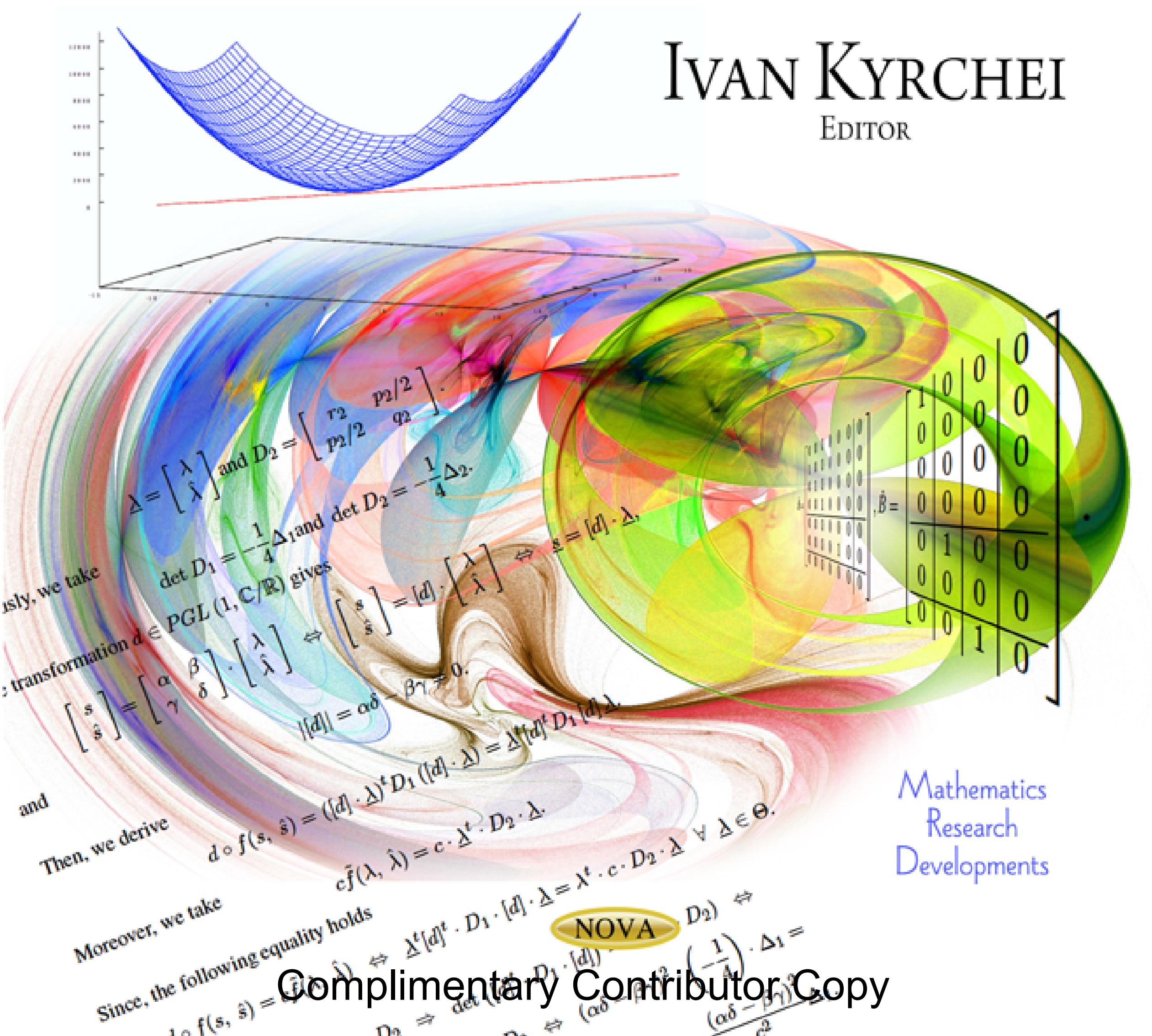
Advances in Linear Algebra Research

Ivan Kyrchei
Editor



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ADVANCES IN LINEAR ALGEBRA RESEARCH

IVAN KYRCHEI
EDITOR

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CONTENTS

PREFACE

This book presents original studies on the leading edge of linear algebra. Each chapter has been carefully selected in an attempt to present substantial research results across a broad spectrum. The main goal of Chapter One is to define and investigate the restricted generalized inverses corresponding to minimization of constrained quadratic form. As stated in Chapter Two, in systems and control theory, Linear Time Invariant (LTI) descriptor (Differential-Algebraic) systems are intimately related to the matrix pencil theory. A review of the most interesting properties of the Projective Equivalence and the Extended Hermite Equivalence classes is presented in the chapter. New determinantal representations of generalized inverse matrices based on their limit representations are introduced in Chapter Three. Using the obtained analogues of the adjoint matrix, Cramer's rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems have been obtained in the chapter. In Chapter Four, a very interesting application of linear algebra of commutative rings to systems theory, is explored. Chapter Five gives a comprehensive investigation to behaviors of a general Hermitian quadratic matrix-valued function by using ranks and inertias of matrices. In Chapter Six, the theory of triangular matrices (tables) is introduced. The main "characters" of the chapter are special triangular tables (which will be called triangular matrices) and their functions paradeterminants and parapermanents. The aim of Chapter Seven is to present the latest developments in iterative methods for solving linear matrix equations. The problems of existence of common eigenvectors and simultaneous triangularization of a pair of matrices over a principal ideal domain with quadratic minimal polynomials are investigated in Chapter Eight. Two approaches to define a noncommutative determinant (a determinant of a matrix with noncommutative elements) are considered in Chapter Nine. The last, Chapter 10, is an example of how the methods of linear algebra are used in natural sciences, particularly in chemistry. In this chapter, it is shown that in a First Order Chemical Kinetics Mechanisms matrix, all columns add to zero, all the diagonal elements are non-positive and all the other matrix entries are non-negative. As a result of this particular structure, the Gershgorin Circles Theorem can be applied to show that all the eigenvalues are negative or zero.

Minimization of a quadratic form $\langle x, Tx\rangle + \langle p, x\rangle + a$ under constraints defined by a linear system is a common optimization problem. In Chapter 1, it is assumed that the operator T is symmetric positive definite or positive semidefinite. Several extensions to different sets of linear matrix constraints are investigated. Solutions of this problem may be given using the Moore-Penrose inverse and/or the Drazin inverse. In addition, several new classes of generalized inverses are defined minimizing the seminorm defined by the quadratic forms, depending on the matrix equation that is used as a constraint.

A number of possibilities for further investigation are considered.

In systems and control theory, Linear Time Invariant (LTI) descriptor (Differential-Algebraic) systems are intimately related to the matrix pencil theory. Actually, a large number of systems are reduced to the study of differential (difference) systems $S(F, G)$ of the form:

$$S(F,G) : F\dot{x}(t) = Gx(t) \ \text{(or the dual } Fx = G\dot{x}(t)\text{)},$$

and

$$S(F,G) : Fx_{k+1} = Gx_k \ \text{(or the dual } Fx_k = Gx_{k+1}\text{)},\ F, G \in \mathbb{C}^{m\times n}$$

and their properties can be characterized by the homogeneous pencil $sF - \hat{s}G$. An essential problem in matrix pencil theory is the study of invariants of $sF - \hat{s}G$ under the *bilinear strict equivalence*. This problem is equivalent to the study of complete *Projective Equivalence* (PE), $\mathcal{E}_{\mathcal{P}}$, defined on the set $\mathbb{C}_r$ of complex homogeneous binary polynomials of fixed homogeneous degree r. For a $f(s, \hat{s}) \in \mathbb{C}_r$, the study of invariants of the PE class $\mathcal{E}_{\mathcal{P}}$ is reduced to a study of invariants of matrices of the set $\mathbb{C}^{k\times 2}$ (for $k \geqslant 3$ with all 2×2-minors non-zero) under the *Extended Hermite Equivalence* (EHE), $\mathcal{E}_{rh}$. In Chapter 2, the authors present a review of the most interesting properties of the PE and the EHE classes. Moreover, the appropriate projective transformation $d \in RGL(1, \mathbb{C}/\mathbb{R})$ is provided analytically ([1]).

By a generalized inverse of a given matrix, the authors mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular. In theory, there are many different generalized inverses that exist. The authors shall consider the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

New determinantal representations of these generalized inverse based on their limit representations are introduced in Chapter 3. Application of this new method allows us to obtain analogues classical adjoint matrix. Using the obtained analogues of the adjoint matrix, the authors get Cramer's rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems. Cramer's rules for the minimum norm least squares solutions and the Drazin inverse solutions of the matrix equations $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ and $\mathbf{AXB} = \mathbf{D}$ are also obtained, where $\mathbf{A}$, $\mathbf{B}$ can be singular matrices of appropriate size. Finally, the authors derive determinantal representations of solutions of the differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where the matrix $\mathbf{A}$ is singular.

Many physical systems in science and engineering can be described at time t in terms of an n-dimensional state vector $x(t)$ and an m-dimensional input vector $u(t)$, governed by an evolution equation of the form $x'(t) = A\cdot x(t) + B\cdot u(t)$, if the time is continuous, or $x(t+1) = A\cdot x(t) + B\cdot u(t)$ in the discrete case. Thus, the system is completely described by the pair of matrices (A, B) of sizes $n\times n$ and $n\times m$ respectively.

In two instances feedback is used to modify the structure of a given system (A, B): first, A can be replaced by $A + BF$, with some characteristic polynomial that ensures stability of the new system $(A + BF, B)$; and second, combining changes of bases with a feedback action $A \mapsto A + BF$ one obtains an equivalent system with a simpler structure.

Given a system (A, B), let (A, B) denote the set of states reachable at finite time when starting with initial condition $x(0) = 0$ and varying $u(t)$, i.e., (A, B) is the right image of the matrix $[B|AB|A^2B|\cdots]$. Also, let $Pols(A, B)$ denote the set of characteristic polynomials of all possible matrices $A + BF$, as F varies.

Classically, (A, B) have their entries in the field of real or complex numbers, but the concept of discrete-time system is generalized to matrix pairs with coefficients in an arbitrary commutative ring R. Therefore, techniques from Linear Algebra over commutative rings are needed.

In Chapter 4, the following problems are studied and solved when R is a commutative von Neumann regular ring:

- A canonical form is obtained for the feedback equivalence of systems (combination of basis changes with a feedback action).
- Given a system (A, B), it is proved that there exist a matrix F and a vector u such that the single-input system $(A + BF, Bu)$ has the same reachable states and the same assignable polynomials as the original system, i.e. $(A + BF, Bu) = (A, B)$ and $Pols(A + BF, Bu) = Pols(A, B)$.

Chapter 5 gives a comprehensive investigation to behaviors of a general Hermitian quadratic matrix-valued function

$$\phi(X) = (\, AXB + C\,)M(\, AXB + C)^{*} + D$$

by using ranks and inertias of matrices. The author first establishes a group of analytical formulas for calculating the global maximal and minimal ranks and inertias of $\phi(X)$. Based on the formulas, the author derives necessary and sufficient conditions for $\phi(X)$ to be a positive definite, positive semi-definite, negative definite, negative semi-definite function, respectively, and then solves two optimization problems of finding two matrices $\widehat{X}$ or $\widetilde{X}$ such that $\phi(X) \succcurlyeq \phi(\widehat{X})$ and $\phi(X) \preccurlyeq \phi(\widetilde{X})$ hold for all X, respectively. As extensions, the author considers definiteness and optimization problems in the Löwner sense of the following two types of multiple Hermitian quadratic matrix-valued function

$$\phi(\, X_1, \ldots, X_k\,) = \left(\sum_{i=1}^{k} A_iX_iB_i + C \right) M \left(\sum_{i=1}^{k} A_iX_iB_i + C \right)^{*} + D,$$

$$\psi(\, X_1, \ldots, X_k\,) = \sum_{i=1}^{k} (\, A_iX_iB_i + C_i\,)M_i(\, A_iX_iB_i + C_i\,)^{*} + D.$$

Some open problems on algebraic properties of these matrix-valued functions are mentioned at the end of Chapter 5.

In Chapter 6, the author considers elements of linear algebra based on triangular tables with entries in some number field and their functions, analogical to the classical notions of a matrix, determinant and permanent. Some properties are investigated and applications in various areas of mathematics are given.

The aim of Chapter 7 is to present the latest developments in iterative methods for solving linear matrix equations. The iterative methods are obtained by extending the methods presented to solve the linear system $Ax = b$. Numerical examples are investigated to confirm the efficiency of the methods.

The problems of existence of common eigenvectors and simultaneous triangularization of a pair of matrices over a principal ideal domain with quadratic minimal polynomials are investigated in Chapter 8. The necessary and sufficient conditions of simultaneous triangularization of a pair of matrices with quadratic minimal polynomials are obtained. As a result, the approach offered provides the necessary and sufficient conditions of simultaneous triangularization of pairs of idempotent matrices and pairs of involutory matrices over a principal ideal domain.

Since product of quaternions is noncommutative, there is a problem how to determine a determinant of a matrix with noncommutative elements (it's called a noncommutative determinant). The authors consider two approaches to define a noncommutative determinant. Primarily, there are row – column determinants that are an extension of the classical definition of the determinant; however, the authors assume predetermined order of elements in each of the terms of the determinant. In Chapter 9, the authors extend the concept of an immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

Properties of the determinant of a Hermitian matrix are established. Based on these properties, analogs of the classical adjont matrix over a quaternion skew field have been obtained. As a result, the authors have a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule by using row–column determinants.

Quasideterminants appeared from the analysis of the procedure of a matrix inversion. By using quasideterminants, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

The common feature in definition of row and column determinants and quasideterminants is that the authors have not one determinant of a quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminants, n row determinants, and n column determinants). The authors have obtained a relation of row-column determinants with quasideterminants of a matrix over a quaternion division algebra.

First order chemical reaction mechanisms are modeled through Ordinary Differential Equations (O.D.E.) systems of the form: , being the chemical species concentrations vector, its time derivative, and the associated system matrix.

A typical example of these reactions, which involves two species, is the Mutarotation of Glucose, which has a corresponding matrix with a null eigenvalue whereas the other one is negative.

A very simple example with three chemical compoundsis grape juice, when it is converted into wine and then transformed into vinegar. A more complicated example,also involving three species, is the adsorption of Carbon Dioxide over Platinum surfaces. Although, in these examples the chemical mechanisms are very different, in both cases the O.D.E. system matrix has two negative eigenvalues and the other one is zero. Consequently, in all these cases that involve two or three chemical species, solutions show a weak stability (i.e., they are stable but not asymptotically). This fact implies that small errors due to measurements in the initial concentrations will remain bounded, but they do not tend to vanish as the reaction proceeds.

In order to know if these results can be extended or not to other chemical mechanisms, a possible general result is studied through an inverse modeling approach, like in previous papers. For this purpose, theoretical mechanisms involving two or more species are proposed and a general type of matrices - so-called First Order Chemical Kinetics Mechanisms (F.O.C.K.M.) matrices - is studied from the eigenvalues and eigenvectors view point.

Chapter 10 shows that in an F.O.C.K.M. matrix all columns add to zero, all the diagonal elements are non-positive and all the other matrix entries are non-negative. Because of this particular structure, the Gershgorin Circles Theorem can be applied to show that all the eigenvalues are negative or zero. Moreover, it can be proved that in the case of the null eigenvalues - under certain conditions - algebraic and geometric multiplicities give the same number.

As an application of these results, several conclusions about the stability of the O.D.E. solutions are obtained for these chemical reactions, and its consequences on the propagation of concentrations and/or surface concentration measurement errors, are analyzed.

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Chapter 3

CRAMER'S RULE FOR GENERALIZED INVERSE SOLUTIONS

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Abstract

By a generalized inverse of a given matrix, we mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular. In theory, there are many different generalized inverses that exist. We shall consider the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

New determinantal representations of these generalized inverse based on their limit representations are introduced in this chapter. Application of this new method allows us to obtain analogues classical adjoint matrix. Using the obtained analogues of the adjoint matrix, we get Cramer's rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems. Cramer's rules for the minimum norm least squares solutions and the Drazin inverse solutions of the matrix equations $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ and $\mathbf{AXB} = \mathbf{D}$ are also obtained, where $\mathbf{A}$, $\mathbf{B}$ can be singular matrices of appropriate size. Finally, we derive determinantal representations of solutions of the differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where the matrix $\mathbf{A}$ is singular.

Keywords: generalized inverse; Drazin inverse; weighted Drazin inverse; Moore-Penrose inverse; weighted Moore-Penrose inverse; system of linear equations; Cramer's Rule; matrix equation; generalized inverse solution; least squares solution; Drazin inverse solution; differential matrix equation

AMS Subject Classification: 15A09; 15A24

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1. Preface

It's well-known in linear algebra, an n-by-n square matrix $\mathbf{A}$ is called invertible (also nonsingular or nondegenerate) if there exists an n-by-n square matrix $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{XA} = \mathbf{I}_n.$$

If this is the case, then the matrix $\mathbf{X}$ is uniquely determined by $\mathbf{A}$ and is called the inverse of $\mathbf{A}$, denoted by $\mathbf{A}^{-1}$.

By a generalized inverse of a given matrix, we mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular.

For any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ consider the following equations in $\mathbf{X}$:

$$\mathbf{AXA} = \mathbf{A}; \tag{1.1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{1.2}$$

$$(\mathbf{AX})^* = \mathbf{AX}; \tag{1.3}$$

$$(\mathbf{XA})^* = \mathbf{XA}. \tag{1.4}$$

and if $m = n$, also

$$\mathbf{AX} = \mathbf{AX}; \tag{1.5}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k. \tag{1.6}$$

For a sequence $\mathcal{G}$ of $\{1, 2, 3, 4, 5\}$ the set of matrices obeying the equations represented in $\mathcal{G}$ is denoted by $\mathbf{A}\{\mathcal{G}\}$. A matrix from $\mathbf{A}\{\mathcal{G}\}$ is called an $\mathcal{G}$-inverse of $\mathbf{A}$ and denoted by $\mathbf{A}^{(\mathcal{G})}$.

Consider some principal cases.

If $\mathbf{X}$ satisfies all the equations (1.1)-(1.4) is said to be **the Moore-Penrose inverse** of $\mathbf{A}$ and denote $\mathbf{A}^+ = \mathbf{A}^{(1,2,3,4)}$. The MoorePenrose inverse was independently described by E. H. Moore [1] in 1920, Arne Bjerhammar [2] in 1951 and Roger Penrose [3] in 1955. R. Penrose introduced the characteristic equations (1.1)-(1.4).

If $\det \mathbf{A} \neq 0$, then $\mathbf{A}^+ = \mathbf{A}^{-1}$.

The group inverse $\mathbf{A}^g$ is the unique $\mathbf{A}^{(1,2,5)}$ inverse of $\mathbf{A}$, and exists if and only if $Ind\,\mathbf{A} = \min\{k : \operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^k\} = 1$.

A matrix $\mathbf{X} = \mathbf{A}^D$ is said to be **the Drazin inverse** of $\mathbf{A}$ if (1.6) (for some positive integer k), (1.2) and (1.5) are satisfied, where $k = Ind\,\mathbf{A}$. It is named after Michael P. Drazin [4]. In particular, when $Ind\mathbf{A} = 1$, then the matrix $\mathbf{X}$ is the group inverse, $\mathbf{X} = \mathbf{A}^g$. If $Ind\mathbf{A} = 0$, then $\mathbf{A}$ is nonsingular, and $\mathbf{A}^D \equiv \mathbf{A}^{-1}$.

Let Hermitian positive definite matrices $\mathbf{M}$ and $\mathbf{N}$ of order m and n, respectively, be given. For any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, **the weighted Moore-Penrose inverse** of $\mathbf{A}$ is the unique solution $\mathbf{X} = \mathbf{A}^+_{M,N}$ of the matrix equations (1.1) and (1.2) and the following equations in $\mathbf{X}$ [5]:

$$(3M)\ (\mathbf{MAX})^* = \mathbf{MAX};\ (4N)\ (\mathbf{NXA})^* = \mathbf{NXA}.$$

In particular, when $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{N} = \mathbf{I}_n$, the matrix $\mathbf{X}$ satisfying the equations (1.1), (1.2), (3M), (4N) is the Moore-Penrose inverse $\mathbf{A}^+$.

The weighted Drazin inverse is being considered as well.

