UDC 512.548.7

TERNARY STRONGLY PARASTROPHIC-ORTHOGONAL QUASIGROUPS WITH 4 DISTINCT PARASTROPHES

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A triplet of ternary operations f_1, f_2, f_3 is called *orthogonal*, if the system of equations

$$\begin{cases} f_1(x_1,x_2,x_3)=a_1, \\ f_2(x_1,x_2,x_3)=a_2, \\ f_3(x_1,x_2,x_3)=a_3 \end{cases}$$

has a unique solution for all $a_1,a_2,a_3\in Q$. A set of ternary operations $\Sigma=\{f_1,f_2,\dots,f_s\}$, $s\geqslant 3$, is called *orthogonal*, if each triplet of distinct operations from Σ is orthogonal. Operations f_1,f_2,f_3 are called *strongly orthogonal* if the set of operations $\{f_1,f_2,f_3,e_1,e_2,e_3\}$ is orthogonal, where e_i defined by the equality $e_i(x_1,x_2,x_3)=x_i$ is called an *i-th selector*, $i\in\{1,2,3\}$.

A ternary quasigroup is called *parastrophic-orthogonal* if it has three orthogonal parastrophes and *totally parastrophic-orthogonal* (*a top quasigroup*) if its all distinct parastrophes are orthogonal.

For each permutation $\sigma\in S_4$, a *σ-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation f is defined by

$${}^{\sigma}f(x_{1\sigma},x_{2\sigma},x_{3\sigma})=x_{4\sigma} :\Longleftrightarrow f(x_1,x_2,x_3)=x_4.$$

The mapping $(\sigma,f)\mapsto {}^{\sigma}f$ is an action of the symmetric group S_4 on the set of all invertible ternary operations defined on a carrier and is called a parastrophic action [2]. The stabilizer group $\mathrm{Ps}(f)$ and the orbit $\mathrm{Po}(f)$ of f

$$\mathrm{Ps}(f):=\{\sigma\in S_4 \mid {}^{\sigma}f=f\}\leqslant S_4, \qquad \mathrm{Po}(f):=\{{}^{\sigma}f \mid \sigma\in S_4\}$$

are called a *parastrophic symmetry group* and a *parastrophic orbit* of the operation f respectively.

Consider $\mathrm{Ps}(f)=S_3$, where $S_3:=\{\iota,\ (12),\ (13),\ (23),\ (123),\ (132)\}$. Then all distinct parastrophes of the quasigroup operation f are representatives from the elements of the set $S_4/S_3=\{S_3,\ (14)S_3,\ (24)S_3,\ (34)S_3\}$.

Theorem 1. *[1] A ternary group isotope $(Q;f)$ has $\mathrm{Ps}(f)=S_3$ if and only if there exists an abelian group $(Q,+,0)$, its bijection α and an element $a\in Q$ such that*

$$f(x,y,z)=\alpha x+\alpha y+\alpha z+a. \qquad (1)$$

Theorem 2. *A triplet of parastrophes* ${}^{\sigma}f$, ${}^{\tau}f$, ${}^{\nu}f$ *of a medial quasigroup* $(Q; f)$ *with the group of parastrophic symmetry* S_3 *is strongly orthogonal if and only if* $\{\sigma, \tau, \nu\} = \{(14), (24), (34)\}$, f *has canonical decomposition* (1), *and* α, $\alpha + \iota$, $2\alpha - \iota$, $\alpha - \iota$ *are automorphisms of* $(Q; +)$.

Corollary 1. *Let* $(Q; f)$ *be a medial quasigroup with* (1) *and* $\mathrm{Ps}(f) = S_3$. *Then* $(Q; f)$ *is not a strongly top quasigroup.*

Theorem 3. *Let* $(Q; +)$ *be an abelian group and* φ *be its automorphism. Then the operations* f_1, f_2, f_3 *defined by*

$$f_1(x, y, z) = \varphi x + y + z, \qquad f_2(x, y, z) = x + \varphi y + z, \qquad f_3(x, y, z) = x + y + \varphi z$$

are strongly orthogonal quasigroup operations if and only if φ, $\varphi - 2\iota$, $\varphi - \iota$, $\varphi + \iota$ *are automorphisms of* $(Q; +)$.

The smallest order of a ternary quasigroup with the parastrophic symmetry group S_3 is 3 (see [3]).

Corollary 2. *Let* $\mathbb{Z}_m$ *be a ring of integers modulo* m. *There does not exist any strongly parastrophic-orthogonal quasigroup* $(\mathbb{Z}_m, f)$ *with the parastrophic symmetry group* S_3 *for even* m. *If* $m = p$ *is prime, then there exists a strongly parastrophic-orthogonal quasigroup* $(\mathbb{Z}_p, f)$ *with the parastrophic symmetry group* S_3 *for each* $p \geqslant 5$.

1. Sokhatsky, F., Pirus, Ye. (2018) *Classification of ternary quasigroups according to their parastrophic symmetry groups, I*, Visnyk DonNu. Series A: Natural Sciences, **1-2**, 70–82.

2. Sokhatsky, F. (2016) *Parastrophic symmetry in quasigroup theory*, Visnyk DonNu. Series A: Natural Sciences, **1-2**, 70–83.

3. McLeish, M. (1979) *On the number of conjugates of n-ary quasigroups*, Can. J. Math. **XXXI** (3), 637–654.

ТЕРНАРНІ СТРОГО ПАРАСТРОФНО-ОРТОГОНАЛЬНІ КВАЗІГРУПИ, ЯКІ МАЮТЬ 4 РІЗНИХ ПАРАСТРОФИ

Розглядаються тернарні квазігрупи з властивістю сильної парастрофної ортогональності, які мають 4 попарно різних парастрофи. Наведено критерій коли медіальна квазігрупа є сильно парастрофно-ортогональною і, як наслідок, один із методів побудови трійки сильного ортогональних тернарних квазігруп. Доведено існування зазначених квазігруп, зокрема, така циклічна квазігрупа існує для кожного простого порядку $p \geqslant 5$.