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DESCENT PROPERTY FOR EXTRAGRADIENT ALGORITHM

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Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Consider the variational inequality problem:

$$\text{find } x \in C : \ \ \langle Ax, y - x \rangle \geq 0 \quad \forall y \in C, \tag{1}$$

where C is a non-empty subset of Hilbert space H, A is an operator mapping from H in H. Denote the solution set of (1) as S.

Assume the following conditions hold:

- the set $C \subseteq H$ is convex and closed;
- the operator $A : H \to H$ is monotone on C, i.e.,

  $$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C,$$

  and Lipschitz continuous on C (with constant $L > 0$), i.e.,

  $$\|Ax - Ay\| \leq C\|x - y\| \quad \forall x, y \in C.$$

Consider the extragradient algorithm for the variational inequality (1)

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right), \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right), \end{cases} \tag{2}$$

where $\lambda > 0$ [1].

We obtain the following result.

Theorem 1 (Descent Property)**.** *Let $C \subseteq H$ be a nonempty, convex, and closed set, and $A : C \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$). Then, for the sequences (x_n) and (y_n) generated by the extragradient algorithm (2), the following inequality holds:*

$$\begin{aligned} &\|x_{n+2} - x_{n+1}\|^2 + \|x_{n+2} - y_{n+1}\|^2 + 2\lambda\langle Ax_{n+1} - Ay_{n+1}, x_{n+1} - x_{n+2}\rangle \\ &\quad \leq \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_n - Ay_n, x_n - x_{n+1}\rangle \\ &\qquad\qquad - (1 - \lambda^2 L^2)\|y_n - x_{n+1}\|^2. \end{aligned}$$

In the absence of constraints ($C = H$), the algorithm takes the form

$$\begin{cases} y_n = x_n - \lambda A x_n, \\ x_{n+1} = x_n - \lambda A y_n, \end{cases} \tag{3}$$

where $\lambda > 0$.

From Theorem 1, we obtain the following results.

Theorem 2 (Descent Property)**.** *Let $A : H \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$). Then, for the sequences (x_n) and (y_n) generated by the extragradient algorithm* (3)*, the following inequality holds:*

$$\|Ax_{n+1}\|^2 \le \|Ax_n\|^2 - (1 - \lambda^2 L^2)\|Ay_n - Ax_n\|^2. \tag{4}$$

Theorem 3. *Let $A : H \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$), $A^{-1}0 \ne \emptyset$. Assume that $\lambda \in \left(0, \frac{1}{L}\right)$. Then, for the sequence (x_n) generated by algorithm* (3)*, the inequality holds.*

$$\|Ax_N\|^2 \le \frac{D^2}{(1 - \lambda^2 L^2)\lambda^2 (N+1)}, \quad N \ge 0, \tag{5}$$

where $D = d(x_0, A^{-1}0) = \min_{z \in A^{-1}0} \|x_0 - z\|$.

The results obtained in this work are new and refine the existing ones [2,3].

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2. Gorbunov E., Loizou N., Gidel G. *Extragradient method: O(1/K) last-iterate convergence for monotone variational inequalities and connections with cocoercivity.* 2021. URL: https://arxiv.org/abs/2110.04261.

3. Cai Y., Oikonomou A., Zheng W. *Tight last-iterate convergence of the extragradient and the optimistic gradient descent-ascent algorithm for constrained monotone variational inequalities.* 2022. URL: https://arxiv.org/abs/2204.09228.

ВЛАСТИВІСТЬ СПУСКУ ДЛЯ ЕКСТРАГРАДІЄНТНОГО АЛГОРИТМУ

У повідомленні розглянуто екстраградієнтний алгоритм для монотонних варіаційних нерівностей та операторних рівнянь. Доведено новий результат щодо властивості спуску екстраградієнтного алгоритму, а для задач без обмежень отримано оцінку швидкості алгоритму в термінах природної нев'язки за слабших умов, ніж відомі.