To determine the inverse and to give its analytic solution, we calculate a matrix of cofactors, known as an adjugate matrix or a classical adjoint matrix. The classical adjoint of $\mathbf{A}$, denote $Adj[\mathbf{A}]$, is the transpose of the cofactor matrix, then $\mathbf{A}^{-1} = \frac{Adj[\mathbf{A}]}{|\mathbf{A}|}$. Representation an inverse matrix by its classical adjoint matrix also plays a key role for Cramer's rule of systems of linear equations or matrices equations.

Obviously, the important question is the following: what are the analogues for the adjoint matrix of generalized inverses and, consequently, for Cramer's rule of generalized inverse solutions of matrix equations?

This is the main goal of the chapter.

In this chapter we shall adopt the following notation. Let $\mathbb{C}^{m\times n}$ be the set of m by n matrices with complex entries, $\mathbb{C}^{m\times n}_r$ be a subset of $\mathbb{C}^{m\times n}$ in which any matrix has rank r, $\mathbf{I}_m$ be the identity matrix of order m, and $\|.\|$ be the Frobenius norm of a matrix.

Denote by $\mathbf{a}_{.j}$ and $\mathbf{a}_{i.}$ the jth column and the ith row of $\mathbf{A} \in \mathbb{C}^{m\times n}$, respectively. Then $\mathbf{a}^*_{.j}$ and $\mathbf{a}^*_{i.}$ denote the jth column and the ith row of a conjugate and transpose matrix $\mathbf{A}^*$ as well. Let $\mathbf{A}_{.j}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its jth column with the vector $\mathbf{b}$, and by $\mathbf{A}_{i.}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its ith row with $\mathbf{b}$.

Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Then $\left|\mathbf{A}^\alpha_\beta\right|$ denotes the minor of $\mathbf{A}$ determined by the rows indexed by α and the columns indexed by β. Clearly, $|\mathbf{A}^\alpha_\alpha|$ denotes a principal minor determined by the rows and columns indexed by α. The cofactor of a_{ij} in $\mathbf{A} \in \mathbb{C}^{n\times n}$ is denoted by $\frac{\partial}{\partial a_{ij}}|\mathbf{A}|$.

For $1 \leq k \leq n$, $L_{k,n} := \{\, \alpha : \alpha = (\alpha_1, \ldots, \alpha_k),\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$ denotes the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$. Let $N_k := L_{k,m} \times L_{k,n}$. For fixed $\alpha \in L_{p,m}$, $\beta \in L_{p,n}$, $1 \leq p \leq k$, let

$$
\begin{gathered}
I_{k,\,m}(\alpha) := \{I :\ I \in L_{k,\,m}, I \supseteq \alpha\},\\
J_{k,\,n}(\beta) := \{J :\ J \in L_{k,\,n}, J \supseteq \beta\},\\
N_k(\alpha, \beta) := I_{k,\,m}(\alpha) \times J_{k,\,n}(\beta)
\end{gathered}
$$

For case $i \in \alpha$ and $j \in \beta$, we denote

$$
\begin{gathered}
I_{k,m}\{i\} := \{\alpha :\ \alpha \in L_{k,m}, i \in \alpha\}, J_{k,n}\{j\} := \{\beta :\ \beta \in L_{k,n}, j \in \beta\},\\
N_k\{i, j\} := I_{k,\,m}\{i\} \times J_{k,\,n}\{j\}.
\end{gathered}
$$

The chapter is organized as follows. In Section 2 determinantal representations by analogues of the classical adjoint matrix for the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses are obtained.

In Section 3 we show that the obtained analogues of the adjoint matrix for the generalized inverse matrices enable us to obtain natural analogues of Cramer's rule for generalized inverse solutions of systems of linear equations and demonstrate it in two examples.

In Section 4, we obtain analogs of the Cramer rule for generalized inverse solutions of the matrix equations, $\mathbf{AX} = \mathbf{B}$, $\mathbf{XA} = \mathbf{B}$ and $\mathbf{AXB} = \mathbf{D}$, namely for the minimum norm least squares solutions and the Drazin inverse solutions. We show numerical examples to illustrate the main results as well.

In Section 5, we use the determinantal representations of the Drazin inverse solution to solutions of the following differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where $\mathbf{A}$ is singular. It is demonstrated in the example.

Facts set forth in Sections 2 and 3 were partly published in [6], in Section 4 were published in [7, 8] and in Sections 5 were published in [8].

Note that we obtained some of the submitted results for matrices over the quaternion skew field within the framework of the theory of the column and row determinants [9, 10, 11, 12, 13, 14].

2. Analogues of the Classical Adjoint Matrix for Generalized Inverse Matrices

For determinantal representations of the generalized inverse matrices as analogues of the classical adjoint matrix, we apply the method, which consists on the limit representation of the generalized inverse matrices, lemmas on rank of some matrices and on characteristic polynomial. We used this method at first in [6] and then in [8]. Liu et al. in [15] deduce the new determinantal representations of the outer inverse $\mathbf{A}^{(2)}_{T,S}$ based on these principles as well. In this chapter we obtain detailed determinantal representations by analogues of the classical adjoint matrix for the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

2.1. Analogues of the Classical Adjoint Matrix for the Moore - Penrose Inverse

Determinantal representation of the Moore - Penrose inverse was studied in [1],[16, 17, 18, 19]. The main result consists in the following theorem.

Theorem 2.1. *The Moore - Penrose inverse* $\mathbf{A}^{+} = (a^{+}_{ij}) \in \mathbb{C}^{n\times m}$ *of* $\mathbf{A} \in \mathbb{C}^{m\times n}_{r}$ *has the following determinantal representation*

$$a^{+}_{ij} = \frac{\sum\limits_{(\alpha,\,\beta)\in N_r\{j,\,i\}} \left|(\mathbf{A}^{*})^{\beta}_{\alpha}\right| \frac{\partial}{\partial a_{j\,i}} \left|\mathbf{A}^{\alpha}_{\beta}\right|}{\sum\limits_{(\gamma,\,\delta)\in N_r} \left|(\mathbf{A}^{*})^{\delta}_{\gamma}\right| \left|\mathbf{A}^{\gamma}_{\delta}\right|}, \quad 1 \le i, j \le n.$$

This determinantal representation of the Moore - Penrose inverse is based on corresponding full-rank representation [16]: if $\mathbf{A} = \mathbf{PQ}$, where $\mathbf{P} \in \mathbb{C}^{m\times r}_{r}$ and $\mathbf{Q} \in \mathbb{C}^{r\times n}_{r}$, then

$$\mathbf{A}^{+} = \mathbf{Q}^{*}(\mathbf{P}^{*}\mathbf{AQ}^{*})^{-1}\mathbf{P}^{*}.$$

For a better understanding of the structure of the Moore - Penrose inverse we consider it by singular value decomposition of $\mathbf{A}$. Let

$$\begin{array}{l} \mathbf{AA}^{*}\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i, \quad i = \overline{1, m} \\ \mathbf{A}^{*}\mathbf{A}\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i, \quad i = \overline{1, n}, \end{array}$$

$$\sigma_1 \le \sigma_2 \le ...\sigma_r > 0 = \sigma_{r+1} = \sigma_{r+2} = ...$$

and the singular value decomposition (SVD) of $\mathbf{A}$ is $\mathbf{A} = \mathbf{U\Sigma V}^*$, where

$$\begin{array}{ll} \mathbf{U} = [\mathbf{u}_1\,\mathbf{u}_2...\mathbf{u}_m] \in \mathbb{C}^{m\times m}, & \mathbf{U}^*\mathbf{U} = \mathbf{I}_m, \\ \mathbf{V} = [\mathbf{v}_1\,\mathbf{v}_2...\mathbf{v}_n] \in \mathbb{C}^{n\times n}, & \mathbf{V}^*\mathbf{V} = \mathbf{I}_n, \end{array}$$

$$\mathbf{\Sigma} = \operatorname{diag}(\sigma_1, \sigma_2, ..., \sigma_r) \in \mathbb{C}^{m\times n}.$$

Then [3], $\mathbf{A}^+ = \mathbf{V\Sigma}^+\mathbf{U}^*$, where $\mathbf{\Sigma}^+ = \operatorname{diag}(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_r^{-1})$.

We need the following limit representation of the Moore-Penrose inverse.

Lemma 2.2. *[20] If $\mathbf{A} \in \mathbb{C}^{m\times n}$, then*

$$\mathbf{A}^+ = \lim_{\lambda\to 0}\mathbf{A}^*\left(\mathbf{AA}^* + \lambda\mathbf{I}\right)^{-1} = \lim_{\lambda\to 0}\left(\mathbf{A}^*\mathbf{A} + \lambda\mathbf{I}\right)^{-1}\mathbf{A}^*,$$

where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is the set of positive real numbers.

Corollary 2.3. *[21] If $\mathbf{A} \in \mathbb{C}^{m\times n}$, then the following statements are true.*

i) If $\operatorname{rank}\mathbf{A} = \mathrm{n}$, *then* $\mathbf{A}^+ = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*$.

ii) If $\operatorname{rank}\mathbf{A} = \mathrm{m}$, *then* $\mathbf{A}^+ = \mathbf{A}^*\left(\mathbf{AA}^*\right)^{-1}$.

iii) If $\operatorname{rank}\mathbf{A} = \mathrm{n} = \mathrm{m}$, *then* $\mathbf{A}^+ = \mathbf{A}^{-1}$.

We need the following well-known theorem about the characteristic polynomial and lemmas on rank of some matrices.

Theorem 2.4. *[22] Let d_r be the sum of principal minors of order r of $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then its characteristic polynomial $p_{\mathbf{A}}(t)$ can be expressed as $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A}) = t^n - d_1t^{n-1} + d_2t^{n-2} - \ldots + (-1)^n d_n$.*

Lemma 2.5. *If $\mathbf{A} \in \mathbb{C}_r^{m\times n}$, then* $\operatorname{rank}\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^*\right) \leq r$.

Proof. Let $\mathbf{P}_{i\,k}\left(-a_{j\,k}\right) \in \mathbb{C}^{n\times n}$, $(k \neq i)$, be the matrix with $-a_{j\,k}$ in the (i, k) entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. It follows that

$$\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^*\right)\cdot\prod_{k\neq i}\mathbf{P}_{i\,k}\left(-a_{j\,k}\right) = \begin{pmatrix} \sum\limits_{k\neq j} a_{1k}^*a_{k1} & \ldots & a_{1j}^* & \ldots & \sum\limits_{k\neq j} a_{1k}^*a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\neq j} a_{nk}^*a_{k1} & \ldots & a_{nj}^* & \ldots & \sum\limits_{k\neq j} a_{nk}^*a_{kn} \end{pmatrix}.$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i-th$$

The obtained above matrix has the following factorization.

$$\begin{pmatrix} \sum\limits_{k\neq j} a_{1k}^*a_{k1} & \ldots & a_{1j}^* & \ldots & \sum\limits_{k\neq j} a_{1k}^*a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\neq j} a_{nk}^*a_{k1} & \ldots & a_{nj}^* & \ldots & \sum\limits_{k\neq j} a_{nk}^*a_{kn} \end{pmatrix} =$$
$$\qquad\qquad i-th$$

$$= \begin{pmatrix} a^*_{11} & a^*_{12} & \dots & a^*_{1m} \\ a^*_{21} & a^*_{22} & \dots & a^*_{2m} \\ \dots & \dots & \dots & \dots \\ a^*_{n1} & a^*_{n2} & \dots & a^*_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{n1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & \underset{i-th}{0} & \dots & a_{mn} \end{pmatrix} j-th.$$

Denote by $\tilde{\mathbf{A}} := \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & \underset{i-th}{0} & \dots & a_{mn} \end{pmatrix} j-th.$ The matrix $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by replacing all entries of the jth row and of the ith column with zeroes except that the (j,i) entry equals 1. Elementary transformations of a matrix do not change its rank. It follows that $\operatorname{rank}(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \le \min\left\{\operatorname{rank}\mathbf{A}^*, \operatorname{rank}\tilde{\mathbf{A}}\right\}$. Since $\operatorname{rank}\tilde{\mathbf{A}} \ge \operatorname{rank}\ \mathbf{A} = \operatorname{rank}\mathbf{A}^*$ and $\operatorname{rank}\mathbf{A}^*\mathbf{A} = \operatorname{rank}\mathbf{A}$ the proof is completed. ■ The following lemma can be proved in the same way.

Lemma 2.6. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$, *then* $\operatorname{rank}(\mathbf{A}\mathbf{A}^*)_{i\,.}\left(\mathbf{a}^*_{j\,.}\right) \le r$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

Lemma 2.7. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\lambda \in \mathbb{R}$, *then*

$$\det\left((\lambda\mathbf{I}_n + \mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}, \tag{2.1}$$

where $c_n^{(ij)} = \left|(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right|$ *and* $c_s^{(ij)} = \sum\limits_{\beta\in J_{s,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)^{\beta}_{\beta}\right|$ *for all* $s = \overline{1, n-1}$, $i = \overline{1,n}$, *and* $j = \overline{1,m}$.

Proof. Denote $\mathbf{A}^*\mathbf{A} = \mathbf{V} = (v_{ij}) \in \mathbb{C}^{n\times n}$. Consider $(\lambda\mathbf{I}_n + \mathbf{V})_{.\,i}(\mathbf{v}_{.i}) \in \mathbb{C}^{n\times n}$. Taking into account Theorem 2.4 we obtain

$$|(\lambda\mathbf{I}_n + \mathbf{V})_{.\,i}(\mathbf{v}_{.i})| = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \tag{2.2}$$

where $d_s = \sum\limits_{\beta\in J_{s,\,n}\{i\}} |(\mathbf{V})^{\beta}_{\beta}|$ is the sum of all principal minors of order s that contain the i-th column for all $s = \overline{1, n-1}$ and $d_n = \det\mathbf{V}$. Since $\mathbf{v}_{.\,i} = \sum\limits_{l} \mathbf{a}^*_{.\,l}a_{li}$, where $\mathbf{a}^*_{.l}$ is the lth column-vector of $\mathbf{A}^*$ for all $l = \overline{1,n}$, then we have on the one hand

$$\begin{gathered} |(\lambda\mathbf{I} + \mathbf{V})_{.\,i}(\mathbf{v}_{.i})| = \sum_{l} |(\lambda\mathbf{I} + \mathbf{V})_{.\,l}(\mathbf{a}^*_{.\,l}a_{li})| = \\ \sum_{l} |(\lambda\mathbf{I} + \mathbf{V})_{.\,i}(\mathbf{a}^*_{.\,l})| \cdot a_{li} \end{gathered} \tag{2.3}$$

Having changed the order of summation, we obtain on the other hand for all $s = \overline{1, n-1}$

$$\begin{gathered} d_s = \sum_{\beta\in J_{s,n}\{i\}} \left|(\mathbf{V})^{\beta}_{\beta}\right| = \sum_{\beta\in J_{s,n}\{i\}} \sum_{l} \left|(\mathbf{V}_{.\,i}(\mathbf{a}^*_{.\,l}a_{l\,i}))^{\beta}_{\beta}\right| = \\ \sum_{l} \sum_{\beta\in J_{s,n}\{i\}} \left|(\mathbf{V}_{.\,i}(\mathbf{a}^*_{.l}))^{\beta}_{\beta}\right| \cdot a_{l\,i}. \end{gathered} \tag{2.4}$$

By substituting (2.3) and (2.4) in (2.2), and equating factors at $a_{l\,i}$ when $l = j$, we obtain the equality (2.1). ∎

By analogy can be proved the following lemma.

Lemma 2.8. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\lambda \in \mathbb{R}$*, then*

$$\det\left((\lambda \mathbf{I}_m + \mathbf{A}\mathbf{A}^*)_{j\,.}\left(\mathbf{a}^*_{i.}\right)\right) = r_1^{(ij)}\lambda^{m-1} + r_2^{(ij)}\lambda^{m-2} + \ldots + r_m^{(ij)},$$

where $r_m^{(ij)} = \left|(\mathbf{A}\mathbf{A}^*)_{j\,.}\left(\mathbf{a}^*_{i.}\right)\right|$ *and* $r_s^{(ij)} = \sum\limits_{\alpha\in I_{s,m}\{j\}} \left|\left((\mathbf{A}\mathbf{A}^*)_{j\,.}\left(\mathbf{a}^*_{i.}\right)\right)^{\alpha}_{\alpha}\right|$ *for all* $s = \overline{1, n-1}$*,* $i = \overline{1,n}$*, and* $j = \overline{1,m}$*.*

The following theorem and remarks introduce the determinantal representations of the Moore-Penrose by analogs of the classical adjoint matrix.

Theorem 2.9. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$ *and* $r < min\{m,n\}$*, then the Moore-Penrose inverse* $\mathbf{A}^+ = \left(a^+_{ij}\right) \in \mathbb{C}^{n\times m}$ *possess the following determinantal representations:*

$$a^+_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{\,.\,i}\left(\mathbf{a}^*_{.j}\right)\right)^{\,\beta}_{\,\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})^{\,\beta}_{\,\beta}\right|}, \tag{2.5}$$

or

$$a^+_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left((\mathbf{A}\mathbf{A}^*)_{j\,.}\left(\mathbf{a}^*_{i.}\right)\right)^{\,\alpha}_{\,\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^*)^{\,\alpha}_{\,\alpha}\right|}. \tag{2.6}$$

for all $i = \overline{1,n}$*,* $j = \overline{1,m}$*.*

Proof. At first we shall obtain the representation (2.5). If $\lambda \in \mathbb{R}_+$, then the matrix $(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) \in \mathbb{C}^{n\times n}$ is Hermitian and $\operatorname{rank}(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) = n$. Hence, there exists its inverse

$$(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})^{-1} = \frac{1}{\det(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})}\begin{pmatrix} L_{11} & L_{21} & \ldots & L_{n\,1} \\ L_{12} & L_{22} & \ldots & L_{n\,2} \\ \ldots & \ldots & \ldots & \ldots \\ L_{1\,n} & L_{2\,n} & \ldots & L_{n\,n} \end{pmatrix},$$

where L_{ij} $(\forall i,j = \overline{1,n})$ is a cofactor in $\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}$. By Lemma 2.2, $\mathbf{A}^+ = \lim\limits_{\lambda\to 0}(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$, so that

$$\mathbf{A}^+ = \lim_{\lambda\to 0}\begin{pmatrix} \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.1}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \ldots & \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,1}\left(\mathbf{a}^*_{.m}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} \\ \ldots & \ldots & \ldots \\ \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.1}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \ldots & \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.m}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} \end{pmatrix}. \tag{2.7}$$

From Theorem 2.4 we get

$$\det(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n,$$

where d_r $(\forall r = \overline{1, n-1})$ is a sum of principal minors of $\mathbf{A}^*\mathbf{A}$ of order r and $d_n = \det \mathbf{A}^*\mathbf{A}$. Since $\operatorname{rank} \mathbf{A}^*\mathbf{A} = \operatorname{rank} \mathbf{A} = r$, then $d_n = d_{n-1} = \ldots = d_{r+1} = 0$ and

$$\det(\lambda \mathbf{I} + \mathbf{A}^*\mathbf{A}) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_r\lambda^{n-r}. \tag{2.8}$$

In the same way, we have for arbitrary $1 \le i \le n$ and $1 \le j \le m$ from Lemma 2.7

$$\det(\lambda \mathbf{I} + \mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) = l_1^{(ij)}\lambda^{n-1} + l_2^{(ij)}\lambda^{n-2} + \ldots + l_n^{(ij)},$$

where for an arbitrary $1 \le k \le n-1$, $l_k^{(ij)} = \sum\limits_{\beta \in J_{k,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)_{\beta}^{\beta}\right|$, and $l_n^{(ij)} = \det(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)$. By Lemma 2.5, $\operatorname{rank}(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \le r$ so that if $k > r$, then $\left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)_{\beta}^{\beta}\right| = 0$, $(\forall \beta \in J_{k,\,n}\{i\}, \forall i = \overline{1,n}, \forall j = \overline{1,m})$. Therefore if $r+1 \le k < n$, then $l_k^{(ij)} = \sum\limits_{\beta \in J_{k,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)_{\beta}^{\beta}\right| = 0$ and $l_n^{(ij)} = \det(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) = 0$, $\left(\forall i = \overline{1,n}, \forall j = \overline{1,m}\right)$. Finally we obtain

$$\det(\lambda \mathbf{I} + \mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) = l_1^{(ij)}\lambda^{n-1} + l_2^{(ij)}\lambda^{n-2} + \ldots + l_r^{(ij)}\lambda^{n-r}. \tag{2.9}$$

By replacing the denominators and the numerators of the fractions in entries of matrix (2.7) with the expressions (2.8) and (2.9) respectively, we get

$$\mathbf{A}^+ = \lim_{\lambda \to 0} \begin{pmatrix} \frac{l_1^{(11)}\lambda^{n-1}+\ldots+l_r^{(11)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(1m)}\lambda^{n-1}+\ldots+l_r^{(1m)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \\ \ldots & \ldots & \ldots \\ \frac{l_1^{(n1)}\lambda^{n-1}+\ldots+l_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(nm)}\lambda^{n-1}+\ldots+l_r^{(nm)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{l_r^{(11)}}{d_r} & \ldots & \frac{l_r^{(1m)}}{d_r} \\ \ldots & \ldots & \ldots \\ \frac{l_r^{(n1)}}{d_r} & \ldots & \frac{l_r^{(nm)}}{d_r} \end{pmatrix}.$$

From here it follows (2.5).

