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THE FUNDAMENTAL SEQUENCES METHOD WITH TIME DISCRETIZATION FOR THE ELASTODYNAMIC PROBLEM

Ihor Borachok

Ivan Franko National University of Lviv, ihor.borachok@lnu.edu.ua

Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a simply connected domain with boundary Γ. We consider the following initial boundary value problem for the elastodynamic equation

$$\begin{cases} a\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} + b\dfrac{\partial \boldsymbol{u}}{\partial t} = \Delta^* \boldsymbol{u} & \text{in } D \times (0,T], \\ \mathcal{B}\boldsymbol{u} = \boldsymbol{f} & \text{on } \Gamma \times (0,T], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{0}, \ \ a\dfrac{\partial \boldsymbol{u}}{\partial t}(\cdot,0) = \boldsymbol{0} & \text{in } D, \end{cases} \tag{1}$$

where $\mathcal{B}$ is a vector-valued boundary operator, $\boldsymbol{f}$ is a given sufficiently smooth vector-valued function, $a \geq 0$, $b \geq 0$, $a+b > 0$, T is the final time, the operator Lamé is defined by

$$\Delta^* = c_s^2 \Delta + (c_p^2 - c_s^2)\,\mathrm{grad}\,\mathrm{div}$$

and the wave velocities c_s and c_p are given by

$$c_s = \sqrt{\frac{\mu}{\rho}}, \qquad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

where ρ denotes the density, λ and μ are the Lamé constants.

Problem (**??**) is solved numerically using a two-step method. Firstly, a temporal semi-discretization is carried out using the Houbolt finite difference scheme. The resulting sequence of stationary problems is then fully discretized using the fundamental sequences method.

The unknown vector-valued function $\boldsymbol{u}$ is approximated by the sequence $\{\boldsymbol{u}_n\}$, where $\boldsymbol{u}_n \approx \boldsymbol{u}(\cdot, t_n)$ at the time nodes

$$t_n = (n+3)h_t, \ n = -3, -2, \ldots, N, \ N \in \mathbb{N}, \ h_t = \frac{T}{N+3}.$$

The elements $\boldsymbol{u}_n$, $n = 0, 1, \ldots, N$, satisfy the following sequence of elliptic problems

$$\begin{cases} \Delta^* \boldsymbol{u}_n - \gamma^2 \boldsymbol{u}_n - \displaystyle\sum_{m=0}^{n-1} \beta_{n,m} \boldsymbol{u}_m = 0 & \text{in } \ D, \\ \mathcal{B}\boldsymbol{u}_n = \boldsymbol{f}_n & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\boldsymbol{f}_n = \boldsymbol{f}(\cdot, t_n)$, $\beta_{n,m}$ and γ are known coefficients.

The vector-valued functions $\boldsymbol{u}_n$, $n = 0, 1, \ldots, N$ are approximated by

$$\boldsymbol{u}_n(\boldsymbol{x}) \approx \boldsymbol{u}_{n,M}(\boldsymbol{x}) = \sum_{m=0}^{n} \sum_{j=1}^{M} \mathbf{N}_{n-m}(\boldsymbol{x}, \boldsymbol{y}_j)\boldsymbol{\alpha}_{m,j}, \ \boldsymbol{x} \in \overline{D}, \tag{3}$$

where $M \in \mathbb{N}$, $\boldsymbol{y}_j \in \mathbb{R}^d \setminus \overline{D}$, $j = 1, 2, \ldots, M$, denote the chosen source points, and $\mathbf{N}_n$, $n = 0, \ldots, N$ are known fundamental matrices, see [**?**, **?**]. The vector-coefficients $\boldsymbol{\alpha}_{n,j} \in \mathbb{R}^d$ are determined from the following recurrent linear systems, for $n = 0, \ldots, N$

$$\sum_{j=1}^{M} \big(\mathcal{B}\mathbf{N}_0(\boldsymbol{x}, \boldsymbol{y}_j)\big)\big|_{\boldsymbol{x}=\boldsymbol{x}_i} \boldsymbol{\alpha}_{n,j} = \boldsymbol{f}_n(\boldsymbol{x}_i) - \sum_{m=0}^{n-1} \sum_{j=1}^{M} \big(\mathcal{B}\mathbf{N}_{n-m}(\boldsymbol{x}, \boldsymbol{y}_j)\big)\big|_{\boldsymbol{x}=\boldsymbol{x}_i} \boldsymbol{\alpha}_{m,j},$$

for $i = 1, \ldots, M_1$, $M_1 \geq M$, with the chosen collocation points $\boldsymbol{x}_i \in \Gamma$.

The error of the approximation is given by

$$\max_{1\leq \ell \leq d} \| \, [\boldsymbol{u}(\cdot, t_n)]_\ell - [\boldsymbol{u}_{n,M}]_\ell \, \|_\infty \leq \hat{C} h_t + C e^{C_\beta(n)} \varrho^{-M}, \ n = 0, \ldots, N,$$

where $[\,\cdot\,]_\ell$ denotes the ℓ-th component, $\hat{C}$, C, $C_\beta(n) > 0$ are given constants and $\hat{\varrho} > 1$.

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МЕТОД ФУНДАМЕНТАЛЬНИХ ПОСЛІДОВНОСТЕЙ ІЗ ЧАСОВОЮ ДИСКРЕТИЗАЦІЄЮ ДЛЯ ЗАДАЧІ ЕЛАСТОДИНАМІКИ

Запропоновано двокроковий чисельний метод для розв'язання початково-крайової задачі для рівняння еластодинаміки. На першому кроці здійснюється часткова дискретизація за часом із використанням скінченно-різницевої схеми Хубольта, у результаті чого отримується послідовність еліптичних задач. На другому кроці отриману послідовність стаціонарних задач повністю дискретизовано до послідовності рекурентних систем лінійних алгебраїчних рівнянь за допомогою методу фундаментальних послідовностей.