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ON THE SET OF SUBSUMS OF BI-GEOMETRIC SERIES WITH RATIONAL INITIAL PARAMETERS

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Depending on the parameters x and y, we study the set of subsums of the bi-geometric series

$$\frac{y}{4}+\frac{x}{4}+\frac{y}{4^2}+\frac{x}{4^2}+\frac{y}{4^3}+\frac{x}{4^3}+\cdots+\frac{y}{4^i}+\frac{x}{4^i}+\ldots$$

This set can be represented as

$$E(x,y)=\left\{\sum_{i=1}^{\infty}\frac{\varepsilon_i}{4^i} : (\varepsilon_i)\in\{0,x,y,x+y\}^{\mathbb{N}}\right\}.$$

It is known from [3] that the set of subsums of a convergent positive series is either a finite union of intervals, Cantor-type set, or Cantorval (a set with non-empty interior and fractal boundary). However, necessary and sufficient conditions for the set of subsums to belong to one of the above topological types are still unknown. Despite significant progress for certain classes of series, the problem remains difficult in the general setting. We also refer the reader to [1], where numerous open problems related to this topic are discussed.

We introduce a set of positive integers $\mathcal{V}$ whose last non-zero digit in the quaternary expansion is either 1 or 3. Its complement, denoted by $\mathcal{D}$, consists of all numbers whose last non-zero digit in the quaternary expansion is 2. Clearly, $\mathcal{V}\cap\mathcal{D}=\emptyset$ and $\mathcal{V}\cup\mathcal{D}=\mathbb{N}$. The affiliation of the parameters x and y with these sets plays a crucial role in determining the topological properties of $E(x,y)$. The following theorems provide necessary and sufficient conditions for $E(x,y)$ to take each of the three possible topological forms.

Theorem 1. *The set $E(x,y)$ is a finite union of closed intervals if and only if $y=2\cdot 4^t\cdot x$ for some $t\in\mathbb{Z}$.*

Theorem 2. *If x and y belong to the same set, either $\mathcal{V}$ or $\mathcal{D}$, then $E(x,y)$ is a Cantor-type set.*

Theorem 3. *If one of the parameters belongs to $\mathcal{V}$ and the other to $\mathcal{D}$, and $y\neq 2\cdot 4^t\cdot x$ for any $t\in\mathbb{Z}$, then $E(x,y)$ is a Cantorval.*

Our recent results yield a partial proof of Theorem 3. In [2], it is shown using a probabilistic approach that if $\{0, x, y, x+y\}$ forms a complete residue system modulo 4, then $E(x, y)$ has positive Lebesgue measure. Since $E(x, y)$ is a self-similar set with similarity dimension 1, it follows from [4] that positive Lebesgue measure implies non-empty interior.

We can also show that for arbitrary $x, y \in \mathbb{R}$:

- $E(x, y) \sim E(\eta x, \eta y)$ for any $\eta \in \mathbb{R} \setminus \{0\}$;
- $E(x, y) \sim E(4^m x, 4^l y)$ for any $m, l \in \mathbb{N}$;
- $E(x, y) \sim E_\nu(x, y)$ for any $\nu \in \mathbb{R}$, where

$$E_\nu(x, y) = \left\{ \sum_{i=1}^{\infty} \frac{\varepsilon_i}{4^i} : (\varepsilon_i) \in \{\nu, x+\nu, y+\nu, x+y+\nu\}^{\mathbb{N}} \right\}.$$

Here $\sim$ denotes that the corresponding sets are of the same topological type.

Next we are able to extend the above result to the case where x and y are rational numbers. Indeed, let $x = p_1/q_1$ and $y = p_2/q_2$, where $p_1, p_2 \in \mathbb{Z}$, $q_1, q_2 \in \mathbb{N}$. According to the above, $E(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ and $E_\nu(\eta\frac{p_1}{q_1}, \eta\frac{p_2}{q_2})$ with

$$\eta = q_1 \cdot q_2, \quad \nu = -\min\left(0, \eta \cdot \frac{p_1}{q_1}, \eta \cdot \frac{p_2}{q_2}, \eta \cdot \frac{p_1}{q_1} + \eta \cdot \frac{p_2}{q_2}\right),$$

are of the same topological type. Thus, we reduce the problem to the scenario of natural parameters x and y.

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ПРО МНОЖИНУ НЕПОВНИХ СУМ БІГЕОМЕТРИЧНИХ РЯДІВ З РАЦІОНАЛЬНИМИ ПАРАМЕТРАМИ

Ми вивчаємо множини неповних сум бігеометричних рядів

$$\frac{y}{4} + \frac{x}{4} + \frac{y}{4^2} + \frac{x}{4^2} + \frac{y}{4^3} + \frac{x}{4^3} + \cdots + \frac{y}{4^i} + \frac{x}{4^i} + \ldots,$$

з раціональними параметрами x та y. Зокрема, ми знаходимо необхідні та достатні умови за яких множина неповних сум такого ряду є скінченним об'єднанням відрізків, множиною канторівського типу або канторвалом.