We can prove (2.6) in the same way. ■

Corollary 2.10. *If* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and* $r < \min\{m, n\}$ *or* $r = m < n$*, then the projection matrix* $\mathbf{P} = \mathbf{A}^+\mathbf{A}$ *can be represented as*

$$\mathbf{P} = \left(\frac{p_{ij}}{d_r(\mathbf{A}^*\mathbf{A})}\right)_{n\times n},$$

where $\mathbf{d}_{.j}$ *denotes the* j*th column of* $(\mathbf{A}^*\mathbf{A})$ *and, for arbitrary* $1 \le i, j \le n$, $p_{ij} = \sum\limits_{\beta \in J_{r,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{d}_{.j})\right)_{\beta}^{\beta}\right|$.

Proof. Representing the Moore - Penrose inverse $\mathbf{A}^{+}$ by (2.5), we obtain

$$\mathbf{P}=\frac{1}{d_r\left(\mathbf{A}^*\mathbf{A}\right)}\begin{pmatrix} l_{11} & l_{12} & \ldots & l_{1m}\\ l_{21} & l_{22} & \ldots & l_{2m}\\ \ldots & \ldots & \ldots & \ldots\\ l_{n1} & l_{n2} & \ldots & l_{nm}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{m\,1} & a_{m\,2} & \ldots & a_{m\,n}\end{pmatrix}.$$

Therefore, for arbitrary $1\leq i,j\leq n$ we get

$$p_{i\,j}=\sum_{k}\sum_{\beta\in J_{r,\,n}\{i\}}\left|((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.\,k}))^{\beta}_{\beta}\right|\cdot a_{k\,j}=$$

$$=\sum_{\beta\in J_{r,\,n}\{i\}}\sum_{k}\left|((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.\,k}\cdot a_{k\,j}))^{\beta}_{\beta}\right|=\sum_{\beta\in J_{r,\,n}\{i\}}\left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{d}^*_{.j})\right)^{\beta}_{\beta}\right|.$$

■ Using the representation (2.6) of the Moore - Penrose inverse the following corollary can be proved in the same way.

Corollary 2.11. *If* $\mathbf{A}\in\mathbb{C}^{m\times n}_{r}$*, where* $r<\min\{m,n\}$ *or* $r=n<m$*, then a projection matrix* $\mathbf{Q}=\mathbf{A}\mathbf{A}^{+}$ *can be represented as*

$$\mathbf{Q}=\left(\frac{q_{ij}}{d_r\left(\mathbf{A}\mathbf{A}^*\right)}\right)_{m\times m},$$

where $\mathbf{g}_{i.}$ *denotes the ith row of* $(\mathbf{A}\mathbf{A}^*)$ *and, for arbitrary* $1\leq i,j\leq m$*,* $q_{i\,j}=\sum\limits_{\alpha\in I_{r,m}\{j\}}\left|((\mathbf{A}\mathbf{A}^*)_{j.}\,(\mathbf{g}_{i.}\,))^{\alpha}_{\alpha}\right|$.

Remark 2.12. *If* $\operatorname{rank}\mathbf{A}=n$*, then from Corollary 2.3 we get* $\mathbf{A}^{+}=\left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*$*. Representing* $\left(\mathbf{A}^*\mathbf{A}\right)^{-1}$ *by the classical adjoint matrix, we have*

$$\mathbf{A}^{+}=\frac{1}{\det(\mathbf{A}^*\mathbf{A})}\begin{pmatrix}\det(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.1}\right) & \ldots & \det(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.\,m}\right)\\ \ldots & \ldots & \ldots\\ \det(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.1}\right) & \ldots & \det(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.\,m}\right)\end{pmatrix}. \tag{2.10}$$

If $n<m$*, then (2.5) is valid.*

Remark 2.13. *As above, if* $\operatorname{rank}\mathbf{A}=m$*, then*

$$\mathbf{A}^{+}=\frac{1}{\det(\mathbf{A}\mathbf{A}^*)}\begin{pmatrix}\det(\mathbf{A}\mathbf{A}^*)_{1\,.}\left(\mathbf{a}^*_{1\,.}\right) & \ldots & \det(\mathbf{A}\mathbf{A}^*)_{m\,.}\left(\mathbf{a}^*_{1\,.}\right)\\ \ldots & \ldots & \ldots\\ \det(\mathbf{A}\mathbf{A}^*)_{1\,.}\left(\mathbf{a}^*_{n\,.}\right) & \ldots & \det(\mathbf{A}\mathbf{A}^*)_{m\,.}\left(\mathbf{a}^*_{n\,.}\right)\end{pmatrix}. \tag{2.11}$$

If $n>m$*, then (2.6) is valid as well.*

Remark 2.14. *By definition of the classical adjoint* $Adj(\mathbf{A})$ *for an arbitrary invertible matrix* $\mathbf{A}\in\mathbb{C}^{n\times n}$ *one may put,* $Adj(\mathbf{A})\cdot\mathbf{A}=\det\mathbf{A}\cdot\mathbf{I}_n$*.*

If $\mathbf{A}\in\mathbb{C}^{m\times n}$ *and* $\operatorname{rank}\mathbf{A}=n$*, then by Corollary 2.3,* $\mathbf{A}^{+}\mathbf{A}=\mathbf{I}_n$*. Representing the matrix* $\mathbf{A}^{+}$ *by (2.10) as* $\mathbf{A}^{+}=\frac{\mathbf{L}}{\det(\mathbf{A}^*\mathbf{A})}$*, we obtain* $\mathbf{L}\mathbf{A}=\det\left(\mathbf{A}^*\mathbf{A}\right)\cdot\mathbf{I}_n$*. This means that the matrix* $\mathbf{L}=(l_{ij})\in\mathbb{C}^{n\times m}$ *is a left analogue of* $Adj(\mathbf{A})$*, where* $\mathbf{A}\in\mathbb{C}^{m\times n}_{n}$*, and* $l_{ij}=\det(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}^*_{.j}\right)$ *for all* $i=\overline{1,n}$*,* $j=\overline{1,m}$*.*

If $\operatorname{rank}\mathbf{A} = m$*, then by Corollary 2.3,* $\mathbf{A}\mathbf{A}^+ = \mathbf{I}_m$*. Representing the matrix* $\mathbf{A}^+$ *by (2.11) as* $\mathbf{A}^+ = \frac{\mathbf{R}}{\det(\mathbf{A}\mathbf{A}^*)}$*, we obtain* $\mathbf{A}\mathbf{R} = \mathbf{I}_m \cdot \det(\mathbf{A}\mathbf{A}^*)$*. This means that the matrix* $\mathbf{R} = (r_{ij}) \in \mathbb{C}^{m\times n}$ *is a right analogue of* $Adj(\mathbf{A})$*, where* $\mathbf{A} \in \mathbb{C}^{m\times n}_m$*, and* $r_{ij} = \det(\mathbf{A}\mathbf{A}^*)_{j.}\,(\mathbf{a}^*_{i.})$ *for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*.*

If $\mathbf{A} \in \mathbb{C}^{m\times n}_r$ *and* $r < \min\{m,n\}$*, then by (2.5) we have* $\mathbf{A}^+ = \frac{\mathbf{L}}{d_r(\mathbf{A}^*\mathbf{A})}$*, where* $\mathbf{L} = (l_{ij}) \in \mathbb{C}^{n\times m}$ *and* $l_{ij} = \sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{\,.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}\right|$ *for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*. From Corollary 2.10 we get* $\mathbf{L}\mathbf{A} = d_r(\mathbf{A}^*\mathbf{A})\cdot\mathbf{P}$*. The matrix* $\mathbf{P}$ *is idempotent. All eigenvalues of an idempotent matrix chose from 1 or 0 only. Thus, there exists an unitary matrix* $\mathbf{U}$ *such that*

$$\mathbf{L}\mathbf{A} = d_r(\mathbf{A}^*\mathbf{A})\,\mathbf{U}\mathbf{diag}\,(1,\ldots,1,0,\ldots,0)\mathbf{U}^*,$$

where $\mathbf{diag}\,(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^{n\times n}$ *is a diagonal matrix. Therefore, the matrix* $\mathbf{L}$ *can be considered as a left analogue of* $Adj(\mathbf{A})$*, where* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$*.*

In the same way, if $\mathbf{A} \in \mathbb{C}^{m\times n}_r$ *and* $r < \min\{m,n\}$*, then by (2.5) we have* $\mathbf{A}^+ = \frac{\mathbf{R}}{d_r(\mathbf{A}\mathbf{A}^*)}$*, where* $\mathbf{R} = (r_{ij}) \in \mathbb{C}^{n\times m}$*,* $r_{ij} = \sum\limits_{\alpha\in I_{r,m}\{j\}} |((\mathbf{A}\mathbf{A}^*)_{j.}\,(\mathbf{a}^*_{i.}))\,{}^{\alpha}_{\alpha}|$ *for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*. From Corollary 2.11 we get* $\mathbf{A}\mathbf{R} = d_r(\mathbf{A}\mathbf{A}^*)\cdot\mathbf{Q}$*. The matrix* $\mathbf{Q}$ *is idempotent. There exists an unitary matrix* $\mathbf{V}$ *such that*

$$\mathbf{A}\mathbf{R} = d_r(\mathbf{A}\mathbf{A}^*)\,\mathbf{V}\mathbf{diag}\,(1,\ldots,1,0,\ldots,0)\mathbf{V}^*,$$

where $\mathbf{diag}\,(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^{m\times m}$*. Therefore, the matrix* $\mathbf{R}$ *can be considered as a right analogue of* $Adj(\mathbf{A})$ *in this case.*

Remark 2.15. *To obtain an entry of* $\mathbf{A}^+$ *by Theorem 2.1 one calculates* $(C^r_n C^r_m + C^{r-1}_{n-1}C^{r-1}_{m-1})$ *determinants of order* r*. Whereas by the equation (2.5) we calculate as much as* $(C^r_n + C^{r-1}_{n-1})$ *determinants of order* r *or we calculate the total of* $(C^r_m + C^{r-1}_{m-1})$ *determinants by (2.6). Therefore the calculation of entries of* $\mathbf{A}^+$ *by Theorem 2.9 is easier than by Theorem 2.1.*

2.2. Analogues of the Classical Adjoint Matrix for the Weighted Moore-Penrose Inverse

Let Hermitian positive definite matrices $\mathbf{M}$ and $\mathbf{N}$ of order m and n, respectively, be given. The weighted Moore-Penrose inverse $\mathbf{X} = \mathbf{A}^+_{M,N}$ can be explicitly expressed from the weighted singular value decomposition due to Van Loan [23].

Lemma 2.16. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$*. There exist* $\mathbf{U} \in \mathbb{C}^{m\times m}$*,* $\mathbf{V} \in \mathbb{C}^{n\times n}$ *satisfying* $\mathbf{U}^*\mathbf{M}\mathbf{U} = \mathbf{I}_m$ *and* $\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$ *such that*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*.$$

Then the weighted Moore-Penrose inverse $\mathbf{A}^+_{M,N}$ *can be represented*

$$\mathbf{A}^+_{M,N} = \mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*\mathbf{M},$$

where $\mathbf{D} = diag(\sigma_1, \sigma_2, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ *and* σ_i^2 *is the nonzero eigenvalues of* $\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\mathbf{A}$.

For the weighted Moore-Penrose inverse $\mathbf{X} = \mathbf{A}^+_{M,N}$, we have the following limit representation.

Lemma 2.17. *([24], Corollary 3.4.) Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$. *Then*

$$\mathbf{A}^+_{M,N} = \lim_{\lambda\to 0}(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A})^{-1}\mathbf{A}^\sharp.$$

By analogy to Lemma 2.17 can be proved the following lemma.

Lemma 2.18. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$. *Then*

$$\mathbf{A}^+_{M,N} = \lim_{\lambda\to 0}\mathbf{A}^\sharp(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^\sharp)^{-1}.$$

Denote by $\mathbf{a}^\sharp_{.j}$ and $\mathbf{a}^\sharp_{i.}$ the jth column and the ith row of $\mathbf{A}^\sharp$ respectively. By putting $\mathbf{A}^\sharp$ instead $\mathbf{A}^*$, we obtain the proofs of the following two lemmas and theorem similar to the proofs of Lemmas 2.5, 2.6, 2.7, 2.8 and Theorem 2.9, respectively.

Lemma 2.19. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$ *and* $\mathbf{A}^\sharp$ *is defined as above, then*

$$\operatorname{rank}\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right) \leq \operatorname{rank}\left(\mathbf{A}^\sharp\mathbf{A}\right),$$

$$\operatorname{rank}\left(\mathbf{A}\mathbf{A}^\sharp\right)_{j\,.}\left(\mathbf{a}^\sharp_{i\,.}\right) \leq \operatorname{rank}\left(\mathbf{A}\mathbf{A}^\sharp\right),$$

for all $i = \overline{1, n}$ *and* $j = \overline{1, m}$

Analogues of the characteristic polynomial are considered in the following lemma.

Lemma 2.20. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\lambda \in \mathbb{R}$, *then*

$$\det\left(\left(\lambda\mathbf{I}_n + \mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right)\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)},$$

$$\det\left((\lambda\mathbf{I}_m + \mathbf{A}\mathbf{A}^\sharp)_{j\,.}(\mathbf{a}^\sharp_{i.})\right) = r_1^{(ij)}\lambda^{m-1} + r_2^{(ij)}\lambda^{m-2} + \ldots + r_m^{(ij)},$$

where $c_n^{(ij)} = \left|\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right)\right|$, $r_m^{(ij)} = |(\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i.})|$ *and* $c_s^{(ij)} = \sum_{\beta\in J_{s,\,n}\{i\}}\left|\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right)\right)^\beta_\beta\right|$, $r_t^{(ij)} = \sum_{\alpha\in I_{t,m}\{j\}}\left|\left((\mathbf{A}\mathbf{A}^\sharp)_{j\,.}(\mathbf{a}^\sharp_{i.})\right)^\alpha_\alpha\right|$ *for all* $s = \overline{1, n-1}$, $t = \overline{1, m-1}$, $i = \overline{1, n}$, *and* $j = \overline{1, m}$.

The following theorem introduce the determinantal representations of the weighted Moore-Penrose by analogs of the classical adjoint matrix.

Theorem 2.21. *If* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and* $r < min\{m,n\}$, *then the weighted Moore-Penrose inverse* $\mathbf{A}_{M,N}^{+} = \left(\tilde{a}_{ij}^{+}\right) \in \mathbb{C}^{n\times m}$ *possess the following determinantal representation:*

$$\tilde{a}_{ij}^{+} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right)\right)\,{}_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{r,\;n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)\,{}_{\beta}^{\beta}\right|}, \tag{2.12}$$

or

$$\tilde{a}_{ij}^{+} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left((\mathbf{A}\mathbf{A}^{\sharp})_{j\,.}\,(\mathbf{a}_{i.}^{\sharp})\right)\,{}_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\;m}} \left|(\mathbf{A}\mathbf{A}^{\sharp})\,{}_{\alpha}^{\alpha}\right|}, \tag{2.13}$$

for all $i = \overline{1,n}$, $j = \overline{1,m}$.

2.3. Analogues of the Classical Adjoint Matrix for the Drazin Inverse

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows.

Theorem 2.22. *[25] If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$ *and*

$$\mathbf{A} = \mathbf{P}\begin{pmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}\end{pmatrix}\mathbf{P}^{-1}$$

where $\mathbf{C}$ *is nonsingular and* $\operatorname{rank}\mathbf{C} = \operatorname{rank}\mathbf{A}^k$, *and* $\mathbf{N}$ *is nilpotent of order* k, *then*

$$\mathbf{A}^D = \mathbf{P}\begin{pmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{P}^{-1}. \tag{2.14}$$

Stanimirovic' [26] introduced a determinantal representation of the Drazin inverse by the following theorem.

Theorem 2.23. *The Drazin inverse* $\mathbf{A}^D = \left(a_{ij}^D\right)$ *of an arbitrary matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\mathbf{A} = k$ *possesses the following determinantal representation*

$$a_{ij}^D = \frac{\sum\limits_{(\alpha,\beta)\in N_{r_k}\{j,i\}} \left|(\mathbf{A}^s)_{\alpha}^{\beta}\right| \frac{\partial}{\partial a_{j\,i}}\left|\mathbf{A}_{\beta}^{\alpha}\right|}{\sum\limits_{(\gamma,\delta)\in N_{r_k}} \left|(\mathbf{A}^s)_{\gamma}^{\delta}\right|\,\left|\mathbf{A}_{\delta}^{\gamma}\right|},\ 1 \le i,j \le n; \tag{2.15}$$

where $s \ge k$ *and* $r_k = rank\mathbf{A}^s$.

This determinantal representations of the Drazin inverse is based on a full-rank representation.

We use the following limit representation of the Drazin inverse.

Lemma 2.24. *[27] If $\mathbf{A} \in \mathbb{C}^{n\times n}$, then*

$$\mathbf{A}^D = \lim_{\lambda\to 0}\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k,$$

where $k = Ind\,\mathbf{A}$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Since the equation (1.6) can be replaced by follows

$$\mathbf{X}\mathbf{A}^{k+1} = \mathbf{A}^k,$$

the following lemma can be obtained by analogy to Lemma 2.24.

Lemma 2.25. *If $\mathbf{A} \in \mathbb{C}^{n\times n}$, then*

$$\mathbf{A}^D = \lim_{\lambda\to 0}\mathbf{A}^k\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1},$$

where $k = Ind\,\mathbf{A}$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Denote by $\mathbf{a}_{.j}^{(k)}$ and $\mathbf{a}_{i.}^{(k)}$ the jth column and the ith row of $\mathbf{A}^k$ respectively. We consider the following auxiliary lemma.

