Harnack’s inequality for quasilinear elliptic equations with nonstandard growth conditions

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We consider quasilinear elliptic equations of the form

$$\text{div} \left( g(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0, \quad x \in \Omega, \quad (1)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$.

Throughout the paper we suppose that the function $g(x, v) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ := [0, +\infty)$, satisfies the following assumptions:

(g) $g(\cdot, v) \in L^1(\Omega)$ for all $v \in \mathbb{R}_+$, $g(x, \cdot)$ is continuous and non-decreasing for almost all $x \in \Omega$, $\lim_{v \to +0} g(x, v) = 0$ and $\lim_{v \to +\infty} g(x, v) = +\infty$;

(g$_1$) there exist $c_1 > 0$, $q > 1$ and $b_0 \geq 0$ such that

$$\frac{g(x, w)}{g(x, v)} \leq c_1 \left( \frac{w}{v} \right)^{q-1}, \quad (2)$$

for all $x \in \Omega$ and for all $w \geq v \geq b_0$;

(g$_2$) there exists $p > 1$ such that

$$\frac{g(x, w)}{g(x, v)} \geq \left( \frac{w}{v} \right)^{p-1}, \quad (3)$$

for all $x \in \Omega$ and for all $w \geq v > 0$;

(g$_3$) for any $K > 0$ and for any ball $B_{r}(x_0) \subset \Omega$ there exists $c_2(K) > 0$ such that

$$g(x_1, v/r) \leq c_2(K) e^{\lambda(r)} g(x_2, v/r),$$

for all $x_1, x_2 \in B_r(x_0)$ and for all $r \leq v \leq K$. Here $\lambda(r) : (0, r_*) \to \mathbb{R}_+$ is a continuous, non-increasing function, satisfying the conditions described below.

Our main result is Harnack’s inequality for bounded weak solutions to equation (1).

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Theorem 1 (Harnack inequality). Fix a point \( x_0 \in \Omega \) and consider the ball \( B_{8\rho}(x_0) \subset \Omega \). Let conditions \((g), (g_1)-(g_3)\) be fulfilled, and let \( u \) be a nonnegative bounded weak solution to Eq. (1). Then there exist positive constants \( C, c, \beta \) depending only on the data such that

\[
\text{ess sup}_{B_{\rho}(x_0)} u \leq C \Lambda(c, \beta, \rho) \left( \text{ess inf}_{B_{\rho}(x_0)} u + (1 + b_0)\rho \right),
\]

where \( \Lambda(c, \beta, \rho) := \exp \left( c \exp \left( \beta \lambda(\rho) \right) \right) \).