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Generic boundary-value problems in Sobolev spaces with a parameter

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We arbitrarily choose a finite interval $(a,b) \subset \mathbb{R}$ and the parameters: $n \in \mathbb{N} \cup \{0\}$, $\{m,r\} \subset \mathbb{N}, 1 \le p \le \infty$. Let $W_p^{n+r}\big([a,b];\mathbb{C}\big)$ be a complex Sobolev space $W_p^{n+r} = \big\{y \in C^{n+r-1}([a,b];\mathbb{C})\colon\ y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\big\}$.

Let $\mathcal{M}$ be a metric space, a parameter $\mu \in \mathcal{M}$ runs through the space $\mathcal{M}$, and an arbitrary point $\mu_0 \in \mathcal{M}$ is fixed. We consider a linear boundary-value problem

$$L(\mu)y(t,\mu) := y^{(r)}(t,\mu) + \sum_{j=1}^{r} A_{r-j}(t,\mu)y^{(r-j)}(t,\mu) = f(t,\mu), \qquad (1)$$

$$B(\mu)y(\cdot;\mu) = c(\mu), \quad t \in (a,b). \qquad (2)$$

Here, for each $\mu \in \mathcal{M}$, the unknown vector-valued function $y(\cdot,\mu) \in (W_p^{n+r})^m$, and we arbitrarily choose the matrix-valued functions $A_{r-j}(\cdot,\mu) \in (W_p^{n})^{m\times m}$, with $j \in \{1,\dots,r\}$, the vector-valued function $f(\cdot,\mu) \in (W_p^{n})^m$, the vector $c(\mu) \in \mathbb{C}^{rm}$, and the linear continuous operator $B(\mu)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$.

The boundary condition (2) consists of rm scalar condition for system of m differential equations of r-th order, we representing vectors and vector-valued functions as columns. A solution to the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot,\mu) \in (W_p^{n+r})^m$ that satisfies both equation (1) (everywhere if $n \ge 1$, and almost everywhere if $n = 0$) on (a,b) and equality (2). The boundary conditions may contain derivatives of the unknown functions of integer and fractional orders that exceed the order of the differential system.

We rewrite the problem (1), (2) in the form of a linear operator equation $\big(L(\mu),B(\mu)\big)y(\cdot;\mu) = (f(\mu),c(\mu))$. Here, $\big(L(\mu),B(\mu)\big)$ is a bounded linear operator on the pair of Banach spaces

$$\big(L(\mu),B(\mu)\big)\colon (W_p^{n+r})^m \to (W_p^{n})^m \times \mathbb{C}^{rm}. \qquad (3)$$

According to [1, Theorem 1], the operator (3) is a bounded Fredholm operator with zero index for every $\mu \in \mathcal{M}$.

Definition 1. A solution to the boundary-value problem (1), (2) depends continuously on the parameter μ at point μ_0 if the following two conditions are satisfied:

- $(*)$ For any μ from some neighborhood of the point μ_0 and arbitrary right-hand sides $f(\cdot;\mu) \in (W_p^n)^m$ and $c(\mu) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\mu)$ from the space $(W_p^{n+r})^m$;
- $(**)$ The convergence of the right-hand sides $f(\cdot;\mu) \to f(\cdot;\mu_0)$ in $(W_p^n)^m$ and $c(\mu) \to c(\mu_0)$ in $\mathbb{C}^{rm}$ as $\mu \to \mu_0$ implies the convergence of the solutions $y(\cdot;\mu) \to y(\cdot;\mu_0)$ in $(W_p^{n+r})^m$ as $\mu \to \mu_0$.

Let us consider the following conditions.

Condition (0). The homogeneous boundary-value problem has only the trivial solution $L(\mu_0)y(t;\mu_0) = 0, \quad t \in (a,b), \quad B(\mu_0)y(\cdot;\mu_0) = 0.$

Limit Conditions as $\mu \to \mu_0$:

- (I) $A_{r-j}(\cdot;\mu) \to A_{r-j}(\cdot;\mu_0)$ in the space $(W_p^n)^{m\times m}$ for each number $j \in \{1,\ldots,r\}$;
- (II) $B(\mu)y \to B(\mu_0)y$ in the space $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

Theorem 1. *A solution to the boundary-value problem* (1), (2) *depends continuously in the parameter* μ *at point* μ_0 *if and only if this problem satisfies conditions* (0), (I), *and* (II).

In the case of $r = 1$, $\mathcal{M} = [0, \varepsilon_0)$, $\varepsilon_0 > 0$, $\mu_0 = 0$, Theorem 1 is proved in [2, Theorem 1].

1. Mikhailets V., Atlasiuk O. *The solvability of inhomogeneous boundary-value problems in Sobolev spaces.* Banach J. Math. Anal. 2024 **18(2)** (12), 23p.
2. Atlasiuk O.M., Mikhailets V.A. *Fredholm one-dimensional boundary-value problems with parameter in Sobolev spaces*, Ukrain. Math. J. 2019 **70** (11), 1677–1687.

Загальні крайові задачі з параметром в просторах Соболєва

Досліджено найбільш загальний клас лінійних неоднорідних крайових задач для систем звичайних диференціальних рівнянь довільного порядку в просторах Соболева. Знайдено конструктивні необхідні і достатні умови неперервності їх розв'язків за параметром, що належать деякому метричному простору. Встановлено двосторонню оцінку порядку збіжності цих розв'язків до розв'язку граничної задачі.