Lemma 2.26. *If $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind\,\mathbf{A} = k$, then for all $i, j = \overline{1, n}$*

$$\operatorname{rank}\mathbf{A}_{i.}^{k+1}\left(\mathbf{a}_{j.}^{(k)}\right) \leq \operatorname{rank}\mathbf{A}^{k+1}.$$

Proof. The matrix $\mathbf{A}_{i.}^{k+1}\left(\mathbf{a}_{j.}^{(k)}\right)$ may by represent as follows

$$\begin{pmatrix} \sum\limits_{s=1}^{n} a_{1s}a_{s1}^{(k)} & \ldots & \sum\limits_{s=1}^{n} a_{1s}a_{sn}^{(k)} \\ \ldots & \ldots & \ldots \\ a_{j1}^{(k)} & \ldots & a_{jn}^{(k)} \\ \ldots & \ldots & \ldots \\ \sum\limits_{s=1}^{n} a_{ns}a_{s1}^{(k)} & \ldots & \sum\limits_{s=1}^{n} a_{ns}a_{sn}^{(k)} \end{pmatrix}$$

Let $\mathbf{P}_{l\,i}\left(-a_{l\,j}\right) \in \mathbb{C}^{n\times n}$, $(l \neq i)$, be a matrix with $-a_{l\,j}$ in the (l, i) entry, 1 in all diagonal entries, and 0 in others. It is a matrix of an elementary transformation. It follows that

$$\mathbf{A}_{i.}^{k+1}\left(\mathbf{a}_{j.}^{(k)}\right)\cdot\prod_{l\neq i}\mathbf{P}_{l\,i}\left(-a_{l\,j}\right) = \begin{pmatrix} \sum\limits_{s\neq j}^{n} a_{1s}a_{s1}^{(k)} & \ldots & \sum\limits_{s\neq j}^{n} a_{1s}a_{sn}^{(k)} \\ \ldots & \ldots & \ldots \\ a_{j1}^{(k)} & \ldots & a_{jn}^{(k)} \\ \ldots & \ldots & \ldots \\ \sum\limits_{s\neq j}^{n} a_{ns}a_{s1}^{(k)} & \ldots & \sum\limits_{s\neq j}^{n} a_{ns}a_{sn}^{(k)} \end{pmatrix} \; ith$$

The obtained above matrix has the following factorization.

$$\begin{pmatrix} \sum\limits_{s\neq j}^{n} a_{1s}a_{s1}^{(k)} & \dots & \sum\limits_{s\neq j}^{n} a_{1s}a_{sn}^{(k)} \\ \dots & \dots & \dots \\ a_{j1}^{(k)} & \dots & a_{j\,n}^{(k)} \\ \dots & \dots & \dots \\ \sum\limits_{s\neq j}^{n} a_{ns}a_{s1}^{(k)} & \dots & \sum\limits_{s\neq j}^{n} a_{ns}a_{sn}^{(k)} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & 0 & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} & \dots & a_{1n}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} & \dots & a_{2n}^{(k)} \\ \dots & \dots & \dots & \dots \\ a_{n1}^{(k)} & a_{n2}^{(k)} & \dots & a_{nn}^{(k)} \end{pmatrix}$$

Denote the first matrix by

$$\tilde{\mathbf{A}} := \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \underset{jth}{0} & \dots & a_{nn} \end{pmatrix} ith.$$

The matrix $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by replacing all entries of the ith row and the jth column with zeroes except for 1 in the (i, j) entry. Elementary transformations of a matrix do not change its rank. It follows that $\operatorname{rank} \mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right) \leq \min\left\{\operatorname{rank} \mathbf{A}^{k}, \operatorname{rank} \tilde{\mathbf{A}}\right\}$. Since $\operatorname{rank} \tilde{\mathbf{A}} \geq \operatorname{rank} \mathbf{A}^{k}$ the proof is completed. ■

The following lemma is proved similarly.

Lemma 2.27. *If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$*, then for all* $i, j = \overline{1, n}$

$$\operatorname{rank} \mathbf{A}_{.i}^{k+1}\left(\mathbf{a}_{.j}^{(k)}\right) \leq \operatorname{rank} \mathbf{A}^{k+1}.$$

Lemma 2.28. *If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *and* $\lambda \in \mathbb{R}$*, then*

$$\det\left((\lambda \mathbf{I}_n + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)})\right) = r_1^{(ij)}\lambda^{n-1} + r_2^{(ij)}\lambda^{n-2} + \ldots + r_n^{(ij)}, \tag{2.16}$$

where $r_n^{(ij)} = \left|\mathbf{A}_{j\,.}^{k+1}(\mathbf{a}_{i.}^{(k)})\right|$ *and* $r_s^{(ij)} = \sum\limits_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}_{j\,.}^{k+1}(\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}\right|$ *for all* $s = \overline{1, n-1}$ *and* $i, j = \overline{1, n}$.

Proof. Consider the matrix $\left((\lambda \mathbf{I}_n + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{j.}^{(k)})\right) \in \mathbb{C}^{n\times n}$. Taking into account Theorem 2.4 we obtain

$$\left|\left((\lambda \mathbf{I}_n + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{j.}^{(k)})\right)\right| = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \tag{2.17}$$

where $d_s = \sum\limits_{\alpha \in I_{s,\,n}\{j\}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|$ is the sum of all principal minors of order s that contain the j-th row for all $s = \overline{1, n-1}$ and $d_n = \det \mathbf{A}^{k+1}$. Since $\mathbf{a}^{(k+1)}_{j.} = \sum\limits_{l} a_{jl}\mathbf{a}^{(k)}_{l.}$, where $\mathbf{a}^{(k)}_{l.}$ is the lth row-vector of $\mathbf{A}^k$ for all $l = \overline{1, n}$, then we have on the one hand

$$\left|\left((\lambda \mathbf{I}_n + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}^{(k)}_{j.})\right)\right| = \sum_l \left|(\lambda \mathbf{I} + \mathbf{A}^{k+1})_{l.}\left(a_{jl}\mathbf{a}^{(k)}_{l.}\right)\right| = \sum_l a_{jl} \cdot \left|(\lambda \mathbf{I} + \mathbf{A}^{k+1})_{l.}\left(\mathbf{a}^{(k)}_{l.}\right)\right| \tag{2.18}$$

Having changed the order of summation, we obtain on the other hand for all $s = \overline{1, n-1}$

$$d_s = \sum_{\alpha \in I_{s,\,n}\{j\}} \left|(\mathbf{A}^{k+1})^{\alpha}_{\alpha}\right| = \sum_{\alpha \in I_{s,\,n}\{j\}} \sum_l \left|\left(\mathbf{A}^{k+1}_{j.}\left(a_{jl}\mathbf{a}^{(k)}_{l.}\right)\right)^{\alpha}_{\alpha}\right| = \sum_l a_{jl} \cdot \sum_{\alpha \in I_{s,\,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j.}\left(\mathbf{a}^{(k)}_{l.}\right)\right)^{\alpha}_{\alpha}\right| \tag{2.19}$$

By substituting (2.18) and (2.19) in (2.17), and equating factors at a_{jl} when $l = i$, we obtain the equality (2.16). ■

Theorem 2.29. *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^k = r \leq n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then the Drazin inverse* $\mathbf{A}^D = \left(a^D_{ij}\right) \in \mathbb{C}^{n\times n}$ *possess the following determinantal representations:*

$$a^D_{ij} = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j.}\left(\mathbf{a}^{(k)}_{i.}\right)\right)^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha \in I_{r,n}} \left|(\mathbf{A}^{k+1})^{\alpha}_{\alpha}\right|}, [\tag{2.20}$$

and

$$a^D_{ij} = \frac{\sum\limits_{\beta \in J_{r,n}\{i\}} \left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta \in J_{r,n}} \left|(\mathbf{A}^{k+1})^{\beta}_{\beta}\right|}, \tag{2.21}$$

for all $i, j = \overline{1, n}$*.*

Proof. At first we shall prove the equation (2.20).

If $\lambda \in \mathbb{R}_+$, then $\operatorname{rank}\left(\lambda \mathbf{I} + \mathbf{A}^{k+1}\right) = n$. Hence, there exists the inverse matrix

$$\left(\lambda \mathbf{I} + \mathbf{A}^{k+1}\right)^{-1} = \frac{1}{\det(\lambda \mathbf{I} + \mathbf{A}^{k+1})}\begin{pmatrix} R_{11} & R_{21} & \dots & R_{n\,1} \\ R_{12} & R_{22} & \dots & R_{n\,2} \\ \dots & \dots & \dots & \dots \\ R_{1\,n} & R_{2\,n} & \dots & R_{n\,n} \end{pmatrix},$$

where R_{ij} is a cofactor in $\lambda \mathbf{I} + \mathbf{A}^{k+1}$ for all $i, j = \overline{1, n}$. By Theorem 2.25, $\mathbf{A}^D = \lim\limits_{\lambda \to 0} \mathbf{A}^k \left(\lambda \mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1}$, so that

$$\mathbf{A}^D = \lim_{\lambda \to 0} \frac{1}{\det(\lambda \mathbf{I} + \mathbf{A}^{k+1})}\begin{pmatrix} \sum_{s=1}^n a^{(k)}_{1s} R_{1s} & \dots & \sum_{s=1}^n a^{(k)}_{1s} R_{ns} \\ \dots & \dots & \dots \\ \sum_{s=1}^n a^{(k)}_{ns} R_{1s} & \dots & \sum_{s=1}^n a^{(k)}_{ns} R_{ns} \end{pmatrix} =$$

$$\lim_{\lambda\to 0}\begin{pmatrix} \frac{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{1.}\left(\mathbf{a}_{1.}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \ldots & \frac{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{n.}\left(\mathbf{a}_{n.}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} \\ \ldots & \ldots & \ldots \\ \frac{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{1.}\left(\mathbf{a}_{n.}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \ldots & \frac{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{n.}\left(\mathbf{a}_{n.}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} \end{pmatrix} \tag{2.22}$$

Taking into account Theorem 2.4 , we have

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_n,$$

where $d_s=\sum\limits_{\alpha\in I_{s,n}}\left|\left(\mathbf{A}^{k+1}\right)_{\alpha}^{\alpha}\right|$ is a sum of the principal minors of $\mathbf{A}^{k+1}$ of order s, for all $s=\overline{1,n-1}$, and $d_n=\det\mathbf{A}^{k+1}$. Since $\operatorname{rank}\mathbf{A}^{k+1}=r$, then $d_n=d_{n-1}=\ldots=d_{r+1}=0$ and

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_r\lambda^{n-r}. \tag{2.23}$$

By Lemma 2.28 for all $i,j=\overline{1,n}$,

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{j.}\left(\mathbf{a}_{i.}^{(k)}\right)=l_1^{(ij)}\lambda^{n-1}+l_2^{(ij)}\lambda^{n-2}+\ldots+l_n^{(ij)},$$

where for all $s=\overline{1,n-1}$,

$$l_s^{(ij)}=\sum_{\alpha\in I_{s,n}\{j\}}\left|\left(\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{i.}^{(k)}\right)\right)_{\alpha}^{\alpha}\right|,$$

and $l_n^{(i\,j)}=\det\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{i.}^{(k)}\right)$.

By Lemma 2.26, $\operatorname{rank}\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{i\,.}^{(k)}\right)\leq r$, so that if $s>r$, then for all $\alpha\in I_{s,n}\{i\}$ and for all $i,j=\overline{1,n}$,

$$\left|\left(\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{i.}^{(k)}\right)\right)_{\alpha}^{\alpha}\right|=0.$$

Therefore if $r+1\leq s<n$, then for all $i,j=\overline{1,n}$,

$$l_s^{(ij)}=\sum_{\alpha\in I_{s,n}\{j\}}\left|\left(\mathbf{A}_{j.}^{k+1}\left(\mathbf{a}_{i.}^{(k)}\right)\right)_{\alpha}^{\alpha}\right|=0,$$

and $l_n^{(ij)}=\det\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{i.}^{(k)}\right)=0$. Finally we obtain

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{j.}\left(\mathbf{a}_{i.}^{(k)}\right)=l_1^{(i\,j)}\lambda^{n-1}+l_2^{(i\,j)}\lambda^{n-2}+\ldots+l_r^{(ij)}\lambda^{n-r}. \tag{2.24}$$

By replacing the denominators and the nominators of the fractions in the entries of the matrix (2.22) with the expressions (2.23) and (2.24) respectively, finally we obtain

$$\mathbf{A}^D=\lim_{\lambda\to 0}\begin{pmatrix} \frac{l_1^{(11)}\lambda^{n-1}+\ldots+l_r^{(11)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(1n)}\lambda^{n-1}+\ldots+l_r^{(1n)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \\ \ldots & \ldots & \ldots \\ \frac{l_1^{(n1)}\lambda^{n-1}+\ldots+l_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(nn)}\lambda^{n-1}+\ldots+l_r^{(nn)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \end{pmatrix}=$$

$$= \begin{pmatrix} \frac{l_r^{(11)}}{d_r} & \dots & \frac{l_r^{(1n)}}{d_r} \\ \dots & \dots & \dots \\ \frac{l_r^{(n1)}}{d_r} & \dots & \frac{l_r^{(nn)}}{d_r} \end{pmatrix},$$

where for all $i, j = \overline{1, n}$,

$$l_r^{(ij)} = \sum_{\alpha \in I_{r,n}\{j\}} \left| \left(\mathbf{A}_{j.}^{k+1} \left(\mathbf{a}_{i.}^{(k)} \right) \right)_\alpha^\alpha \right|, \ d_r = \sum_{\alpha \in I_{r,n}} \left| \left(\mathbf{A}^{k+1} \right)_\alpha^\alpha \right|.$$

The equation (2.21) can be proved similarly.

This completes the proof. ■ Using Theorem 2.29 we evidently can obtain determinantal representations of the group inverse and the following determinantal representation of the identities $\mathbf{A}^D\mathbf{A}$ and $\mathbf{A}\mathbf{A}^D$ on $R(\mathbf{A}^k)$

Corollary 2.30. *If* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A} = r \le n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then the group inverse* $\mathbf{A}^g = \left(a_{ij}^g\right) \in \mathbb{C}^{n\times n}$ *possess the following determinantal representations:*

$$a_{ij}^g = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left| \left(\mathbf{A}_{j.}^2 \left(\mathbf{a}_{i.} \right) \right)_\alpha^\alpha \right|}{\sum\limits_{\alpha \in I_{r,n}} |(\mathbf{A}^2)_\alpha^\alpha|}, \tag{2.25}$$

$$a_{ij}^g = \frac{\sum\limits_{\beta \in J_{r,n}\{i\}} \left| \left(\mathbf{A}_{.i}^2 \left(\mathbf{a}_{.j} \right) \right)_\beta^\beta \right|}{\sum\limits_{\beta \in J_{r,n}} \left|(\mathbf{A}^2)_\beta^\beta\right|},$$

for all $i, j = \overline{1, n}$.

Corollary 2.31. *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then the matrix* $\mathbf{A}\mathbf{A}^D = (q_{ij}) \in \mathbb{C}^{n\times n}$ *possess the following determinantal representation*

$$q_{ij} = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left| \left(\mathbf{A}_{j.}^{k+1} \left(\mathbf{a}_{i.}^{(k+1)} \right) \right)_\beta^\beta \right|}{\sum\limits_{\alpha \in I_{r,n}} \left|(\mathbf{A}^{k+1})_\beta^\beta\right|}, \tag{2.26}$$

for all $i, j = \overline{1, n}$.

Corollary 2.32. *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then the matrix* $\mathbf{A}^D\mathbf{A} = (p_{ij}) \in \mathbb{C}^{n\times n}$ *possess the following determinantal representation*

$$p_{ij} = \frac{\sum\limits_{\beta \in J_{r,n}\{i\}} \left| \left(\mathbf{A}_{.i}^{k+1} \left(\mathbf{a}_{.j}{}^{(k+1)} \right) \right)_\beta^\beta \right|}{\sum\limits_{\beta \in J_{r,n}} \left| \left(\mathbf{A}_{.i}^{k+1} \right)_\beta^\beta \right|}, \tag{2.27}$$

for all $i, j = \overline{1, n}$.

2.4. Analogues of the Classical Adjoint Matrix for the W-Weighted Drazin Inverse

Cline and Greville [28] extended the Drazin inverse of square matrix to rectangular matrix and called it as **the weighted Drazin inverse** (WDI). The W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{C}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{C}^{n\times m}$ is defined to be the unique solution $\mathbf{X} \in \mathbb{C}^{m\times n}$ of the following three matrix equations:

$$\begin{array}{l} 1)\ (\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^{k}, \\ 2)\ \mathbf{XWAWX} = \mathbf{X}, \\ 3)\ \mathbf{AWX} = \mathbf{XWA}, \end{array} \tag{2.28}$$

where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$. It is denoted by $\mathbf{X} = \mathbf{A}_{d,W}$. In particular, when $\mathbf{A} \in \mathbb{C}^{m\times m}$ and $\mathbf{W} = \mathbf{I}_m$, then $\mathbf{A}_{d,W}$ reduce to $\mathbf{A}^D$. If $\mathbf{A} \in \mathbb{C}^{m\times m}$ is non-singular square matrix and $\mathbf{W} = \mathbf{I}_m$, then $Ind(A) = 0$ and $\mathbf{A}_{d,W} = \mathbf{A}^D = \mathbf{A}^{-1}$.

The properties of WDI can be found in (e.g.,[29, 30, 31, 32]). We note the general algebraic structures of the W-weighted Drazin inverse [29]. Let for $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{W} \in \mathbb{C}^{n\times m}$ exist $\mathbf{L} \in \mathbb{C}^{m\times m}$ and $\mathbf{Q} \in \mathbb{C}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{L}\begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}\mathbf{Q}^{-1}, \quad \mathbf{W} = \mathbf{Q}\begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix}\mathbf{L}^{-1}.$$

Then

$$\mathbf{A}_{d,W} = \mathbf{L}\begin{pmatrix} (\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}^{-1},$$

where $\mathbf{L}$, $\mathbf{L}$, $\mathbf{A}_{11}$, $\mathbf{W}_{11}$ are non-singular matrices, and $\mathbf{A}_{22}$, $\mathbf{W}_{22}$ are nilpotent matrices. By [27] we have the following limit representations of the W-weighted Drazin inverse,

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\left(\lambda\mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1}(\mathbf{AW})^k\mathbf{A} \tag{2.29}$$

and

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\mathbf{A}(\mathbf{WA})^k\left(\lambda\mathbf{I}_n + (\mathbf{WA})^{k+2}\right)^{-1} \tag{2.30}$$

where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Denote $\mathbf{WA} =: \mathbf{U}$ and $\mathbf{AW} =: \mathbf{V}$. Denote by $\mathbf{v}_{.j}^{(k)}$ and $\mathbf{v}_{i.}^{(k)}$ the jth column and the ith row of $\mathbf{V}^k$ respectively. Denote by $\bar{\mathbf{V}}^k := (\mathbf{AW})^k\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\bar{\mathbf{W}} = \mathbf{WAW} \in \mathbb{C}^{n\times m}$.

Lemma 2.33. *If $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{C}^{m\times m}$ with $Ind\mathbf{V} = k$, then*

$$\operatorname{rank}\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) \le \operatorname{rank}\left(\mathbf{V}^{k+2}\right). \tag{2.31}$$

Proof. We have $\mathbf{V}^{k+2} = \bar{\mathbf{V}}^k\bar{\mathbf{W}}$. Let $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) \in \mathbb{C}^{m\times m}$, $(s \ne i)$, be a matrix with $-\bar{w}_{j\,s}$ in the (i, s) entry, 1 in all diagonal entries, and 0 in others. The matrix $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s})$, $(s \ne i)$, is a matrix of an elementary transformation. It follows that

$$\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\cdot\prod_{s\ne i}\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) = \begin{pmatrix} \sum\limits_{s\ne j}\bar{v}_{1s}^{(k)}\bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum\limits_{s\ne j}\bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{s\ne j}\bar{v}_{ms}^{(k)}\bar{w}_{s1} & \dots & \underbrace{\bar{v}_{mj}^{(k)}}_{i-th} & \dots & \sum\limits_{s\ne j}\bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix}.$$

We have the next factorization of the obtained matrix.

$$\begin{pmatrix} \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{s1} & \ldots & \bar{v}_{1j}^{(k)} & \ldots & \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{s1} & \ldots & \bar{v}_{mj}^{(k)} & \ldots & \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix} =$$
$$i-th$$

$$= \begin{pmatrix} \bar{v}_{11}^{(k)} & \bar{v}_{12}^{(k)} & \ldots & \bar{v}_{1n}^{(k)} \\ \bar{v}_{21}^{(k)} & \bar{v}_{22}^{(k)} & \ldots & \bar{v}_{2n}^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ \bar{v}_{m1}^{(k)} & \bar{v}_{m2}^{(k)} & \ldots & \bar{v}_{mn}^{(k)} \end{pmatrix} \begin{pmatrix} \bar{w}_{11} & \ldots & 0 & \ldots & \bar{w}_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{w}_{n1} & \ldots & 0 & \ldots & \bar{w}_{nm} \end{pmatrix} j-th.$$
$$i-th$$

Denote $\tilde{\mathbf{W}} := \begin{pmatrix} \bar{w}_{11} & \ldots & 0 & \ldots & \bar{w}_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{w}_{n1} & \ldots & 0 & \ldots & \bar{w}_{nm} \end{pmatrix} j-th$ (column $i-th$). The matrix $\tilde{\mathbf{W}}$ is obtained from $\bar{\mathbf{W}} = \mathbf{WAW}$ by replacing all entries of the jth row and the ith column with zeroes except for 1 in the (i, j) entry. Since elementary transformations of a matrix do not change a rank, then $\operatorname{rank} \mathbf{V}^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right) \leq \min\left\{\operatorname{rank} \bar{\mathbf{V}}^k, \operatorname{rank} \tilde{\mathbf{W}}\right\}$. It is obvious that

$$\operatorname{rank} \bar{\mathbf{V}}^k = \operatorname{rank} (\mathbf{AW})^k\mathbf{A} \geq \operatorname{rank} (\mathbf{AW})^{k+2},$$
$$\operatorname{rank} \tilde{\mathbf{W}} \geq \operatorname{rank}\ \mathbf{WAW} \geq \operatorname{rank} (\mathbf{AW})^{k+2}.$$

From this the inequality (2.31) follows immediately. ■

The next lemma is proved similarly.

