Let us discuss the laws for the growth of the cells. The external scale $l(\tau)$ increases because of adsorption of single cells by other cells. The law for its growth is determined by the nature of the initial perturbations $\Phi(\mathbf{r})$. Let the spectral density $g_\Phi(k)$ of these perturbations decrease rapidly to zero in the limit $k\to\infty$. This assumption is, in the framework of the adiabatic scenario, justified by the suppression of small-scale modes in the course of the recombination.[5]

Let us determine the characteristic size of the region in which the absolute minima are found from the condition that the increase of the paraboloid and the initial action are equal:

$$l^2(\tau)/2\tau \sim \sqrt{D(l(\tau))}, \quad D(\rho) = \langle [\Phi(\rho) - \Phi(0)]^2 \rangle. \tag{14}$$

From Eq. (14) it follows that the laws for the increase of $l(\tau)$ are qualitatively different if the dispersion of the initial action is limited, $\langle\Phi\rangle^2 = \Phi_\Phi^2 < \infty$, and if

$$g_\Phi(k) \sim \kappa^2 k^{-\theta}, \quad 3<\theta<5, \quad k\to 0. \tag{15}$$

In the first case, $D(\rho \gg l_0) \approx 2\sigma_\Phi^2$ and from Eq. (14) we find

$$l(\tau) \sim \sqrt{\sigma_\Phi \tau} \sim l_0 (\tau/\tau^*)^{1/2}; \tag{16}$$

here the increase in $l(\tau)$ is determined only by the integral characteristics of the spectrum of the initial action and is independent of its fine structure. In the second case, in which $D(\rho) \sim \kappa^2 \rho^{\theta-3}$, Eq. (14) gives

$$l(\tau) \sim \sqrt{\kappa}\, \tau^{2/(7-\theta)}. \tag{17}$$

The physical difference between these two cases of the initial perturbations is the absence (in the first case) and the presence (in the second case) of slowly decreasing spatial correlations of the initial perturbations.

Which case is realized in the universe depends on the initial density fluctuations. We assume that the spectrum of these fluctuations is $g_\rho(k) \sim k^n$. The gravitational instability leads to velocity fluctuations whose action spectrum, found in a linear approximation, is $g_\Phi \sim g_\rho k^{-4}$. We thus find that $\theta = 4 - n$ in Eq. (15) and if $n \geq 1$, Eq. (16) will be valid.

In the case $\sigma_\Phi^2 < \infty$, the velocity field and the cell structure can be analyzed in detail statistically because of the presence of the small parameter $\epsilon = l_0/l$ at $\tau \gg \tau^*$ (Ref. 9). The one-point and two-point probability densities, the spectra, and more accurate laws for the increase of $l(\tau)$ have been found for the velocity field.[9]

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[1]J. H. Oort, Ann. Rev. Astron. Astrophys. 21, 373 (1983).
[2]Ya. B. Zeldovich, J. Einasto, and S. F. Shandarin, Nature 300, 407 (1982).
[3]Ya. B. Zeldovich, Astrofizika 6, 319 (1970).
[4]Ya. B. Zel'dovich, Pis'ma Astron. Zh. 8, 195 (1982).
[5]S. F. Shandarin, A. G. Doroshkevich, and Ya. B. Zel'dovich, Usp. Fiz. Nauk 139, 83 (1983) [Sov. Phys. Usp. 26, 46 (1983)].
[6]A. A. Klypin and S. F. Shandarin, Mon. Not. R. Astron. Soc. 204, 891 (1983).
[7]S. F. Shandarin and Ya. B. Zel'dovich, Phys. Rev. Lett. 52, 1488 (1984).
[8]Ya. B. Zel'dovich, Proc. Nat. Acad. Sci. USA 80, 2410, (1983).
[9]S. N. Gurbatov and A. I. Saichev, Izv. Vyssh. Uchebn. Zaved., Radiofiz. 27, 456 (1984).
[10]J. M. Burgers, The Non-Linear Diffusion Equation, Reidel, Dordrecht (1974).
[11]Sh. Kida, J. Fluid Mech. 93, 337 (1979).
[12]S. N. Gurbatov, A. I. Saichev, and I. G. Yakushkin, Usp. Fiz. Nauk 141, 221 (1983) [Sov. Phys. Usp. 26, 857 (1983)].
[13]N. N. Kuznetsov and B. L. Rozhdestvenskii, Zh. Vyschisl. Mat. Mat. Fiz. 1, 417 (1961).

