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Continuity Properties of Solutions to the Anisotropic N–Laplacian with L^1 Lower-Order Term

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For an open bounded set $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, $N \geqslant 2$ we consider the anisotropic elliptic equation

$$-\sum_{i=1}^{N}(a_i(x,\nabla u))_{x_i} = f(x), \quad x \in \Omega, \quad f(x) \in L^1(\Omega), \tag{1}$$

where $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are measurable functions satisfying the conditions

$$\begin{cases} \sum\limits_{i=1}^{N} a_i(x,\xi)\,\xi_i \geqslant K_1 \sum\limits_{i=1}^{N} |\xi_i|^{p_i}, \quad 1 < p_1 \leqslant p_2 \leqslant ... \leqslant p_N, \quad \sum\limits_{i=1}^{N} \frac{1}{p_i} = 1, \\ |a_i(x,\xi)| \leqslant K_2 \Big(\sum\limits_{j=1}^{N} |\xi_j|^{p_j} \Big)^{1-\frac{1}{p_i}}, \quad i = 1, ..., N, \end{cases} \tag{2}$$

where K_1 and K_2 are positive constants, and $f(x)$ satisfies conditions that will be specified below.

Equation (1) without lower order term was studied in [1], using the strategy from [2], under the condition $\sum\limits_{i=1}^{N} \frac{1}{p_i} = 1$, with no additional restrictions on the values p_i. In this paper, we establish the continuity of solutions to (1)-(2) by following the similar strategy but employing a different iteration, specifically, the Kilpeläinen–Malý technique, properly adapted to the anisotropic equations.

Before formulating the main results, let us remind the reader of the definition of a weak solution to equation (1). We extend the function f by zero outside Ω, i.e., on $\mathbb{R}^N \setminus \Omega$, and, for any $\rho > 0$, set

$$\mathcal{W}^{|f|}_{1,N}(\rho) := \sup_{x\in\Omega} \mathcal{W}^{|f|}_{1,N}(x,\rho) = \int\limits_0^{\rho} \Big(\int\limits_{B_r(x)} |f(z)| dz \Big)^{\frac{1}{N-1}} \frac{dr}{r}. \tag{3}$$

Define the anisotropic Sobolev spaces $W^{1,\boldsymbol{p}}(\Omega)$ and $W_0^{1,\boldsymbol{p}}(\Omega)$ as follows

$$W^{1,\boldsymbol{p}}(\Omega) := \{u \in W^{1,1}(\Omega), \quad u_{x_i} \in L^{p_i}(\Omega), \quad i = 1, ..., N\},$$
$$W_0^{1,\boldsymbol{p}}(\Omega) := \{u \in W_0^{1,1}(\Omega), \quad u_{x_i} \in L^{p_i}(\Omega), \quad i = 1, ..., N\}.$$

Definition 1. We say that a function $u \in W^{1,\boldsymbol{p}}(\Omega) \cap L^{\infty}(\Omega)$ is a bounded weak solution of equation (1) under conditions (2) if the following identity holds

$$\sum_{i=1}^{N} \int\limits_{\Omega} a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i} \, dx = \int\limits_{\Omega} f(x) \, \varphi dx, \tag{4}$$

for arbitrary $\varphi \in W_0^{1,\boldsymbol{p}}(\Omega) \cap L^{\infty}(\Omega), \ \varphi \geqslant 0$.

Our main result in this paper reads as follows.

Theorem 1. *[3] Let u be a bounded weak solution of* (1)-(2)*, and assume also that*

$$\lim_{\rho \to 0} \mathcal{W}_{1,N}^{|f|}(\rho) = 0, \tag{5}$$

then $u \in C(\Omega)$.

Remark 1. In the case $p_1 = ... = p_N = N$ and $f = 0$ or $f \neq 0$, we cover the known results, see, for example [4–7].

1. Ciani S., Henriquez E., Skrypnik I.I. *On the continuity of solutions to anisotropic elliptic operators in the limiting case*, Bull. of the London Math. Soc. https://doi.org/10.1112/blms.70047

2. Skrypnik I.I., Voitovych M. V. *On the generalized $U_{m,p}^{f}$ classes of De Giorgi-Ladyzhenskaya- Ural'tseva and pointwise estimates of solutions to high-order elliptic equations via Wolff potentials*, J. Diff. Equations. 2020, **268(11)**, 6778-6820.

3. Savchenko, M. O., Skrypnik, I. I., Yevgenieva, Y. A. *On the continuity of solutions to the anisotropic N-Laplacian with L^1 lower order term*, arXiv:2505.03381.

4. Albercio A., Cianchi A., Sbordone C. *On the modulus of continuity of solutions to the n-Laplace equation*, J. Elliptic Parabolic Equations. 2015, **1(1)**, 1-12.

5. Albercio A., Ferone V. *Regularity properties of solutions of elliptic equation in $\mathbb{R}^n$ in limit cases*, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni. 1995, **6(4)**, 237-250.

6. Iwaniec T., Onninnen J. *Continuity estimates for n-harmonic equations*, Indiana Univ. Math. J. 2007, **56(2)**, 805-824.

7. Jiang R., Koskela P., Yang D. *Continuity of solutions to n- harmonic equations*, Manuscripta Math. 2012, **139(1-2)** , 237-248.

Властивості неперервності розв'язків анізотропного N–Лапласіана з молодшим членом у L^1

Ми встановили неперервність обмежених розв'язків анізотропного еліптичного рівняння

$$-\sum_{i=1}^{N}\Big(|u_{x_i}|^{p_i-2}u_{x_i}\Big)_{x_i} = f(x), \quad x\in\Omega, \quad f(x)\in L^1(\Omega)$$

за умов

$$\min_{1\leqslant i\leqslant N} p_i > 1, \quad \sum_{i=1}^{N}\frac{1}{p_i} = 1$$

та

$$\lim_{\rho\to 0}\sup_{x\in\Omega}\int\limits_0^{\rho}\Big(\int\limits_{B_r(x)}|f(y)|\,dy\Big)^{\frac{1}{N-1}}\frac{dr}{r} = 0.$$

У стандартному випадку $p_1 = ... = p_N = N$ *ці умови покривають відомі результати для* N*-Лапласіана.*