Lemma 2.34. *If* $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{C}^{n\times n}$ *with* $Ind\mathbf{U} = k$, *then*

$$\operatorname{rank}\left(\mathbf{U}^{k+2}\right)_{i\,.}\left(\bar{\mathbf{u}}^{(k)}_{j\,.}\right) \leq \operatorname{rank}\left(\mathbf{U}^{k+2}\right),$$

where $\bar{\mathbf{U}}^k := \mathbf{A}(\mathbf{WA})^k \in \mathbb{C}^{m\times n}$

Analogues of the characteristic polynomial are considered in the following two lemmas.

Lemma 2.35. *If* $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{C}^{m\times m}$ *with* $Ind\mathbf{V} = k$ *and* $\lambda \in \mathbb{R}$, *then*

$$\left|\left(\lambda\mathbf{I}_m + \mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right| = c_1^{(ij)}\lambda^{m-1} + c_2^{(ij)}\lambda^{m-2} + \ldots + c_m^{(ij)}, \tag{2.32}$$

where $c_m^{(ij)} = \det\left(\mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)$ *and* $c_s^{(ij)} = \sum\limits_{\beta\in J_{s,\,m}\{i\}} \det\left(\left(\mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$ *for all* $s = \overline{1, m-1}$, $i = \overline{1, m}$, *and* $j = \overline{1, n}$.

Proof. Consider the matrix $\left(\lambda\mathbf{I}+\mathbf{V}^{k+2}\right)_{.i}(\mathbf{v}_{.i}^{(k+2)})\in\mathbb{C}^{m\times m}$. Taking into account Theorem 2.4 we obtain

$$\left|\left(\lambda\mathbf{I}+\mathbf{V}^{k+2}\right)_{.i}\left(\mathbf{v}_{.i}^{(k+2)}\right)\right|=d_1\lambda^{m-1}+d_2\lambda^{m-2}+\ldots+d_m, \tag{2.33}$$

where $d_s=\sum\limits_{\beta\in J_{s,\,m}\{i\}}|\left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta}|$ is the sum of all principal minors of order s that contain the i-th column for all $s=\overline{1,m-1}$ and $d_m=\det\left(\mathbf{V}^{k+2}\right)$. Since $\mathbf{v}_{.i}^{(k+2)}=\begin{pmatrix}\sum\limits_l\bar{v}_{1l}^{(k)}\bar{w}_{li}\\ \sum\limits_l\bar{v}_{2l}^{(k)}\bar{w}_{li}\\ \vdots\\ \sum\limits_l\bar{v}_{nl}^{(k)}\bar{w}_{li}\end{pmatrix}=\sum\limits_l\bar{\mathbf{v}}_{.l}^{(k)}\bar{w}_{li}$, where $\bar{\mathbf{v}}_{.l}^{(k)}$ is the lth column-vector of $\bar{\mathbf{V}}^k=(\mathbf{AW})^k\mathbf{A}$ and $\mathbf{WAW}=\bar{\mathbf{W}}=(\bar{w}_{li})$ for all $l=\overline{1,n}$, then we have on the one hand

$$\begin{gathered}\left|\left(\lambda\mathbf{I}+\mathbf{V}^{k+2}\right)_{.i}\left(\mathbf{v}_{.i}^{(k+2)}\right)\right|=\sum_l\left|\left(\lambda\mathbf{I}+\mathbf{V}^{k+2}\right)_{.l}\left(\bar{\mathbf{v}}_{.l}^{(k)}\bar{w}_{li}\right)\right|=\\ \sum_l\left|\left(\lambda\mathbf{I}+\mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}_{.l}^{(k)}\right)\right|\cdot\bar{w}_{li}\end{gathered} \tag{2.34}$$

Having changed the order of summation, we obtain on the other hand for all $s=\overline{1,m-1}$

$$\begin{gathered}d_s=\sum_{\beta\in J_{s,\,m}\{i\}}\left|\left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta}\right|=\sum_{\beta\in J_{s,\,m}\{i\}}\sum_l\left|\left(\left(\mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}_{.l}^{(k)}\bar{w}_{l\,i}\right)\right)_{\beta}^{\beta}\right|=\\ \sum_l\sum_{\beta\in J_{s,\,m}\{i\}}\left|\left(\left(\mathbf{V}^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}_{.l}^{(k)}\right)\right)_{\beta}^{\beta}\right|\cdot\bar{w}_{l\,i}.\end{gathered} \tag{2.35}$$

By substituting (2.34) and (2.35) in (2.33), and equating factors at $\bar{w}_{l\,i}$ when $l=j$, we obtain the equality (2.32). ■ By analogy can be proved the following lemma.

Lemma 2.36. *If* $\mathbf{WA}=\mathbf{U}=(u_{ij})\in\mathbb{C}^{n\times n}$ *with* $Ind\mathbf{U}=k$ *and* $\lambda\in\mathbb{R}$*, then*

$$\left|(\lambda\mathbf{I}+\mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)})\right|=r_1^{(ij)}\lambda^{n-1}+r_2^{(ij)}\lambda^{n-2}+\ldots+r_n^{(ij)},$$

where $r_n^{(ij)}=\left|(\mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)})\right|$ *and* $r_s^{(ij)}=\sum\limits_{\alpha\in I_{s,n}\{j\}}\left|\left((\mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)})\right)_{\alpha}^{\alpha}\right|$ *for all* $s=\overline{1,n-1}$*,* $i=\overline{1,m}$*, and* $j=\overline{1,n}$*.*

Theorem 2.37. *If* $\mathbf{A}\in\mathbb{C}^{m\times n}$*,* $\mathbf{W}\in\mathbb{C}^{n\times m}$ *with* $k=\max\{Ind(\mathbf{AW}),Ind(\mathbf{WA})\}$ *and* $\operatorname{rank}(\mathbf{AW})^k=r$*, then the W-weighted Drazin inverse* $\mathbf{A}_{d,W}=\left(a_{ij}^{d,W}\right)\in\mathbb{C}^{m\times n}$ *with respect to* $\mathbf{W}$ *possess the following determinantal representations:*

$$a_{ij}^{d,W}=\frac{\sum\limits_{\beta\in J_{r,\,m}\{i\}}\left|\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,m}}\left|(\mathbf{AW})^{k+2}{}_{\beta}^{\beta}\right|}, \tag{2.36}$$

or

$$a_{ij}^{d,W} = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left| \left((\mathbf{WA})_{j\,.}^{k+2} (\bar{\mathbf{u}}_{i.}^{(k)}) \right) {}_{\alpha}^{\alpha} \right|}{\sum\limits_{\alpha \in I_{r,\,n}} \left| (\mathbf{WA})^{k+2} {}_{\alpha}^{\alpha} \right|}. \tag{2.37}$$

where $\bar{\mathbf{v}}_{.j}^{(k)}$ *is the* j*th column of* $\bar{\mathbf{V}}^k = (\mathbf{AW})^k \mathbf{A}$ *for all* $j = 1, ..., m$ *and* $\bar{\mathbf{u}}_{i.}^{(k)}$ *is the* i*th row of* $\bar{\mathbf{U}}^k = \mathbf{A}(\mathbf{WA})^k$ *for all* $i = 1, ..., n$.

Proof. At first we shall prove (2.36). By (2.29),

$$\mathbf{A}_{d,W} = \lim_{\lambda \to 0} \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)^{-1} (\mathbf{AW})^k \mathbf{A}.$$

Let

$$\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)^{-1} = \frac{1}{\det \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{m1} \\ L_{12} & L_{22} & \dots & L_{m2} \\ \dots & \dots & \dots & \dots \\ L_{1m} & L_{2m} & \dots & L_{mm} \end{pmatrix},$$

where L_{ij} is a left ij-th cofactor of a matrix $\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}$. Then we have

$$\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)^{-1} (\mathbf{AW})^k \mathbf{A} =$$
$$= \frac{1}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)} \begin{pmatrix} \sum\limits_{s=1}^{m} L_{s1} \bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{s1} \bar{v}_{s2}^{(k)} & \dots & \sum\limits_{s=1}^{m} L_{s1} \bar{v}_{sn}^{(k)} \\ \sum\limits_{s=1}^{m} L_{s2} \bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{s2} \bar{v}_{s2}^{(k)} & \dots & \sum\limits_{s=1}^{m} L_{s2} \bar{v}_{sn}^{(k)} \\ \dots & \dots & \dots & \dots \\ \sum\limits_{s=1}^{m} L_{sm} \bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{sm} \bar{v}_{s2}^{(k)} & \dots & \sum\limits_{s=1}^{m} L_{sm} \bar{v}_{sn}^{(k)} \end{pmatrix}.$$

By (2.29), we obtain

$$\mathbf{A}_{d,W} = \lim_{\lambda \to 0} \begin{pmatrix} \frac{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)_{.1} \left(\bar{\mathbf{v}}_{.1}^{(k)} \right) \right|}{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) \right|} & \dots & \frac{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)_{.1} \left(\bar{\mathbf{v}}_{.n}^{(k)} \right) \right|}{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) \right|} \\ \dots & \dots & \dots \\ \frac{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)_{.n} \left(\bar{\mathbf{v}}_{.1}^{(k)} \right) \right|}{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) \right|} & \dots & \frac{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right)_{.m} \left(\bar{\mathbf{v}}_{.n}^{(k)} \right) \right|}{\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) \right|} \end{pmatrix}. \tag{2.38}$$

By Theorem 2.4 we have

$$\left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) \right| = \lambda^m + d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \ldots + d_m,$$

where $d_s = \sum\limits_{\beta \in J_{s,\,m}} \left| \left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2} \right) {}_{\beta}^{\beta} \right|$ is a sum of principal minors of $(\mathbf{AW})^{k+2}$ of order s for all $s = \overline{1, m-1}$ and $d_m = \left| (\mathbf{AW})^{k+2} \right|$.

Since

$$\operatorname{rank}(\mathbf{AW})^{k+2} = \operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^k = r,$$

then

$$d_m = d_{m-1} = \ldots = d_{r+1} = 0.$$

It follows that $\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right) = \lambda^m + d_1\lambda^{m-1} + d_2\lambda^{m-2} + \ldots + d_r\lambda^{m-r}$.

By Lemma 2.35

$$\left|\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right| = c_1^{(ij)}\lambda^{m-1} + c_2^{(ij)}\lambda^{m-2} + \ldots + c_m^{(ij)}$$

for $i = \overline{1,m}$ and $j = \overline{1,n}$, where $c_s^{(ij)} = \sum\limits_{\beta \in J_{s,\,m}\{i\}} \left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right|$ for all $s = \overline{1, m-1}$ and $c_m^{(ij)} = \left|(\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right|$.

We shall prove that $c_k^{(ij)} = 0$, when $k \geq r+1$ for $i = \overline{1,m}$ and $j = \overline{1,n}$. By Lemma 2.33 $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right) \leq r$, then the matrix $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)$ has no more r linearly independent columns.

Consider $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$, when $\beta \in J_{s,m}\{i\}$. It is a principal submatrix of $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)$ of order $s \geq r+1$. Deleting both its i-th row and column, we obtain a principal submatrix of order $s-1$ of $(\mathbf{AW})^{k+2}$. We denote it by $\mathbf{M}$. The following cases are possible.

- Let $s = r+1$ and $\det \mathbf{M} \neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$ keeps their right-linear independence. Hence, they are basis in a matrix $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$, and the i-th column is the right linear combination of its basis columns. From this, $\left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right| = 0$, when $\beta \in J_{s,n}\{i\}$ and $s = r+1$.

- If $s = r+1$ and $\det \mathbf{M} = 0$, than p, $(p \leq r)$, columns are basis in $\mathbf{M}$ and in $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$. Then $\left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right| = 0$ as well.

- If $s > r+1$, then $\det \mathbf{M} = 0$ and p, $(p < r)$, columns are basis in the both matrices $\mathbf{M}$ and $\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$. Therefore, $\left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right| = 0$.

Thus in all cases we have $\left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right| = 0$, when $\beta \in J_{s,m}\{i\}$ and $r+1 \leq s < m$. From here if $r+1 \leq s < m$, then

$$c_s^{(ij)} = \sum_{\beta \in J_{s,\,m}\{i\}} \left|\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right| = 0,$$

and $c_m^{(ij)} = \det\left((\mathbf{AW})^{k+2}_{.i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right) = 0$ for $i = \overline{1,m}$ and $j = \overline{1,n}$.

Hence, $\left|\left(\lambda \mathbf{I} + (\mathbf{AW})^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right| = c_1^{(ij)}\lambda^{m-1} + \ldots + c_r^{(ij)}\lambda^{m-r}$ for $i = \overline{1,m}$ and $j = \overline{1,n}$. By substituting these values in the matrix from (2.38), we obtain

$$\mathbf{A}_{d,W} = \lim_{\lambda \to 0} \begin{pmatrix} \frac{c_1^{(11)}\lambda^{m-1}+...+c_r^{(11)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} & \dots & \frac{c_1^{(1n)}\lambda^{m-1}+...+c_r^{(1n)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} \\ \dots & \dots & \dots \\ \frac{c_1^{(m1)}\lambda^{m-1}+...+c_r^{(m1)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} & \dots & \frac{c_1^{(mn)}\lambda^{m-1}+...+c_r^{(mn)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} \end{pmatrix} = \begin{pmatrix} \frac{c_r^{(11)}}{d_r} & \dots & \frac{c_r^{(1n)}}{d_r} \\ \dots & \dots & \dots \\ \frac{c_r^{(m1)}}{d_r} & \dots & \frac{c_r^{(mn)}}{d_r} \end{pmatrix}.$$

where $c_r^{(ij)} = \sum_{\beta \in J_{r,\,m}\{i\}} \left| \left(\left(\mathbf{A}^{k+1}\right)_{.\,i} \left(\mathbf{a}_{.j}^{(k)}\right)\right) {}_{\beta}^{\beta} \right|$ and $d_r = \sum_{\beta \in J_{r,\,m}} \left| \left(\mathbf{A}^{k+1}\right) {}_{\beta}^{\beta} \right|$. Thus, we have obtained the determinantal representation of $\mathbf{A}_{d,W}$ by (2.36).

By analogy can be proved (2.37). ■

3. Cramer's Rules for Generalized Inverse Solutions of Systems of Linear Equations

An obvious consequence of a determinantal representation of the inverse matrix by the classical adjoint matrix is the Cramer rule. As we know, Cramer's rule gives an explicit expression for the solution of nonsingular linear equations. In [33], Robinson gave an elegant proof of Cramer's rule which aroused great interest in finding determinantal formulas for solutions of some restricted linear equations both consistent and nonconsistent. It has been widely discussed by Robinson [33], Ben-Israel [34], Verghese [35], Werner [36], Chen [37], Ji [38] ,Wang [39], Wei [31].

In this section we demonstrate that the obtained analogues of the adjoint matrix for the generalized inverse matrices enable us to obtain natural analogues of Cramer's rule for generalized inverse solutions of systems of linear equations.

3.1. Cramer's Rule for the Least Squares Solution with the Minimum Norm

Definition 3.1. *Suppose in a complex system of linear equations:*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{3.1}$$

the coefficient matrix $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and a column of constants* $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{C}^m$. *The least squares solution with the minimum norm of (3.1) is the vector* $\mathbf{x}^0 \in \mathbb{C}^n$ *satisfying*

$$\left\|\mathbf{x}^0\right\| = \min_{\tilde{\mathbf{x}} \in \mathbb{C}^n} \left\{ \|\tilde{\mathbf{x}}\| \mid \|\mathbf{A} \cdot \tilde{\mathbf{x}} - \mathbf{y}\| = \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{A} \cdot \mathbf{x} - \mathbf{y}\| \right\},$$

where $\mathbb{C}^n$ *is an* n*-dimension complex vector space.*

If the equation (3.1) has no precision solutions, then $\mathbf{x}^0$ is its optimal approximation.

The following important proposition is well-known.

Theorem 3.2. *[21] The vector* $\mathbf{x} = \mathbf{A}^+\mathbf{y}$ *is the least squares solution with the minimum norm of the system (3.1).*

Theorem 3.3. *The following statements are true for the system of linear equations (3.1).*

i) If $\operatorname{rank}\mathbf{A} = n$*, then the components of the least squares solution with the minimum norm* $\mathbf{x}^0 = \left(x_1^0, \ldots, x_n^0\right)^T$ *are obtained by the formula*

$$x_j^0 = \frac{\det(\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{f})}{\det \mathbf{A}^*\mathbf{A}}, \quad \left(\forall j = \overline{1,n}\right), \tag{3.2}$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{y}$.

ii) If $\operatorname{rank}\mathbf{A} = r \leq m < n$*, then*

$$x_j^0 = \frac{\sum\limits_{\beta \in J_{r,n}\{j\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{f})\right)_\beta^\beta\right|}{d_r\left(\mathbf{A}^*\mathbf{A}\right)}, \quad \left(\forall j = \overline{1,n}\right). \tag{3.3}$$

Proof. i) If $\operatorname{rank}\mathbf{A} = n$, then we can represent $\mathbf{A}^+$ by (2.10). By multiplying $\mathbf{A}^+$ into $\mathbf{y}$ we get (3.2).

ii) If $\operatorname{rank}\mathbf{A} = k \leq m < n$, then $\mathbf{A}^+$ can be represented by (2.5). By multiplying $\mathbf{A}^+$ into $\mathbf{y}$ the least squares solution with the minimum norm of the linear system (3.1) is given by components as in (3.3). ■ Using (2.7) and (2.11), we can obtain another representation of the Cramer rule for the least squares solution with the minimum norm of a linear system.

Theorem 3.4. *The following statements are true for a system of linear equations written in the form* $\mathbf{x} \cdot \mathbf{A} = \mathbf{y}$.

i) If $\operatorname{rank}\mathbf{A} = m$*, then the components of the least squares solution* $\mathbf{x}^0 = \mathbf{y}\mathbf{A}^+$ *are obtained by the formula*

$$x_i^0 = \frac{\det(\mathbf{A}\mathbf{A}^*)_{i.}(\mathbf{g})}{\det \mathbf{A}\mathbf{A}^*}, \quad \left(\forall i = \overline{1,m}\right),$$

where $\mathbf{g} = \mathbf{y}\mathbf{A}^*$.

ii) If $\operatorname{rank}\mathbf{A} = r \leq n < m$*, then*

$$x_i^0 = \frac{\sum\limits_{\alpha \in I_{r,m}\{i\}} \left|\left((\mathbf{A}\mathbf{A}^*)_{i.}(\mathbf{g})\right)_\alpha^\alpha\right|}{d_r\left(\mathbf{A}\mathbf{A}^*\right)}, \quad \left(\forall i = \overline{1,m}\right).$$

Proof. The proof of this theorem is analogous to that of Theorem 3.3. ■

Remark 3.5. *The obtained formulas of the Cramer rule for the least squares solution differ from similar formulas in [34, 36, 37, 38, 39]. They give a closer analogue to usual Cramer's rule for consistent nonsingular systems of linear equations.*

3.2. Cramer's Rule for the Drazin Inverse Solution

In some situations, however, people pay more attention to the Drazin inverse solution of singular linear systems [40, 41, 42, 43].

Consider a general system of linear equations (3.1), where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{x}$, $\mathbf{y}$ are vectors in $\mathbb{C}^n$. $R(\mathbf{A})$ denotes the range of $\mathbf{A}$ and $N(\mathbf{A})$ denotes the null space of $\mathbf{A}$.

The characteristic of the Drazin inverse solution $\mathbf{A}^D\mathbf{y}$ is given in [24] by the following theorem.

Theorem 3.6. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then $\mathbf{A}^D\mathbf{y}$ is both the unique solution in $R(\mathbf{A}^k)$ of*

$$\mathbf{A}^{k+1}\mathbf{x} = \mathbf{A}^{\mathrm{k}}\mathbf{y}, \tag{3.4}$$

and the unique minimal $\mathbf{P}$-norm least squares solution of (3.1).

