\ {}^{l}\mathfrak{A},\ {}^{r}\mathfrak{A}\}.$$

Variety	*Identity*
$\mathfrak{A} =^{s} \mathfrak{A}$	$x \circ (x \circ y) = (x \circ x) \circ y,$ $(x \circ y) \circ y = x \circ (y \circ y)$
${}^{l}\mathfrak{A} =^{sl} \mathfrak{A}$	$((xy \overset{l}{\circ} x) \circ y) \circ xy = xy,$ $xy \circ y = x \circ (y \overset{l}{\circ} y)$
${}^{r}\mathfrak{A} =^{sr} \mathfrak{A}$	$x \circ xy = (x \overset{r}{\circ} x) \circ y,$ $xy \circ (x \circ (y \overset{r}{\circ} xy)) = xy$

Example 1. The quasigroup $(\mathbb{Z}_7, \circ)$ defined by $x \circ y = x + y + c \pmod 7$ serves as an example illustrating Theorem 1. That is, it satisfies one of the identities corresponding to the varieties $\mathfrak{A}$, ${}^{l}\mathfrak{A}$, or ${}^{r}\mathfrak{A}$, depending on the parameter $c \in \mathbb{Z}_7$.

The quasigroup $(\mathbb{Z}_7, \circ)$ defined by $x \circ y = x + y \pmod 7$ is alternative and satisfies the identities of the variety $\mathfrak{A}$.

The quasigroup $(\mathbb{Z}_7, \circ)$ defined by $x \circ y = x - y \pmod 7$ satisfies the identities corresponding to the parastrophe ${}^{l}\mathfrak{A}$.

The quasigroup $(\mathbb{Z}_7, \circ)$ defined by $x \circ y = -x + y \pmod 7$ satisfies the identities corresponding to the parastrophe ${}^{r}\mathfrak{A}$.

In this class of examples, the identities from Theorem 1 are distributed among the three varieties $\mathfrak{A}$, ${}^{l}\mathfrak{A}$, ${}^{r}\mathfrak{A}$.

1. Sokhatsky F., Lutsenko A. *Classification of quasigroups according to directions of translations II.* Commentationes Mathematicae Universitatis Carolinae 2021, **62** (3), 309–323.
2. Sokhatsky F.M., Lutsenko A.V., Fryz I.V. *Construction of Quasigroups with Invertibility Properties.* Journal of Mathematical Sciences 2024, **279**, 115–132.
3. Sokhatsky F.M. *Some linear conditions and their application to describing group isotopes.* Quasigroups and Related Systems 1999, **6**, 43–59.

ПАРАСТРОФНА ОРБІТА АЛЬТЕРНАТИВНИХ КВАЗІГРУП

Знайдено парастрофну орбіту альтернативних квазігруп, яка складається з трьох многовидів. Для кожного з отриманих многовидів встановлено відповідні тотожності. Також наведено приклади квазігруп, які демонструють виконання тотожностей, що відповідають різним парастрофним класам альтернативних квазігруп.