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Lagrangian analysis of invariant third-order equations of motion in relativistic mechanics of classical particles

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Interest has recently increased in the description of motion of classical scalar and spin particles in gauge and gravitational fields. The corresponding equations generalize the Lorentz and Papapetrou–Pirani equations.[1,2] At the same time, the Mathisson–Papapetrou equations with Pirani's condition $u_\beta S^{\alpha\beta} = 0$ ($u^\alpha = \dot{x}^\alpha$) lead to the third-order equation of motion[3]

$$m_0 \frac{Du^\alpha}{ds} = S^{\alpha\beta} \frac{D^2 u_\beta}{ds^2} - \frac{1}{2} R^\alpha{}_{\beta\gamma\sigma} u^\beta S^{\gamma\sigma}, \tag{1}$$

$$u_\beta u^\beta = 1.$$

The Mathisson–Papapetrou equations, like the earlier equations of Frenkel, are derived by many authors from a variational principle. However, until now the problem of the existence of a Lagrangian for the Mathisson equation (1) was not investigated. On the other hand, if it is not assumed that the particle is a test particle, the force of radiation friction in the equation of motion will also contain a third-order term,[4,5] like, say, in the Lorentz–Dirac equations

$$m_0 \left[\frac{\dot{u}^\alpha}{\|u\|^2} - \frac{\dot{u}_\beta u^\beta}{\|u\|^4} u^\alpha \right]$$

$$= \frac{2}{3} e^2 \left[\frac{\ddot{u}^\alpha}{\|u\|^3} - \frac{\ddot{u}_\beta u^\beta}{\|u\|^5} u^\alpha - 3 \frac{\dot{u}_\beta u^\beta}{\|u\|^5} \dot{u}^\alpha + 3 \frac{(\dot{u}_\beta u^\beta)^2}{\|u\|^7} u^\alpha \right]$$

$$+ \frac{e}{\|u\|} F^{\alpha\beta} u_\beta. \tag{2}$$

The question of whether there exists a Lagrangian for the Lorentz–Dirac equation has not yet been settled.[6] Equation (2) is invariant under changes of parametrization of

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integral curves. Upon setting $m_0=0$ and $F^{\alpha\beta}=0$, this equation becomes the equation of geodesic circles, which defines the hyperbolic motion of relativistic particles. Equation (2) was investigated in connection with the study of reference frames which are accelerated equally.[7]

A parameter-invariant form of the Mathisson equations (1) may be obtained directly from the Dirac equations

$$\frac{Dp^\alpha}{d\lambda} = \frac{1}{2} R^\alpha{}_{\beta\gamma\sigma} u^\beta S^{\gamma\delta}, \qquad \frac{DS^{\alpha\beta}}{d\lambda} = 2p^{[\alpha}u^{\beta]},$$

where λ is an arbitrary parameter along the world line. Here we propose an equivalent formulation in terms of the spin four-vector

$$\|u\|\sigma_\alpha = \frac{\sqrt{|g|}}{2}\,\epsilon_{\alpha\beta\gamma\delta}u^\beta S^{\gamma\delta}: \tag{3}$$

namely,

$$m_0\|u\|\left[(u\cdot u)^2\frac{Du_\alpha}{dz} - (u\cdot u)u_\beta\frac{Du^\beta}{dz}u_\alpha\right]$$
$$= \sqrt{|g|}\,\|u\|\sigma^\nu\left[(u\cdot u)\epsilon_{\alpha\beta\gamma\nu}\frac{D^2u^\beta}{dz}u^\gamma\right.$$
$$\left. -3u_\delta\frac{Du^\delta}{dz}\epsilon_{\alpha\beta\gamma\nu}\frac{Du^\beta}{dz}u^\gamma - \frac{(u\cdot u)^2}{2}\epsilon_{\gamma\delta\mu\nu}R_{\alpha\beta}{}^{\gamma\delta}u^\beta u^\mu\right]. \tag{4}$$

We must add to Eq. (4) the condition $\|u\|u_\beta\sigma^\beta=0$ and the the spin part of Dixon's equation

$$\|u\|^3\frac{D\sigma^\alpha}{d\lambda} + \|u\|\sigma_\beta\frac{Du^\beta}{d\lambda}u^\alpha = 0. \tag{5}$$

If the world line of the particle is a null geodesic, then expression $\|u\|\sigma_\alpha$ in Eqs. (4) and (5) must be understood in the context of (3). If the particle has no rest mass, we must set $\|u\|m_0=0$ (Ref. 9) in Eq. (4). For the motion of a particle with nonzero rest mass in flat space–time, the spin four-vector remains constant and Eq. (5) drops out.