Remark 3.7. *The $\mathbf{P}$-norm is defined as $\|\mathbf{x}\|_{\mathbf{P}} = \|\mathbf{P}^{-1}\mathbf{x}\|$ for $\mathbf{x} \in \mathbb{C}^n$, where $\mathbf{P}$ is a nonsingular matrix that transforms $\mathbf{A}$ into its Jordan canonical form (2.14).*

In other words, the the Drazin inverse solution $\mathbf{x} = \mathbf{A}^D\mathbf{y}$ is the unique solution of the problem: for a given $\mathbf{A}$ and a given vector $\mathbf{y} \in R(\mathbf{A}^k)$, find a vector $\mathbf{x} \in R(\mathbf{A}^k)$ satisfying $\mathbf{A}\mathbf{x} = \mathbf{y}$ with $Ind\,\mathbf{A} = k$.

In general, unlike $\mathbf{A}^{+}\mathbf{y}$, the Drazin inverse solution $\mathbf{A}^D\mathbf{y}$ is not a true solution of a singular system (3.1), even if the system is consistent. However, Theorem 3.6 means that $\mathbf{A}^D\mathbf{y}$ is the unique minimal P-norm least squares solution of (3.1).

A determinantal representation of the $\mathbf{P}$-norm least squares solution of a system of linear equations (3.1) by the determinantal representation (2.15) of the Drazin inverse has been obtained in [44].

We give Cramer's rule for the $\mathbf{P}$-norm least squares solution (the Drazin inverse solution) of (3.1) in the following theorem.

Theorem 3.8. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind(\mathbf{A}) = k$ and $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r$. Then the unique minimal $\mathbf{P}$-norm least squares solution $\widehat{\mathbf{x}} = (\widehat{x}_1, \ldots, \widehat{x}_n)^T$ of the system (3.1) is given by*

$$\widehat{x}_i = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left| \left(\mathbf{A}^{k+1}_{.i}\left(\mathbf{f}\right)\right)^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r,n}} \left| \left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta} \right|} \quad \forall i = \overline{1, n}, \tag{3.5}$$

where $\mathbf{f} = \mathbf{A}^k\mathbf{y}$.

Proof. Representing the Drazin inverse by (2.21) and by virtue of Theorem 3.6, we have

$$\widehat{\mathbf{x}} = \begin{pmatrix} \widehat{x}_1 \\ \ldots \\ \widehat{x}_n \end{pmatrix} = \mathbf{A}^D\mathbf{y} = \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \begin{pmatrix} \sum\limits_{s=1}^{n} d_{1s}y_s \\ \ldots \\ \sum\limits_{s=1}^{n} d_{ns}y_s \end{pmatrix}.$$

Therefore,

$$\widehat{x}_i = \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{s=1}^{n} \sum_{\beta \in J_{r,n}\{i\}} \left| \left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.\,s} \right) \right)^{\beta}_{\beta} \right| \cdot y_s =$$

$$= \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{\beta \in J_{r,\,n}\{i\}} \sum_{s=1}^{n} \left| \left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.\,s} \right) \right)^{\beta}_{\beta} \right| \cdot y_s =$$

$$= \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{\beta \in J_{r,\,n}\{i\}} \sum_{s=1}^{n} \left| \left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.\,s} \cdot y_s \right) \right)^{\beta}_{\beta} \right|.$$

From this (3.5) follows immediately. ■ If we shall present a system of linear equations as,

$$\mathbf{x}\mathbf{A} = \mathbf{y}, \tag{3.6}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$ and $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r$, then by using the Drazin inverse determinantal representation (2.20) we have the following analog of Cramer's rule for the Drazin inverse solution of (3.6):

$$\widehat{x}_i = \frac{\sum\limits_{\alpha \in I_{r,\,n}\{i\}} \left| \left(\mathbf{A}^{k+1}_{i\,.}\left(\mathbf{g} \right) \right)^{\alpha}_{\alpha} \right|}{\sum\limits_{\alpha \in I_{r,n}} \left| \left(\mathbf{A}^{k+1} \right)^{\alpha}_{\alpha} \right|}, \quad \forall i = \overline{1,n},$$

where $\mathbf{g} = \mathbf{y}\mathbf{A}^{k}$.

3.3. Cramer's Rule for the W-Weighted Drazin Inverse Solution

Consider restricted linear equations

$$\mathbf{WAW}\mathbf{x} = \mathbf{y}, \tag{3.7}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W} \in \mathbb{C}^{n\times m}$, $k_1 = Ind(\mathbf{AW})$, $k_2 = Ind(\mathbf{WA})$ with $\mathbf{y} \in R((\mathbf{WA})^{k_2})$ and $\operatorname{rank}(\mathbf{WA})^{k_2} = \operatorname{rank}(\mathbf{AW})^{k_1} = r$.

In [31], Wei has showed that there exists an unique solution $\mathbf{A}_{d,W}\mathbf{y}$ of the linear equations (3.7) and given a Cramer rule for the W-weighted Drazin inverse solution of (3.7) by the following theorem.

Theorem 3.9. *Let* $\mathbf{A}$, $\mathbf{W}$ *be the same as in (3.7). Suppose that* $\mathbf{U} \in \mathbb{C}^{n\times(n-r)}_{n-r}$ *and* $\mathbf{V}^* \in \mathbb{C}^{m\times(m-r)}_{m-r}$ *be matrices whose columns form bases for* $N((\mathbf{WA})^{k_2})$ *and* $N((\mathbf{AW})^{k_1})$*, respectively. Then the unique W-weighted Drazin inverse solution* $\mathbf{x} = (x_1, ..., x_m$ *of (3.7) satisfies*

$$x_i = \det\begin{pmatrix} \mathbf{WAW}(i \to \mathbf{y}) & \mathbf{U} \\ \mathbf{V}(i \to \mathbf{0}) & \mathbf{0} \end{pmatrix} \Big/ \det\begin{pmatrix} \mathbf{WAW} & \mathbf{U} \\ \mathbf{V} & \mathbf{0} \end{pmatrix}$$

where $i = \overline{1,m}$.

Let $k = \max\{k_1, k_2\}$. Denote $\mathbf{f} = (\mathbf{AW})^k\mathbf{A}\cdot\mathbf{y}$. Then by Theorem 2.37 using the determinantal representation (2.36) of the W-weighted Drazin inverse $\mathbf{A}_{d,W}$, we evidently obtain the following Cramer's rule of the W-weighted Drazin inverse solution of (3.7),

$$x_i = \frac{\sum\limits_{\beta\in J_{r,\,m}\{i\}} \left|\left((\mathbf{AW})^{k+2}_{.\,i}(\mathbf{f})\right){}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,m}} \left|(\mathbf{AW})^{k+2}{}^{\beta}_{\beta}\right|}, \tag{3.8}$$

where $i = \overline{1, m}$.

Remark 3.10. *Note that for (3.8) unlike Theorem 3.9, we do not need auxiliary matrices* $\mathbf{U}$ *and* $\mathbf{V}$.

3.4. Examples

1. Let us consider the system of linear equations.

$$\begin{cases} 2x_1 - 5x_3 + 4x_4 = 1, \\ 7x_1 - 4x_2 - 9x_3 + 1.5x_4 = 2, \\ 3x_1 - 4x_2 + 7x_3 - 6.5x_4 = 3, \\ x_1 - 4x_2 + 12x_3 - 10.5x_4 = 1. \end{cases} \tag{3.9}$$

The coefficient matrix of the system is $\mathbf{A} = \begin{pmatrix} 2 & 0 & -5 & 4 \\ 7 & -4 & -9 & 1.5 \\ 3 & -4 & 7 & -6.5 \\ 1 & -4 & 12 & -10.5 \end{pmatrix}$. The rank of $\mathbf{A}$ is equal to 3. We have

$$\mathbf{A}^* = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ -5 & -9 & 7 & 12 \\ 4 & 1.5 & -6.5 & -10.5 \end{pmatrix}, \mathbf{A}^*\mathbf{A} = \begin{pmatrix} 63 & -44 & -40 & -11.5 \\ -44 & 48 & -40 & 62 \\ -40 & -40 & 299 & -205 \\ -11.5 & 62 & -205 & 170.75 \end{pmatrix}.$$

At first we obtain entries of $\mathbf{A}^+$ by (2.10):

$$d_3(\mathbf{A}^*\mathbf{A}) = \begin{vmatrix} 63 & -44 & -40 \\ -44 & 48 & -40 \\ -40 & -40 & 299 \end{vmatrix} + \begin{vmatrix} 63 & -44 & -11.5 \\ -44 & 48 & 62 \\ -11.5 & 62 & 170.75 \end{vmatrix} +$$
$$+ \begin{vmatrix} 63 & -40 & -11.5 \\ -40 & 299 & -205 \\ -11.5 & -205 & 170.75 \end{vmatrix} + \begin{vmatrix} 48 & -40 & 62 \\ -40 & 299 & -205 \\ 62 & -205 & 170.75 \end{vmatrix} = 102060,$$

$$l_{11} = \begin{vmatrix} 2 & -44 & -40 \\ 0 & 48 & -40 \\ -5 & -40 & 299 \end{vmatrix} + \begin{vmatrix} 2 & -44 & -11.5 \\ 0 & 48 & 62 \\ 4 & 62 & 170.75 \end{vmatrix} + \begin{vmatrix} 2 & -40 & -11.5 \\ -5 & 299 & -205 \\ 4 & -205 & 170.75 \end{vmatrix} =$$
$$= 25779,$$

and so forth. Continuing in the same way, we get

$$\mathbf{A}^{+} = \frac{1}{102060}\begin{pmatrix} 25779 & -4905 & 20742 & -5037 \\ -3840 & -2880 & -4800 & -960 \\ 28350 & -17010 & 22680 & -5670 \\ 39558 & -18810 & 26484 & -13074 \end{pmatrix}.$$

Now we obtain the least squares solution of the system (3.9) by the matrix method.

$$\mathbf{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \end{pmatrix} = \frac{1}{102060}\begin{pmatrix} 25779 & -4905 & 20742 & -5037 \\ -3840 & -2880 & -4800 & -960 \\ 28350 & -17010 & 22680 & -5670 \\ 39558 & -18810 & 26484 & -13074 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} =$$

$$= \frac{1}{102060}\begin{pmatrix} 73158 \\ -24960 \\ 56700 \\ 68316 \end{pmatrix} = \begin{pmatrix} \frac{12193}{17010} \\ -\frac{416}{1071} \\ \frac{5}{9} \\ \frac{5693}{8505} \end{pmatrix}$$

Next we get the least squares solution with minimum norm of the system (3.9) by the Cramer rule (3.3), where

$$\mathbf{f} = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ -5 & -9 & 7 & 12 \\ 4 & 1.5 & -6.5 & -10.5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 26 \\ -24 \\ 10 \\ -23 \end{pmatrix}.$$

Thus we have

$$x_1^0 = \frac{1}{102060}\left(\begin{vmatrix} 26 & -44 & -40 \\ -24 & 48 & -40 \\ 10 & -40 & 299 \end{vmatrix} + \begin{vmatrix} 26 & -44 & -11.5 \\ -24 & 48 & 62 \\ -23 & 62 & 170.75 \end{vmatrix} + \right.$$

$$\left. + \begin{vmatrix} 26 & -40 & -11.5 \\ 10 & 299 & -205 \\ 23 & -205 & 170.75 \end{vmatrix} \right) = \frac{73158}{102060} = \frac{12193}{17010};$$

$$x_2^0 = \frac{1}{102060}\left(\begin{vmatrix} 63 & 26 & -40 \\ -44 & -24 & -40 \\ -40 & 10 & 299 \end{vmatrix} + \begin{vmatrix} 63 & 26 & -11.5 \\ -44 & -24 & 62 \\ -11.5 & -23 & 170.75 \end{vmatrix} + \right.$$

$$\left. + \begin{vmatrix} -24 & -40 & 62 \\ 10 & 299 & -205 \\ -23 & -205 & 170.75 \end{vmatrix} \right) = \frac{-24960}{102060} = -\frac{416}{1071};$$

$$x_3^0 = \frac{1}{102060}\left(\begin{vmatrix} 63 & -44 & 26 \\ -44 & 48 & -24 \\ -40 & -40 & 10 \end{vmatrix} + \begin{vmatrix} 63 & 26 & -11.5 \\ -40 & 10 & -205 \\ -11.5 & -23 & 170.75 \end{vmatrix} + \right.$$

$$+\begin{vmatrix} 48 & -24 & 62 \\ -40 & 10 & -205 \\ 62 & -23 & 170.75 \end{vmatrix}\Bigg) = \frac{56700}{102060} = \frac{5}{9};$$

$$x_4^0 = \frac{1}{102060}\left(\begin{vmatrix} 63 & -44 & 26 \\ -44 & 48 & -24 \\ -11.5 & 62 & -23 \end{vmatrix} + \begin{vmatrix} 63 & -40 & 26 \\ -40 & 299 & 10 \\ -11.5 & -205 & -23 \end{vmatrix} + \right.$$

$$\left. + \begin{vmatrix} 48 & -40 & -24 \\ -40 & 299 & 10 \\ 62 & -205 & -23 \end{vmatrix}\right) = \frac{68316}{102060} = \frac{5693}{8505}.$$

2. Let us consider the following system of linear equations.

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 1, \\ x_2 - x_3 + x_4 = 2, \\ x_1 - x_2 + x_3 + 2x_4 = 3, \\ x_1 - x_2 + x_3 + x_4 = 1. \end{cases} \tag{3.10}$$

The coefficient matrix of the system is the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{pmatrix}$. It is easy to verify the following:

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -4 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 4 & -5 & 5 & 4 \\ 3 & -4 & 4 & 3 \end{pmatrix}, \ \mathbf{A}^3 = \begin{pmatrix} 10 & -14 & 14 & 10 \\ -1 & 2 & -2 & -1 \\ 13 & -18 & 18 & 13 \\ 10 & -14 & 14 & 10 \end{pmatrix},$$

and $\operatorname{rank}\mathbf{A} = 3$, $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A}^3 = 2$. This implies $k = Ind(\mathbf{A}) = 2$. We obtain entries of $\mathbf{A}^D$ by (2.21).

$$d_2(\mathbf{A}^3) = \begin{vmatrix} 10 & -14 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 14 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix}$$
$$+ \begin{vmatrix} 2 & -2 \\ -18 & 18 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -14 & 10 \end{vmatrix} + \begin{vmatrix} 18 & 13 \\ 14 & 10 \end{vmatrix} = 8,$$

$$d_{11} = \begin{vmatrix} 3 & -14 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 14 \\ 4 & 18 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ 3 & 10 \end{vmatrix} = 4,$$

and so forth.

Continuing in the same way, we get $\mathbf{A}^D = \begin{pmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ 1.75 & 2.5 & -2.5 & 1.75 \\ 1.25 & 1.5 & -1.5 & 1.25 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{pmatrix}$. Now we obtain the Drazin inverse solution $\widehat{\mathbf{x}}$ of the system (3.10) by the Cramer rule (3.5), where

$$\mathbf{g} = \mathbf{A}^2\mathbf{y} = \begin{pmatrix} 3 & -4 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 4 & -5 & 5 & 4 \\ 3 & -4 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 13 \\ 10 \end{pmatrix}.$$

Thus we have

$$\widehat{x}_1 = \frac{1}{8}\left(\begin{vmatrix} 10 & -14 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 14 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix} \right) = \frac{1}{2},$$

$$\widehat{x}_2 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 10 & 10 \end{vmatrix} \right) = 1,$$

$$\widehat{x}_3 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ 13 & 13 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -18 & 13 \end{vmatrix} + \begin{vmatrix} 13 & 13 \\ 10 & 10 \end{vmatrix} \right) = 1,$$

$$\widehat{x}_4 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -14 & 10 \end{vmatrix} + \begin{vmatrix} 18 & 13 \\ 14 & 10 \end{vmatrix} \right) = \frac{1}{2}.$$

4. Cramer's Rule of the Generalized Inverse Solutions of Some Matrix Equations

Matrix equation is one of the important study fields of linear algebra. Linear matrix equations, such as

$$\mathbf{AX} = \mathbf{C}, \tag{4.1}$$

$$\mathbf{XB} = \mathbf{D}, \tag{4.2}$$

and

$$\mathbf{AXB} = \mathbf{D}, \tag{4.3}$$

play an important role in linear system theory therefore a large number of papers have presented several methods for solving these matrix equations [45, 46, 47, 48, 49]. In [50], Khatri and Mitra studied the Hermitian solutions to the matrix equations (4.1) and (4.3) over the complex field and the system of the equations (4.1) and (4.2). Wang, in [51, 52], and Li and Wu, in [53] studied the bisymmetric, symmetric and skew-antisymmetric least squares solution to this system over the quaternion skew field. Extreme ranks of real matrices in least squares solution of the equation (4.3) was investigated in [54] over the complex field and in [55] over the quaternion skew field.

As we know, the Cramer rule gives an explicit expression for the solution of nonsingular linear equations. Robinson's result ([33]) aroused great interest in finding determinantal representations of a least squares solution as some analogs of Cramer's rule for the matrix equations (for example, [56, 57, 58]). Cramer's rule for solutions of the restricted matrix equations (4.1), (4.2) and (4.3) was established in [59, 60, 61].

In this section, we obtain analogs of the Cramer rule for generalized inverse solutions of the aforementioned equations without any restriction.

We shall show numerical examples to illustrate the main results as well.

4.1. Cramer's Rule for the Minimum Norm Least Squares Solution of Some Matrix Equations

Definition 4.1. *Consider a matrix equation*

$$\mathbf{AX} = \mathbf{B}, \tag{4.4}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{m\times s}$ *are given,* $\mathbf{X} \in \mathbb{C}^{n\times s}$ *is unknown. Suppose*

$$S_1 = \{\mathbf{X}|\mathbf{X} \in \mathbb{C}^{n\times s}, \|\mathbf{AX} - \mathbf{B}\| = \min\}.$$

Then matrices $\mathbf{X} \in \mathbb{C}^{n\times s}$ *such that* $\mathbf{X} \in S_1$ *are called least squares solutions of the matrix equation (4.4). If* $\mathbf{X}_{LS} = \min_{\mathbf{X}\in S_1}\|\mathbf{X}\|$*, then* $\mathbf{X}_{LS}$ *is called the minimum norm least squares solution of (4.4).*

If the equation (4.4) has no precision solutions, then $\mathbf{X}_{LS}$ is its optimal approximation. The following important proposition is well-known.

Lemma 4.2. *([38]) The least squares solutions of (4.4) are*

$$\mathbf{X} = \mathbf{A}^{+}\mathbf{B} + (\mathbf{I}_n - \mathbf{A}^{+}\mathbf{A})\mathbf{C},$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{m\times s}$ *are given, and* $\mathbf{C} \in \mathbb{C}^{n\times s}$ *is an arbitrary matrix. The least squares minimum norm solution is* $\mathbf{X}_{LS} = \mathbf{A}^{+}\mathbf{B}$.

We denote $\mathbf{A}^*\mathbf{B} =: \hat{\mathbf{B}} = (\hat{b}_{ij}) \in \mathbb{C}^{n\times s}$.

Theorem 4.3. *(i) If* $\operatorname{rank}\mathbf{A} = r \leq m < n$*, then we have for the minimum norm least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n\times s}$ *of (4.4) for all* $i = \overline{1, n}$*,* $j = \overline{1, s}$

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{\,.\,i}\left(\hat{\mathbf{b}}_{.j}\right)\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|}. \tag{4.5}$$

(ii) If $\operatorname{rank}\mathbf{A} = n$*, then for all* $i = \overline{1, n}$*,* $j = \overline{1, s}$ *we have*

$$x_{i\,j} = \frac{\det(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\hat{\mathbf{b}}_{.j}\right)}{\det(\mathbf{A}^*\mathbf{A})}, \tag{4.6}$$

where $\hat{\mathbf{b}}_{.j}$ *is the* j*th column of* $\hat{\mathbf{B}}$ *for all* $j = \overline{1, s}$.

Proof. i) If $\operatorname{rank}\mathbf{A} = r \leq m < n$, then by Theorem 2.9 we can represent $\mathbf{A}^{+}$ by (2.5). Therefore, we obtain for all $i = \overline{1, n}$, $j = \overline{1, s}$

$$x_{ij} = \sum_{k=1}^{m} a_{ik}^{+} b_{kj} = \sum_{k=1}^{m} \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{\,.\,i}\left(\mathbf{a}^*_{.k}\right)\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|} \cdot b_{kj} =$$

$$\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \sum_{k=1}^{m} \left|\left((\mathbf{A}^*\mathbf{A})_{\,.\,i}\left(\mathbf{a}^*_{.k}\right)\right)\,{}^{\beta}_{\beta}\right| \cdot b_{kj}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|}.$$

Since $\sum\limits_k \mathbf{a}^*_{.\,k} b_{kj} = \begin{pmatrix} \sum\limits_k a^*_{1k} b_{kj} \\ \sum\limits_k a^*_{2k} b_{kj} \\ \vdots \\ \sum\limits_k a^*_{nk} b_{kj} \end{pmatrix} = \hat{\mathbf{b}}_{.j}$, then it follows (4.5).

