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Normal subgroups of iterated wreath products of symmetric groups

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Normal subgroups and there properties for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded. Special classes of normal subgroups are investigated, their generators are found and presented as in the form of Kaloujnine [1] tables as well as in the form of wreath recursion.

Inverse limit of wreath product of permutation groups is found.

Definition 1. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0$, $[a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \tag{1}$$

be called set of type $\widetilde{A}_n^{(1)}$ and denote this set by $e \wr \widetilde{A}_n$.

Definition 2. The permutational subwreath product $G \wr\wr H$ is the semidirect product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [2] $\tilde{H}^X$ by the respective permutations of the subdirect factors.

Definition 3. *The subgroup $E \wr \widetilde{A}_n^{(2)}$ be denoted by $\widetilde{A}_n^{(3)}$.*

Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$.

Definition 4. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [1] form:
$[e]_0$, $[e, e, \ldots, e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{2}$$

be denoted by $\widetilde{A}_{n^2}^{(3)}$.

Definition 5. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

1. *elements of* $E \wr A_n$,
2. *elements with the tableau [1] presentation* $[e]_0,\ [\pi_1, \ldots, \pi_n]_1$, *that* $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T_n} \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ *of a subdirect product is subject of item 1) and 2).*

Theorem 1. *The monolith of* $S_n \wr S_m$ *is* $e \wr A_m$.

Proposition 1. *The order of* $\widetilde{T_n}$ *is* $\frac{(n!)^n}{2^{n-1}}$.

Theorem 2. *The subgroup* $\widetilde{A}_3^{(1)}$ *of* $S_3 \wr S_3$ *has the structure* $\widetilde{A}_3^{(1)} \simeq (C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)$.
The structure of subgroup $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ *is* $\widetilde{A}_n^{(1)} \simeq (\prod_{i=1}^{n} A_n) \rtimes (\prod_{i=1}^{n-1} C_2)$.

Theorem 3. *Proper normal subgroups in* $S_n \wr S_m$, *where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

1. *subgroups that act only on the second level are*

$$\widetilde{A}_m^{(1)}, \widetilde{T_m}, E \wr S_m, E \wr A_m,$$

2. *subgroups that act on both levels are* $A_n \wr\wr \widetilde{A}_m^{(1)},\ S_n \wr\wr \widetilde{B}_m^{(1)}$, $A_n \wr S_m$,

wherein the subgroup $S_n \wr\wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (1).

The subgroup $E \wr \widetilde{A}_n^{(1)}$ be denoted by $\widetilde{A}_n^{(2)}$.
Proposition 1. The subgroup $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$.

Theorem 4. *The subgroup* $\widetilde{A}_n^{(1)}$ *has* ***normal rank*** $n-1$ *[3] in* $S_n \wr S_n$, $n \geqslant 3$ *provided* $n \equiv 1 (mod 2)$ *and* ***normal rank*** n *iff* $n \equiv 0 (mod 2)$ *and* $n \geqslant 3$.

Definition 5. *A subgroup in* $S_n \wr S_n \wr S_n$ *is denoted by* $\widetilde{T}_{n^2}^{(3)}$ *if it consists of:*

1. *elements of the form* $E \wr E \wr A_n$,

2. *elements with the tableau [1] presentation* $[e]_1,\ [e\ldots,e]_2,\ [\pi_1\ldots,\pi_n,\pi_{n+1}\ldots,\pi_{n^2}]$ *wherein* $\forall i=1,\ldots,n$*:* $\pi_i \in S_n \setminus A_n$.

Definition 6. The set of elements from $\wr_{i=1}^{k} S_{n_i}, n_i \geqslant 3$ with depth m satisfying the following condition

$$\sum_{i=1}^{n^j} dec([a_i]_j) = 2t, t \in \mathbb{N},\ m \le j \le k,\quad [a_i]_j = e,\ whenever\ j = \overline{1, m-1} \quad (3)$$

be called $\widetilde{A}_{n^j}^{(m,k)}$, where $m<k$.

Theorem 5. *The order of normal subgroup* $\widetilde{A}_{n^j}^{(0,k-1)}$ *is* $(\frac{1}{2})^k \cdot (n!)^{(\frac{n^{(k+1)}-1}{n-1})}$ *and the order of the quotient* $\wr_{i=1}^{k} S_{n_i} \Big/ \widetilde{A}_{n^j}^{(0,k-1)}$ *is* 2^k. *The order of generalized alternating group of* k*-th level* $\widetilde{A}_{n^k}^{(k)}$ *is* 2^{n^k-1}.

Theorem 6. *The quotient* $\wr_{i=1}^{k} S_{n_i}$ *by* $\widetilde{A}_{n^j}^{(0,k-1)}$ *is the following group* $\prod_{i=1}^{k} \mathbb{Z}_2$.

The set of normal subgroup of $S_n \wr S_n$ is denoted by $N(S_n \wr S_n)$. Subgroup with number i from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$.

Theorem 2. The full list of normal subgroups of $S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:

1 **Type** T_{023} contains: $E \wr \tilde{A}_n \wr H$, $\widetilde{T_n} \wr H$, where $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. There are 6 subgroups.

2 **The second type of subgroups is subclass in** T_{023} with new base of wreath product subgroup $\tilde{A}_{n^2}$: $E \wr S_n \wr \tilde{A}_{n^2}$, $E \wr N_i(S_n \wr S_n)$. Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in **Type** T_{023} is 18.

3 **Type** T_{003}: $A^{(3)}_{00(n^2)} = E \wr E \wr \tilde{A}_{n^2}$, $\widetilde{T_{n^2}}$, $\widetilde{T_n}^{(3)}$. Hence, here are 3 new subgroups.

4 **Type** T_{123}: $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ and $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$. Thus, there are 29 new normal subgroups in T_{123}, taking into account repetition.

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