Lagrange functions with higher derivatives have been used for a long time in mechanics of classical particles.[10,11] The corresponding ordinary higher-order Euler–Lagrange equations are also called Euler–Poisson equations. In this article we investigate the question of the existence in pseudo-Euclidean space of invariant Euler–Poisson third-order equations. The variational problem is locally defined by a Lagrangian density $L(t, \mathbf{x}, \mathbf{v}, \mathbf{v}')dt$ on the space $J_2(\mathbf{R}, \mathbf{R}^{n-1})$ of second-order jets. Let $\mathbf{p}: T_r(\mathbf{R}^n) \to J_r(\mathbf{R}, \mathbf{R}^{n-1})$ be the local expression of the canonical projection of the space $T_n(\mathbf{R}^n) = J_r(\mathbf{R}, \mathbf{R}^n)\cdot(0)$ of 1^r-velocities on the manifold $C_r(\mathbf{R}^n, 1)$ of one-dimensional contact elements in $\mathbf{R}^n$. Identifying $\mathbf{R}^n$ with the direct product $\mathbf{R}\times\mathbf{R}^{n-1}$, we denote by $x=(t, \mathbf{x})$, $u = x = (u^0, \mathbf{u})$, $\dot{u}$, ..., $\overset{(r)}{u}$ the canonical coordinates on $T_r(\mathbf{R}^n)$. In the space $\mathbf{R}^n$, there arises a parameter-invariant variational problem with Lagrangian $\mathcal{L}(x, u, \dot{u}) = u^0L(t, \mathbf{x}, \mathbf{v}\circ\mathbf{p}, \mathbf{v}'\circ\mathbf{p})$, defined locally on $T_2(\mathbf{R}^n)$. We denote by $\mathbf{E}$ and $\mathcal{E} = (\mathcal{E}_0, \underline{\mathcal{E}})$ the Euler–Poisson expressions generated by the Langrangians L and $\mathcal{L}$, respectively. Then $\mathcal{E}_0u^0 + \underline{\mathcal{E}}.\mathbf{u} = 0$ and the relation $\underline{\mathcal{E}}(x, u, \dot{u}, \ddot{u}) = u^0\mathbf{E}(t, \mathbf{x}, \mathbf{v}\circ\mathbf{p}, \mathbf{v}'\circ\mathbf{p}, \mathbf{v}''\circ\mathbf{p})$. holds. The arbitrary expression $\mathbf{E}(t, \mathbf{x}, \mathbf{v}', \mathbf{v}'')$ is an Euler–Poisson expression if and only if

$$\mathbf{E} = \mathbf{A}.\mathbf{v}'' + (\mathbf{v}'.\vec{\partial}_v)\mathbf{A}.\mathbf{v}' + \mathbf{B}.\mathbf{v}' + \mathbf{c}, \tag{6}$$

where the skew-symmetric matrix $\mathbf{A}$, the matrix $\mathbf{B}$, and the row $\mathbf{c}$ depend only on the variables t, $\mathbf{x}$, and $\mathbf{v}'$ and satisfy the known Lagrangianity condition.[12] The Euler morphism[13] $E: J_3(\mathbf{R}, \mathbf{R}^{n-1}) \to T^*(\mathbf{R}^{n-1})\otimes_{\mathbf{R}^n} T^*(\mathbf{R})$, $E(t, \mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{v}'') = \mathbf{E}(t, \mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{v}'')\otimes dt$, is an affine mapping over $J_2(\mathbf{R}, \mathbf{R}^{n-1})$. The Pfaff form with values in the vector space $\mathbf{R}^{n-1*}$ is therefore naturally associated with it: it:

$$\underline{\epsilon} = (\epsilon_i) = \mathbf{A}.d\mathbf{v}' + \underline{\kappa}\,dt, \qquad \underline{\kappa} = (\mathbf{v}'.\vec{\partial}_v)\mathbf{A}.\mathbf{v}' + \mathbf{B}.\mathbf{v}' + \mathbf{c}.$$