(ii) The proof of this case is similarly to that of (i) by using Corollary 2.3. ■

Definition 4.4. *Consider a matrix equation*

$$\mathbf{X}\mathbf{A} = \mathbf{B}, \tag{4.7}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{s\times n}$ *are given,* $\mathbf{X} \in \mathbb{C}^{s\times m}$ *is unknown. Suppose*

$$S_2 = \{\mathbf{X} |\, \mathbf{X} \in \mathbb{C}^{s\times m}, \|\mathbf{X}\mathbf{A} - \mathbf{B}\| = \min\}.$$

Then matrices $\mathbf{X} \in \mathbb{C}^{s\times m}$ *such that* $\mathbf{X} \in S_2$ *are called least squares solutions of the matrix equation (4.7). If* $\mathbf{X}_{LS} = \min_{\mathbf{X}\in S_2} \|\mathbf{X}\|$*, then* $\mathbf{X}_{LS}$ *is called the minimum norm least squares solution of (4.7).*

The following lemma can be obtained by analogy to Lemma 4.2.

Lemma 4.5. *The least squares solutions of (4.7) are*

$$\mathbf{X} = \mathbf{B}\mathbf{A}^+ + \mathbf{C}(\mathbf{I}_m - \mathbf{A}\mathbf{A}^+),$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{s\times n}$ *are given, and* $\mathbf{C} \in \mathbb{C}^{s\times m}$ *is an arbitrary matrix. The minimum norm least squares solution is* $\mathbf{X}_{LS} = \mathbf{B}\mathbf{A}^+$.

We denote $\mathbf{B}\mathbf{A}^* =: \check{\mathbf{B}} = (\check{b}_{ij}) \in \mathbb{C}^{s\times m}$.

Theorem 4.6. *(i) If* $\operatorname{rank}\mathbf{A} = r \leq n < m$*, then we have for the minimum norm least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{s\times m}$ *of (4.7) for all* $i = \overline{1,s}$*,* $j = \overline{1,m}$

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left((\mathbf{A}\mathbf{A}^*)_{j\,.}\left(\check{\mathbf{b}}_{i\,.}\right)\right)\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}\right|}. \tag{4.8}$$

(ii) If $\operatorname{rank}\mathbf{A} = m$*, then for all* $i = \overline{1,s}$*,* $j = \overline{1,m}$ *we have*

$$x_{ij} = \frac{\det(\mathbf{A}\mathbf{A}^*)_{j.}\left(\check{\mathbf{b}}_{i\,.}\right)}{\det(\mathbf{A}\mathbf{A}^*)}, \tag{4.9}$$

where $\check{\mathbf{b}}_{i.}$ *is the ith row of* $\check{\mathbf{B}}$ *for all* $i = \overline{1,s}$*.*

Proof. (i) If $\operatorname{rank}\mathbf{A} = r \leq n < m$, then by Theorem 2.9 we can represent $\mathbf{A}^+$ by (2.6). Therefore, for all $i = \overline{1,s}$, $j = \overline{1,m}$ we obtain

$$x_{ij} = \sum_{k=1}^{n} b_{ik} a^+_{kj} = \sum_{k=1}^{n} b_{ik} \cdot \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left((\mathbf{A}\mathbf{A}^*)_{j\,.}\left(\mathbf{a}^*_{k\,.}\right)\right)\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}\right|} =$$

$$\frac{\sum_{k=1}^{n} b_{ik} \sum\limits_{\alpha \in I_{r,m}\{j\}} \left| \left((\mathbf{AA}^*)_{j\,.} \left(\mathbf{a}^*_{k\,.}\right) \right) {}^{\alpha}_{\alpha} \right|}{\sum\limits_{\alpha \in I_{r,\,m}} |(\mathbf{AA}^*) {}^{\alpha}_{\alpha}|}$$

Since for all $i = \overline{1, s}$

$$\sum_k b_{ik}\mathbf{a}^*_{k\,.} = \left(\sum_k b_{ik} a^*_{k1} \quad \sum_k b_{ik} a^*_{k2} \quad \cdots \quad \sum_k b_{ik} a^*_{km} \right) = \check{\mathbf{b}}_{i.},$$

then it follows (4.8).

(ii) The proof of this case is similarly to that of (i) by using Corollary 2.3. ■

Definition 4.7. *Consider a matrix equation*

$$\mathbf{AXB} = \mathbf{D}, \tag{4.10}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}_{r_1}, \mathbf{B} \in \mathbb{C}^{p\times q}_{r_2}, \mathbf{D} \in \mathbb{C}^{m\times q}$ *are given,* $\mathbf{X} \in \mathbb{C}^{n\times p}$ *is unknown. Suppose*

$$S_3 = \{\mathbf{X} |\, \mathbf{X} \in \mathbb{C}^{n\times p}, \|\mathbf{AXB} - \mathbf{D}\| = \min\}.$$

Then matrices $\mathbf{X} \in \mathbb{C}^{n\times p}$ *such that* $\mathbf{X} \in S_3$ *are called least squares solutions of the matrix equation (4.10). If* $\mathbf{X}_{LS} = \min_{\mathbf{X}\in S_3} \|\mathbf{X}\|$*, then* $\mathbf{X}_{LS}$ *is called the minimum norm least squares solution of (4.10).*

The following important proposition is well-known.

Lemma 4.8. *([36]) The least squares solutions of (4.10) are*

$$\mathbf{X} = \mathbf{A}^+\mathbf{DB}^+ + (\mathbf{I}_n - \mathbf{A}^+\mathbf{A})\mathbf{V} + \mathbf{W}(\mathbf{I}_p - \mathbf{BB}^+),$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}_{r_1}, \mathbf{B} \in \mathbb{C}^{p\times q}_{r_2}, \mathbf{D} \in \mathbb{C}^{m\times q}$ *are given, and* $\{\mathbf{V}, \mathbf{W}\} \subset \mathbb{C}^{n\times p}$ *are arbitrary quaternion matrices. The minimum norm least squares solution is* $\mathbf{X}_{LS} = \mathbf{A}^+\mathbf{DB}^+$.

We denote $\widetilde{\mathbf{D}} = \mathbf{A}^*\mathbf{DB}^*$.

Theorem 4.9. *(i) If* $\operatorname{rank} \mathbf{A} = r_1 < n$ *and* $\operatorname{rank} \mathbf{B} = r_2 < p$*, then for the minimum norm least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n\times p}$ *of (4.10) we have*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,\,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} \left(\mathbf{d}^{\mathbf{B}}_{\,.\,j} \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^*\mathbf{A})^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{BB}^*)^{\alpha}_{\alpha}|}, \tag{4.11}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \left| (\mathbf{BB}^*)_{j\,.} \left(\mathbf{d}^{\mathbf{A}}_{i\,.} \right) {}^{\alpha}_{\alpha} \right|}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^*\mathbf{A})^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{BB}^*)^{\alpha}_{\alpha}|}, \tag{4.12}$$

where

$$\mathbf{d}_{.j}^{\mathbf{B}} = \left[\sum_{\alpha \in I_{r_2,p}\{j\}} \left| (\mathbf{BB}^*)_{j.} \left(\tilde{\mathbf{d}}_{1.} \right) {}_{\alpha}^{\alpha} \right|, ..., \sum_{\alpha \in I_{r_2,p}\{j\}} \left| (\mathbf{BB}^*)_{j.} \left(\tilde{\mathbf{d}}_{n.} \right) {}_{\alpha}^{\alpha} \right| \right]^T, \quad (4.13)$$

$$\mathbf{d}_{i.}^{\mathbf{A}} = \left[\sum_{\beta \in J_{r_1,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} \left(\tilde{\mathbf{d}}_{.1} \right) {}_{\beta}^{\beta} \right|, ..., \sum_{\alpha \in I_{r_1,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} \left(\tilde{\mathbf{d}}_{.p} \right) {}_{\beta}^{\beta} \right| \right] \quad (4.14)$$

are the column-vector and the row-vector, respectively. $\tilde{\mathbf{d}}_{i\,.}$ *is the* i*-th row of* $\widetilde{\mathbf{D}}$ *for all* $i = \overline{1,n}$*, and* $\tilde{\mathbf{d}}_{.\,j}$ *is the* j*-th column of* $\widetilde{\mathbf{D}}$ *for all* $j = \overline{1,p}$*.*

(ii) If $\operatorname{rank} \mathbf{A} = n$ *and* $\operatorname{rank} \mathbf{B} = p$*, then for the least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ *of (4.10) we have for all* $i = \overline{1,n}$*,* $j = \overline{1,p}$*,*

$$x_{i\,j} = \frac{\det\left((\mathbf{A}^*\mathbf{A})_{.\,i} \left(\mathbf{d}_{.j}^{\mathbf{B}} \right) \right)}{\det(\mathbf{A}^*\mathbf{A}) \cdot \det(\mathbf{BB}^*)}, \quad (4.15)$$

or

$$x_{i\,j} = \frac{\det\left((\mathbf{BB}^*)_{j.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}} \right) \right)}{\det(\mathbf{A}^*\mathbf{A}) \cdot \det(\mathbf{BB}^*)}, \quad (4.16)$$

where

$$\mathbf{d}_{.j}^{\mathbf{B}} := \left[\det\left((\mathbf{BB}^*)_{j.} \left(\tilde{\mathbf{d}}_{1\,.} \right) \right), \ldots, \det\left((\mathbf{BB}^*)_{j.} \left(\tilde{\mathbf{d}}_{n\,.} \right) \right) \right]^T, \quad (4.17)$$

$$\mathbf{d}_{i\,.}^{\mathbf{A}} := \left[\det\left((\mathbf{A}^*\mathbf{A})_{.\,i} \left(\tilde{\mathbf{d}}_{.1} \right) \right), \ldots, \det\left((\mathbf{A}^*\mathbf{A})_{.\,i} \left(\tilde{\mathbf{d}}_{.p} \right) \right) \right] \quad (4.18)$$

are respectively the column-vector and the row-vector.

(iii) If $\operatorname{rank} \mathbf{A} = n$ *and* $\operatorname{rank} \mathbf{B} = r_2 < p$*, then for the least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ *of (4.10) we have*

$$x_{ij} = \frac{\det\left((\mathbf{A}^*\mathbf{A})_{.\,i} \left(\mathbf{d}_{.j}^{\mathbf{B}} \right) \right)}{\det(\mathbf{A}^*\mathbf{A}) \sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{BB}^*)_{\alpha}^{\alpha}|}, \quad (4.19)$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \left| (\mathbf{BB}^*)_{j\,.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}} \right) {}_{\alpha}^{\alpha} \right|}{\det(\mathbf{A}^*\mathbf{A}) \sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{BB}^*)_{\alpha}^{\alpha}|}, \quad (4.20)$$

where $\mathbf{d}_{.j}^{\mathbf{B}}$ *is (4.13) and* $\mathbf{d}_{i\,.}^{\mathbf{A}}$ *is (4.18).*

(iiii) If $\operatorname{rank} \mathbf{A} = r_1 < m$ *and* $\operatorname{rank} \mathbf{B} = p$*, then for the least squares solution* $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ *of (4.10) we have*

$$x_{i\,j} = \frac{\det\left((\mathbf{BB}^*)_{j.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}} \right) \right)}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^*\mathbf{A})_{\beta}^{\beta} \right| \cdot \det(\mathbf{BB}^*)}, \quad (4.21)$$

or

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} \left(\mathbf{d}^{\mathbf{B}}_{\,.\,j} \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^*\mathbf{A})^{\beta}_{\beta} \right| \det(\mathbf{B}\mathbf{B}^*)}, \tag{4.22}$$

where $\mathbf{d}^{\mathbf{B}}_{\,.\,j}$ *is (4.17) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (4.14).*

Proof. (i) If $\mathbf{A} \in \mathbb{C}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{C}^{p\times q}_{r_2}$ and $r_1 < n$, $r_2 < p$, then by Theorem 2.9 the Moore-Penrose inverses $\mathbf{A}^+ = \left(a^+_{ij} \right) \in \mathbb{C}^{n\times m}$ and $\mathbf{B}^+ = \left(b^+_{ij} \right) \in \mathbb{C}^{q\times p}$ possess the following determinantal representations respectively,

$$a^+_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} \left(\mathbf{a}^*_{.j} \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A})\, {}^{\beta}_{\beta} \right|},$$

$$b^+_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\,.} (\mathbf{b}^*_{i.}) \, {}^{\alpha}_{\alpha}|}{\sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*) \, {}^{\alpha}_{\alpha}|}. \tag{4.23}$$

Since by Theorem 4.8 $\mathbf{X}_{LS} = \mathbf{A}^+\mathbf{D}\mathbf{B}^+$, then an entry of $\mathbf{X}_{LS} = (x_{ij})$ is

$$x_{ij} = \sum_{s=1}^{q} \left(\sum_{k=1}^{m} a^+_{ik} d_{ks} \right) b^+_{sj}. \tag{4.24}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the sth column of $\mathbf{A}^*\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{C}^{n\times q}$ for all $s = \overline{1,q}$. It follows from $\sum\limits_{k} \mathbf{a}^*_{.\,k} d_{ks} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{k=1}^{m} a^+_{ik} d_{ks} = \sum_{k=1}^{m} \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} (\mathbf{a}^*_{.k}) \, {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) \, {}^{\beta}_{\beta} \right|} \cdot d_{ks} =$$

$$\frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \sum\limits_{k=1}^{m} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} (\mathbf{a}^*_{.k}) \, {}^{\beta}_{\beta} \right| \cdot d_{ks}}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) \, {}^{\beta}_{\beta} \right|} = \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} \left(\hat{\mathbf{d}}_{.\,s} \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) \, {}^{\beta}_{\beta} \right|} \tag{4.25}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are respectively the unit row-vector and the unit column-vector whose components are 0, except the sth components, which are 1. Substituting (4.25) and (4.23) in (4.24), we obtain

$$x_{ij} = \sum_{s=1}^{q} \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{\,.\,i} \left(\hat{\mathbf{d}}_{.\,s} \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) \, {}^{\beta}_{\beta} \right|} \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\,.} (\mathbf{b}^*_{s.}) \, {}^{\alpha}_{\alpha}|}{\sum\limits_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*) \, {}^{\alpha}_{\alpha}|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{l=1}^{n} \mathbf{e}_{.\,l} \hat{d}_{ls}, \quad \mathbf{b}^*_{s.} = \sum_{t=1}^{p} b^*_{st} \mathbf{e}_{t.}, \quad \sum_{s=1}^{q} \hat{d}_{ls} b^*_{st} = \widetilde{d}_{lt}, \tag{4.26}$$

then we have

$$x_{ij} = \frac{\sum\limits_{s=1}^{q}\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{e}_{.\,l})\,{}^{\beta}_{\beta}\right| \hat{d}_{ls} b^*_{st} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^*)_{j\,.}(\mathbf{e}_{t.})\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^*)\,{}^{\alpha}_{\alpha}\right|} =$$

$$\frac{\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{e}_{.\,l})\,{}^{\beta}_{\beta}\right| \widetilde{d}_{lt} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^*)_{j\,.}(\mathbf{e}_{t.})\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^*)\,{}^{\alpha}_{\alpha}\right|}. \tag{4.27}$$

Denote by

$$d^{\mathbf{A}}_{it} := \sum_{\beta\in J_{r_1,\,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\widetilde{\mathbf{d}}_{.\,t}\right){}^{\beta}_{\beta}\right| = \sum_{l=1}^{n}\sum_{\beta\in J_{r_1,\,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{e}_{.\,l})\,{}^{\beta}_{\beta}\right| \widetilde{d}_{lt}$$

the t-th component of a row-vector $\mathbf{d}^{\mathbf{A}}_{i\,.} = (d^{\mathbf{A}}_{i1}, ..., d^{\mathbf{A}}_{ip})$ for all $t = \overline{1,p}$. Substituting it in (4.27), we have

$$x_{ij} = \frac{\sum\limits_{t=1}^{p} d^{\mathbf{A}}_{it} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^*)_{j\,.}(\mathbf{e}_{t.})\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^*)\,{}^{\alpha}_{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{p} d^{\mathbf{A}}_{it}\mathbf{e}_{t.} = \mathbf{d}^{\mathbf{A}}_{i\,.}$, then it follows (4.12).

If we denote by

$$d^{\mathbf{B}}_{lj} := \sum_{t=1}^{p} \widetilde{d}_{lt} \sum_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^*)_{j\,.}(\mathbf{e}_{t.})\,{}^{\alpha}_{\alpha}\right| = \sum_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^*)_{j\,.}(\widetilde{\mathbf{d}}_{l.})\,{}^{\alpha}_{\alpha}\right| \tag{4.28}$$

the l-th component of a column-vector $\mathbf{d}^{\mathbf{B}}_{.j} = (d^{\mathbf{B}}_{1j}, ..., d^{\mathbf{B}}_{jn})^T$ for all $l = \overline{1,n}$ and substitute it in (4.27), we obtain

$$x_{ij} = \frac{\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{e}_{.\,l})\,{}^{\beta}_{\beta}\right| d^{\mathbf{B}}_{lj}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^*)\,{}^{\alpha}_{\alpha}\right|}.$$

Since $\sum\limits_{l=1}^{n} \mathbf{e}_{.l} d^{\mathbf{B}}_{lj} = \mathbf{d}^{\mathbf{B}}_{.j}$, then it follows (4.11).

(ii) If $\operatorname{rank}\mathbf{A} = n$ and $\operatorname{rank}\mathbf{B} = p$, then by Corollary 2.3 $\mathbf{A}^{+} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ and $\mathbf{B}^{+} = \mathbf{B}^*(\mathbf{B}\mathbf{B}^*)^{-1}$. Therefore, we obtain

$$\mathbf{X}_{LS} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{D}\mathbf{B}^*(\mathbf{B}\mathbf{B}^*)^{-1} =$$

$$= \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} = \frac{1}{\det(\mathbf{A}^*\mathbf{A})} \begin{pmatrix} L_{11}^{\mathbf{A}} & L_{21}^{\mathbf{A}} & \dots & L_{n1}^{\mathbf{A}} \\ L_{12}^{\mathbf{A}} & L_{22}^{\mathbf{A}} & \dots & L_{n2}^{\mathbf{A}} \\ \dots & \dots & \dots & \dots \\ L_{1n}^{\mathbf{A}} & L_{2n}^{\mathbf{A}} & \dots & L_{nn}^{\mathbf{A}} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \dots & \tilde{d}_{1m} \\ \tilde{d}_{21} & \tilde{d}_{22} & \dots & \tilde{d}_{2m} \\ \dots & \dots & \dots & \dots \\ \tilde{d}_{n1} & \tilde{d}_{n2} & \dots & \tilde{d}_{nm} \end{pmatrix} \frac{1}{\det(\mathbf{B}\mathbf{B}^*)} \begin{pmatrix} R_{11}^{\mathbf{B}} & R_{21}^{\mathbf{B}} & \dots & R_{p1}^{\mathbf{B}} \\ R_{12}^{\mathbf{B}} & R_{22}^{\mathbf{B}} & \dots & R_{p2}^{\mathbf{B}} \\ \dots & \dots & \dots & \dots \\ R_{1p}^{\mathbf{B}} & R_{2p}^{\mathbf{B}} & \dots & R_{pp}^{\mathbf{B}} \end{pmatrix},$$

where $\tilde{d}_{ij}$ is ij-th entry of the matrix $\widetilde{\mathbf{D}}$, $L_{ij}^{\mathbf{A}}$ is the ij-th cofactor of $(\mathbf{A}^*\mathbf{A})$ for all $i, j = \overline{1, n}$ and $R_{ij}^{\mathbf{B}}$ is the ij-th cofactor of $(\mathbf{B}\mathbf{B}^*)$ for all $i, j = \overline{1, p}$. This implies

$$x_{ij} = \frac{\sum\limits_{k=1}^{n} L_{ki}^{\mathbf{A}} \left(\sum\limits_{s=1}^{p} \tilde{d}_{ks} R_{js}^{\mathbf{B}} \right)}{\det(\mathbf{A}^*\mathbf{A}) \cdot \det(\mathbf{B}\mathbf{B}^*)}, \tag{4.29}$$

for all $i = \overline{1, n}$, $j = \overline{1, p}$. We obtain the sum in parentheses and denote it as follows

$$\sum_{s=1}^{p} \tilde{d}_{k\,s} R_{j\,s}^{\mathbf{B}} = \det(\mathbf{B}\mathbf{B}^*)_{j.}\left(\tilde{\mathbf{d}}_{k.}\right) := d_{k\,j}^{\mathbf{B}},$$

where $\tilde{\mathbf{d}}_{k.}$ is the k-th row-vector of $\tilde{\mathbf{D}}$ for all $k = \overline{1, n}$. Suppose $\mathbf{d}_{.j}^{\mathbf{B}} := \left(d_{1\,j}^{\mathbf{B}}, \dots, d_{n\,j}^{\mathbf{B}}\right)^T$ is the column-vector for all $j = \overline{1, p}$. Reducing the sum $\sum\limits_{k=1}^{n} L_{ki}^{\mathbf{A}} d_{k\,j}^{\mathbf{B}}$, we obtain an analog of Cramer's rule for (4.10) by (4.15).