The form $\underline{\epsilon}$ and the $T(\mathbf{R}^{n-1})$-valued contact form $\theta = \frac{\partial}{\partial x^i}\otimes(dx^i - v^i dt) + \frac{\partial}{\partial v^i}\otimes(dv^i - v'^i dt)$ generate a module $\mathfrak{M}(\underline{\epsilon}, \underline{\theta})$ over the algebra of differential forms on the space $J_2(\mathbf{R}, \mathbf{R}^{n-1})$ with values in the vector space End $[\mathbf{R}^{n-1*}\oplus T(\mathbf{R}^{n-1})]$. We assume that the pseudogroup Γ of transformations, which is generated by the vector field $X = \tau\partial_t + \xi\cdot\partial_x$, acts in $\mathbf{R}^n$ and we also assume that $X_{(r)} = X + \Sigma_{l=0}^{r-1}\underline{\xi}^{(l+1)}.\vec{\partial}_{v(l)}$ is the extension of the generator X on the space $J_r(\mathbf{R}, \mathbf{R}^{n-1})$. The infinitesimal condition of invariance of the Euler–Poisson equations under the pseudogroup Γ is expressed by the invariance of the module $\mathfrak{M}(\underline{\epsilon}, \underline{\theta})$ under the action of the vector field $X_{(2)}$. The defining equations have the form

$$X_{(1)}(\mathbf{A}) = \underline{\Lambda}.\mathbf{A} - \mathbf{A}.\underline{\xi}^{(2)}_{,\mathbf{v}},$$
$$X_{(2)}(\underline{\kappa}) = \underline{\Lambda}.\underline{\kappa} - \mathbf{A}.(\partial_t + \mathbf{v}.\vec{\partial}_x + \mathbf{v}'.\vec{\partial}_v)\underline{\xi}^{(2)} - \underline{\kappa}(\partial_t + \mathbf{v}.\vec{\partial}_x)\tau. \tag{7}$$

The matrix Λ is used as an undefined multiplier. In the following by invariance we mean the invariance under pseudo-orthogonal transformations.

Suppose the Lagrangian $\mathcal{L}$ is translation-invariant. Solving the partial differential equation (7) in conjunction with the Lagrangianity condition,[12] we can prove the following assertions:

1. In three-dimensional pseudo-Euclidean space there exists only a one-parameter family of third-order invariant Euler–Poisson equations:

$$\frac{m}{\|u\|^3}[(u\cdot u)\dot{u} - (\dot{u}\cdot u)u] = \frac{\ddot{u}\times u}{\|u\|^3} - 3\frac{\dot{u}\cdot u}{\|u\|^5}\dot{u}\times u. \tag{8}$$

If $m=0$, Eq. (8) is algebraically equivalent to the equation of geodesic circles.

2. In four-dimensional pseudo-Euclidean space there are no invariant third-order Euler–Poisson equations. In this case however, we can, find an invariant family of Euler–Poisson equations

$$\frac{m}{\|\sigma\|^3}\left[\frac{\dot{u}}{\|u\|} - \frac{\dot{u}\cdot u}{\|u\|^3}u\right] = -\frac{*\ddot{u}\wedge u\wedge\sigma}{\|\sigma\wedge u\|^3}$$
$$+ 3\frac{*\dot{u}\wedge u\wedge\sigma}{\|\sigma\wedge u\|^5}(\sigma\wedge u)\cdot(\sigma\wedge\dot{u}). \tag{9}$$

which depend on the four-vector parameter $\sigma = (\sigma^0, \underline{\sigma})$. Comparing it with (4), we can show that Eq. (9) describes the motion of free particles with rest mass $m_0 = m\cdot\left[1 - \frac{(\sigma\cdot u)^2}{(\sigma\cdot\sigma)(u\cdot u)}\right]^{3/2}$ and constant spin four-vector σ.

If $m=0$, Eq. (9) with the supplementary condition $\sigma\cdot u=0$ describes the motion of massless time-like particles with constant spin four-vector.

3. If m = 0, the set of integral lines of Eq. (9) contains those geodesic circles along which the unit four-velocity vector in the motion makes an arbitrary constant angle with a singled-out direction σ: $\left(\frac{\sigma \cdot u}{\|u\|}\right)^{\cdot} = 0$. The geodesic circles are singled-out in the set of integral curves of Eq. (9) by the conditions m = 0 and

$$\left(\frac{\sigma \cdot u}{\|u\|}\right)^{\cdot} = 0 \text{ and } (\sigma \cdot u)\dot{u} \wedge u = 0.$$

4. Equation (8) also has a physical meaning. It describes the planar motion of a free-particle with rest mass m $\|\sigma\|$ and spin $\underline{\sigma}$, orthogonal to the plane of motion. Such motions were analyzed in Ref. 14.

Making use of the Lagrangianity condition,[12] we can verify that there are no Euler–Poisson equations equivalent to the Lorentz–Dirac equations (2).

The Lagrangian for Eq. (8) is not invariant and has the following general form:

$$\mathcal{L} = \frac{1}{2\|u\|}\left[\frac{u_2(\dot{u}_1 u_0 - \dot{u}_0 u_1)}{u_0 u^0 + u_1 u^1} - \frac{u_1(\dot{u}_2 u_0 - \dot{u}_0 u_2)}{u_0 u^0 + u_2 u^2}\right] + (\dot{u}.\partial_u)f + c.u - m\|u\|,$$

where the arbitrary function satisfies the condition $u.\partial_u f$. In a more general approach, let us assume that the pseudo-Euclidean space has dimension larger than two and that the metric's signature is different from two. There will then be no invariant Lagrangian on it, for which the Euler–Poisson equation is of the third order.

Let X be the generator of Lorentz transformations, specified by the vector $\underline{\omega}$ and $\underline{\beta}$, and let expressions $\mathcal{E}$, and $\mathbf{E}$ correspond to Eq. (9). Then $X_{(3)}(\mathbf{E}) = \underline{\omega} \times \mathbf{E} + (\underline{\beta} \cdot \mathbf{v})\mathbf{E} - (\mathbf{E} \cdot \mathbf{v})\underline{\beta}$. Let X be the generator of pseudo-orthogonal transformations of the three-dimensional space, specified by the vector parameter, w, $X^T_{(3)}$ its prolongation to the space $T_3(\mathbf{R}^3)$, and let expression $\mathcal{E}$ correspond to Eq. (8). Then $X^T_3(\mathcal{E}) = w \times \mathcal{E}$. We thus conclude that neither the pseudo-orthogonal transformations nor the Lorentz transformations are generalized invariance transformations[13] of the corresponding variational problems. Accordingly, the proposed method of finding invariant Euler–Poisson equations is essentially more general than the methods proceeding from the Langrangian.

We have shown that in certain cases the Mathisson equation and the equation of geodesic circles can be considered in the context of Ostrogradskii's mechanics and Kawaguci's geometry. The case of two-dimensional space was considered in Ref. 15.

[1]H. J. Wospakrik, Phys. Rev. D. 26, No. 2, 523 (1982).
[2]F. J. Vanhecke, Rev. Bras. Fiz., 126, No. 2, 343 (1982).
[3]M. Mathisson, Acta Phys. Pol., 6, 163 (1937).
[4]P. Havas and J. N. Goldberg, Phys. Rev., 128, 398 (1962).
[5]M. E. Brachet and E. Tirapegni, Nuovo Cim., 47, A, 210 (1978).
[6]A. A. Sokolov and I. M. Ternov, The Relativistic Electron [in Russian], Nauka, Moscow (1983).
[7]E. L. Hill, Phys. Rev., 72, 143 (1947).
[8]W. G. Dixon, Proc. R. Soc. London, Ser. A 314, 499(1970).
[9]M. Baylin and S. Ragusa, Phys. Rev., D 15, 3543 (1977).
[10]F. Bopp, Z. Naturforsch., Bd. 3a, H. 8-11, 564 (1948).
[11]H. Honl, Ibid., Bd. 3a, H. 8-11, 573 (1948).
[12]R. Ya. Matsyuk, in: Mat. Methods Phys.-Mech. Mech. Fields 20, 16-19 (1984).
[13]I. Kolar, in: Proc. Conf. on Diff. Geom. and Its Appl., CSSR-GDR-Pol., Prague (1981), pp. 117-123.
[14]R. M. Plyatsko, "Effects of Einstein's theory of gravitation, conditioned by the oscillations and spin of a test body," Author's Abstract of Candidate's Dissertation, Minsk (1979).
[15]R. Ya. Matsyuk, in: Math. Methods Phys. Mech. Fields 16, 84-88 (1982).

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