Interchanging the order of summation in (4.29), we have

$$x_{ij} = \frac{\sum\limits_{s=1}^{p} \left(\sum\limits_{k=1}^{n} L_{ki}^{\mathbf{A}} \tilde{d}_{ks} \right) R_{js}^{\mathbf{B}}}{\det(\mathbf{A}^*\mathbf{A}) \cdot \det(\mathbf{B}\mathbf{B}^*)}.$$

We obtain the sum in parentheses and denote it as follows

$$\sum_{k=1}^{n} L_{ki}^{\mathbf{A}} \tilde{d}_{k\,s} = \det(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\tilde{\mathbf{d}}_{.\,s}\right) =: d_{i\,s}^{\mathbf{A}},$$

where $\tilde{\mathbf{d}}_{.\,s}$ is the s-th column-vector of $\tilde{\mathbf{D}}$ for all $s = \overline{1, p}$. Suppose $\mathbf{d}_{i.}^{\mathbf{A}} := \left(d_{i\,1}^{\mathbf{A}}, \dots, d_{i\,p}^{\mathbf{A}}\right)$ is the row-vector for all $i = \overline{1, n}$. Reducing the sum $\sum\limits_{s=1}^{n} d_{i\,s}^{\mathbf{A}} R_{js}^{\mathbf{B}}$, we obtain another analog of Cramer's rule for the least squares solutions of (4.10) by (4.16).

(iii) If $\mathbf{A} \in \mathbb{C}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{C}^{p\times q}_{r_2}$ and $r_1 = n$, $r_2 < p$, then by Remark 2.12 and Theorem 2.9 the Moore-Penrose inverses $\mathbf{A}^+ = \left(a^+_{ij}\right) \in \mathbb{C}^{n\times m}$ and $\mathbf{B}^+ = \left(b^+_{ij}\right) \in \mathbb{C}^{q\times p}$ possess the following determinantal representations respectively,

$$a^+_{ij} = \frac{\det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)}{\det\left(\mathbf{A}^*\mathbf{A}\right)},$$

$$b^+_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{BB}^*)_{j\,.}\left(\mathbf{b}^*_{i.}\right)^{\,\alpha}_{\,\alpha}\right|}{\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{BB}^*)^{\,\alpha}_{\,\alpha}\right|}. \tag{4.30}$$

Since by Theorem 4.8 $\mathbf{X}_{LS} = \mathbf{A}^+\mathbf{D}\mathbf{B}^+$, then an entry of $\mathbf{X}_{LS} = (x_{ij})$ is (4.24). Denote by $\hat{\mathbf{d}}_{.s}$ the s-th column of $\mathbf{A}^*\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{C}^{n\times q}$ for all $s = \overline{1,q}$. It follows from $\sum\limits_k \mathbf{a}^*_{.\,k}d_{ks} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{k=1}^{m} a^+_{ik}d_{ks} = \sum_{k=1}^{m} \frac{\det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.k}\right)}{\det\left(\mathbf{A}^*\mathbf{A}\right)}\cdot d_{ks} = \frac{\det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)}{\det\left(\mathbf{A}^*\mathbf{A}\right)} \tag{4.31}$$

Substituting (4.31) and (4.30) in (4.24), and using (4.26) we have

$$x_{ij} = \sum_{s=1}^{q} \frac{\det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)}{\det\left(\mathbf{A}^*\mathbf{A}\right)} \frac{\sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{BB}^*)_{j\,.}\left(\mathbf{b}^*_{s.}\right)^{\,\alpha}_{\,\alpha}\right|}{\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{BB}^*)^{\,\alpha}_{\,\alpha}\right|} =$$

$$\frac{\sum\limits_{s=1}^{q}\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n} \det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\hat{d}_{ls}b^*_{st} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{BB}^*)_{j\,.}\left(\mathbf{e}_{t.}\right)^{\,\alpha}_{\,\alpha}\right|}{\det\left(\mathbf{A}^*\mathbf{A}\right)\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{BB}^*)^{\,\alpha}_{\,\alpha}\right|} =$$

$$\frac{\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n} \det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\ \widetilde{d}_{lt} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \left|(\mathbf{BB}^*)_{j\,.}\left(\mathbf{e}_{t.}\right)^{\,\alpha}_{\,\alpha}\right|}{\det\left(\mathbf{A}^*\mathbf{A}\right)\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{BB}^*)^{\,\alpha}_{\,\alpha}\right|}. \tag{4.32}$$

If we substitute (4.28) in (4.32), then we get

$$x_{ij} = \frac{\sum\limits_{l=1}^{n} \det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\ d^{\mathbf{B}}_{lj}}{\det\left(\mathbf{A}^*\mathbf{A}\right)\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{BB}^*)^{\,\alpha}_{\,\alpha}\right|}.$$

Since again $\sum\limits_{l=1}^{n} \mathbf{e}_{.l}d^{\mathbf{B}}_{lj} = \mathbf{d}^{\mathbf{B}}_{.j}$, then it follows (4.19), where $\mathbf{d}^{\mathbf{B}}_{.j}$ is (4.13).

If we denote by

$$d^{\mathbf{A}}_{it} :=$$

$$\sum_{l=1}^{n} \det \left(\mathbf{A}^{*}\mathbf{A}\right)_{.\,i} \left(\widetilde{\mathbf{d}}_{.\,t}\right) = \sum_{l=1}^{n} \det \left(\mathbf{A}^{*}\mathbf{A}\right)_{.\,i} \left(\mathbf{e}_{.\,l}\right) \widetilde{d}_{lt}$$

the t-th component of a row-vector $\mathbf{d}^{\mathbf{A}}_{i\,.} = (d^{\mathbf{A}}_{i1}, ..., d^{\mathbf{A}}_{ip})$ for all $t = \overline{1,p}$ and substitute it in (4.32), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{p} d^{\mathbf{A}}_{it} \sum\limits_{\alpha \in I_{r_2,p}\{j\}} \left|(\mathbf{B}\mathbf{B}^{*})_{j\,.}\left(\mathbf{e}_{t.}\right) {}^{\alpha}_{\alpha}\right|}{\det\left(\mathbf{A}^{*}\mathbf{A}\right) \sum\limits_{\alpha \in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{*}) {}^{\alpha}_{\alpha}\right|}.$$

Since again $\sum\limits_{t=1}^{p} d^{\mathbf{A}}_{it}\mathbf{e}_{t.} = \mathbf{d}^{\mathbf{A}}_{i\,.}$, then it follows (4.20), where $\mathbf{d}^{\mathbf{A}}_{i\,.}$ is (4.18).

(iiii) The proof is similar to the proof of (iii). ■

4.2. Cramer's Rule of the Drazin Inverse Solutions of Some Matrix Equations

Consider a matrix equation

$$\mathbf{A}\mathbf{X} = \mathbf{B}, \tag{4.33}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind\,\mathbf{A} = k$, $\mathbf{B} \in \mathbb{C}^{n\times m}$ are given and $\mathbf{X} \in \mathbb{C}^{n\times m}$ is unknown.

Theorem 4.10. *([62], Theorem 1) If the range space $R(\mathbf{B}) \subset R(\mathbf{A}^k)$, then the matrix equation (4.33) with constrain $R(\mathbf{X}) \subset R(\mathbf{A}^k)$ has a unique solution*

$$\mathbf{X} = \mathbf{A}^{D}\mathbf{B}.$$

We denote $\mathbf{A}^k\mathbf{B} =: \hat{\mathbf{B}} = (\hat{b}_{ij}) \in \mathbb{C}^{n\times m}$.

Theorem 4.11. *If* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^{D}\mathbf{B} = (x_{ij}) \in \mathbb{C}^{n\times m}$ *of (4.33) we have for all* $i = \overline{1,n}$, $j = \overline{1,m}$,

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\hat{\mathbf{b}}_{.j}\right)\right) {}^{\beta}_{\beta}\right|}{\sum\limits_{\beta \in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right) {}^{\beta}_{\beta}\right|}. \tag{4.34}$$

Proof. By Theorem 2.29 we can represent $\mathbf{A}^{D}$ by (2.21). Therefore, we obtain for all $i = \overline{1,n}$, $j = \overline{1,m}$,

$$x_{ij} = \sum_{s=1}^{n} a^{D}_{is} b_{sj} = \sum_{s=1}^{n} \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.s}\right)\right) {}^{\beta}_{\beta}\right|}{\sum\limits_{\beta \in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right) {}^{\beta}_{\beta}\right|} \cdot b_{sj} =$$

$$\frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \sum_{s=1}^{n} \left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.s}\right)\right) {}^{\beta}_{\beta}\right| \cdot b_{sj}}{\sum\limits_{\beta \in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right) {}^{\beta}_{\beta}\right|}.$$

Since $\sum\limits_{s} \mathbf{a}_{.s}^{(k)} b_{sj} = \begin{pmatrix} \sum\limits_{s} a_{1s}^{(k)} b_{sj} \\ \sum\limits_{s} a_{2s}^{(k)} b_{sj} \\ \vdots \\ \sum\limits_{s} a_{ns}^{(k)} b_{sj} \end{pmatrix} = \hat{\mathbf{b}}_{.j}$, then it follows (4.34). ■

Consider a matrix equation

$$\mathbf{XA} = \mathbf{B}, \tag{4.35}$$

where $\mathbf{A} \in \mathbb{C}^{m\times m}$ with $Ind\mathbf{A} = k$, $\mathbf{B} \in \mathbb{C}^{n\times m}$ are given and $\mathbf{X} \in \mathbb{C}^{n\times m}$ is unknown.

Theorem 4.12. *([62], Theorem 2) If the null space $N(\mathbf{B}) \supset N(\mathbf{A}^k)$, then the matrix equation (4.35) with constrain $N(\mathbf{X}) \supset N(\mathbf{A}^k)$ has a unique solution*

$$\mathbf{X} = \mathbf{BA}^D.$$

We denote $\mathbf{BA}^k =: \check{\mathbf{B}} = (\check{b}_{ij}) \in \mathbb{C}^{n\times m}$.

Theorem 4.13. *If* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le m$ *for* $\mathbf{A} \in \mathbb{C}^{m\times m}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{BA}^D = (x_{ij}) \in \mathbb{C}^{n\times m}$ *of (4.35), we have for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*,*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left(\mathbf{A}_{j\,.}^{k+1}\left(\check{\mathbf{b}}_{i\,.}\right)\right)\,{}_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{A}^{k+1}\right)\,{}_{\alpha}^{\alpha}\right|}. \tag{4.36}$$

Proof. By Theorem 2.29 we can represent $\mathbf{A}^D$ by (2.20). Therefore, we obtain for all $i = \overline{1,n}$, $j = \overline{1,m}$,

$$x_{ij} = \sum_{s=1}^{m} b_{is} a_{sj}^D = \sum_{s=1}^{m} b_{is} \cdot \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left(\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{s\,.}^{(k)}\right)\right)\,{}_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{A}^{k+1}\right)\,{}_{\alpha}^{\alpha}\right|} =$$

$$\frac{\sum_{s=1}^{m} b_{ik} \sum\limits_{\alpha\in I_{r,m}\{j\}} \left|\left(\mathbf{A}_{j\,.}^{k+1}\left(\mathbf{a}_{s\,.}^{(k)}\right)\right)\,{}_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{A}^{k+1}\right)\,{}_{\alpha}^{\alpha}\right|}$$

Since for all $i = \overline{1,n}$

$$\sum_{s} b_{is}\mathbf{a}_{s\,.}^{(k)} = \begin{pmatrix} \sum\limits_{s} b_{is} a_{s1}^{(k)} & \sum\limits_{s} b_{is} a_{s2}^{(k)} & \cdots & \sum\limits_{s} b_{is} a_{sm}^{(k)} \end{pmatrix} = \check{\mathbf{b}}_{i.},$$

then it follows (4.36). ■

Consider a matrix equation

$$\mathbf{AXB} = \mathbf{D}, \tag{4.37}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$ with $Ind\mathbf{A} = k_1$, $\mathbf{B} \in \mathbb{C}^{m\times m}$ with $Ind\mathbf{B} = k_2$ and $\mathbf{D} \in \mathbb{C}^{n\times m}$ are given, and $\mathbf{X} \in \mathbb{C}^{n\times m}$ is unknown.

Theorem 4.14. *([62], Theorem 3) If* $R(\mathbf{D}) \subset R(\mathbf{A}^{k_1})$ *and* $N(\mathbf{D}) \supset N(\mathbf{B}^{k_2})$, $k = max\{k_1, k_2\}$, *then the matrix equation (4.37) with constrain* $R(\mathbf{X}) \subset R(\mathbf{A}^k)$ *and* $N(\mathbf{X}) \supset N(\mathbf{B}^k)$ *has a unique solution*

$$\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D.$$

We denote $\mathbf{A}^{k_1}\mathbf{D}\mathbf{B}^{k_2} =: \widetilde{\mathbf{D}} = (\widetilde{d}_{ij}) \in \mathbb{C}^{n\times m}$.

Theorem 4.15. *If* $\operatorname{rank}\mathbf{A}^{k_1+1} = \operatorname{rank}\mathbf{A}^{k_1} = r_1 \leq n$ *for* $\mathbf{A} \in \mathbb{C}^{n\times n}$, *and* $\operatorname{rank}\mathbf{B}^{k_2+1} = \operatorname{rank}\mathbf{B}^{k_2} = r_2 \leq m$ *for* $\mathbf{B} \in \mathbb{C}^{m\times m}$, *then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D =: (x_{ij}) \in \mathbb{C}^{n\times m}$ *of (4.37) we have*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \left|\mathbf{A}^{k_1+1}_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.j}\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r_1,n}} \left|(\mathbf{A}^{k_1+1})^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})^{\alpha}_{\alpha}\right|}, \tag{4.38}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \left|\mathbf{B}^{k_2+1}_{j\,.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,n}} \left|(\mathbf{A}^{k_1+1})^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})^{\alpha}_{\alpha}\right|}, \tag{4.39}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left[\sum_{\alpha\in I_{r_2,m}\{j\}} \left|\mathbf{B}^{k_2+1}_{j.}\left(\widetilde{\mathbf{d}}_{1.}\right)\,{}^{\alpha}_{\alpha}\right|, \ldots, \sum_{\alpha\in I_{r_2,m}\{j\}} \left|\mathbf{B}^{k_2+1}_{j.}\left(\widetilde{\mathbf{d}}_{n.}\right)\,{}^{\alpha}_{\alpha}\right|\right]^T, \tag{4.40}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left[\sum_{\beta\in J_{r_1,n}\{i\}} \left|\mathbf{A}^{k_1+1}_{.i}\left(\widetilde{\mathbf{d}}_{.1}\right)\,{}^{\beta}_{\beta}\right|, \ldots, \sum_{\alpha\in I_{r_1,n}\{i\}} \left|\mathbf{A}^{k_1+1}_{.i}\left(\widetilde{\mathbf{d}}_{.\,m}\right)\,{}^{\beta}_{\beta}\right|\right]$$

are the column-vector and the row-vector. $\widetilde{\mathbf{d}}_{i.}$ *and* $\widetilde{\mathbf{d}}_{.j}$ *are respectively the* i*-th row and the* j*-th column of* $\widetilde{\mathbf{D}}$ *for all* $i = \overline{1,n}$, $j = \overline{1,m}$.

Proof. By (2.21) and (2.20) the Drazin inverses $\mathbf{A}^D = \left(a^D_{ij}\right) \in \mathbb{C}^{n\times n}$ and $\mathbf{B}^D = \left(b^D_{ij}\right) \in \mathbb{C}^{m\times m}$ possess the following determinantal representations, respectively,

$$a^D_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \left|\mathbf{A}^{k_1+1}_{.\,i}\left(\mathbf{a}^{(k_1)}_{.j}\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)\,{}^{\beta}_{\beta}\right|},$$

$$b^D_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \left|\mathbf{B}^{k_2+1}_{j\,.}(\mathbf{b}^{(k_2)}_{i.})\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})\,{}^{\alpha}_{\alpha}\right|}. \tag{4.41}$$

Then an entry of the Drazin inverse solution $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D =: (x_{ij}) \in \mathbb{C}^{n\times m}$ is

$$x_{ij}=\sum_{s=1}^{m}\left(\sum_{t=1}^{n}a_{it}^{D}d_{ts}\right)b_{sj}^{D}. \tag{4.42}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the s-th column of $\mathbf{A}^{k}\mathbf{D}=:\hat{\mathbf{D}}=(\hat{d}_{ij})\in\mathbb{C}^{n\times m}$ for all $s=\overline{1,m}$. It follows from $\sum\limits_{t}\mathbf{a}_{.t}^{D}d_{ts}=\hat{\mathbf{d}}_{.s}$ that

$$\sum_{t=1}^{n}a_{it}^{D}d_{ts}=\sum_{t=1}^{n}\frac{\sum\limits_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\mathbf{a}_{.t}^{(k_1)}\right)\,_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|}\cdot d_{ts}=$$

$$\frac{\sum\limits_{\beta\in J_{r_1,n}\{i\}}\sum\limits_{t=1}^{n}\left|\mathbf{A}_{.i}^{k_1+1}\left(\mathbf{a}_{.t}^{(k_1)}\right)\,_{\beta}^{\beta}\right|\cdot d_{ts}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|}=\frac{\sum\limits_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\hat{\mathbf{d}}_{.s}\right)\,_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|} \tag{4.43}$$

Substituting (4.43) and (4.41) in (4.42), we obtain

$$x_{ij}=\sum_{s=1}^{m}\frac{\sum\limits_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\hat{\mathbf{d}}_{.s}\right)\,_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|}\frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}}\left|\mathbf{B}_{j.}^{k_2+1}(\mathbf{b}_{s.}^{(k_2)})\,_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)\,_{\alpha}^{\alpha}\right|}.$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.s}$ are respectively the unit row-vector and the unit column-vector whose components are 0, except the sth components, which are 1. Since

$$\hat{\mathbf{d}}_{.s}=\sum_{l=1}^{n}\mathbf{e}_{.l}\hat{d}_{ls},\quad \mathbf{b}_{s.}^{(k_2)}=\sum_{t=1}^{m}b_{st}^{(k_2)}\mathbf{e}_{t.},\quad \sum_{s=1}^{m}\hat{d}_{ls}b_{st}^{(k_2)}=\widetilde{d}_{lt},$$

then we have

$$x_{ij}=$$

$$\frac{\sum\limits_{s=1}^{m}\sum\limits_{t=1}^{m}\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\mathbf{e}_{.l}\right)\,_{\beta}^{\beta}\right|\hat{d}_{ls}b_{st}^{(k_2)}\sum\limits_{\alpha\in I_{r_2,m}\{j\}}\left|\mathbf{B}_{j.}^{k_2+1}(\mathbf{e}_{t.})\,_{\alpha}^{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)\,_{\alpha}^{\alpha}\right|}=$$

$$\frac{\sum\limits_{t=1}^{m}\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\mathbf{e}_{.l}\right)\,_{\beta}^{\beta}\right|\widetilde{d}_{lt}\sum\limits_{\alpha\in I_{r_2,m}\{j\}}\left|\mathbf{B}_{j.}^{k_2+1}(\mathbf{e}_{t.})\,_{\alpha}^{\alpha}\right|}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)\,_{\alpha}^{\alpha}\right|}. \tag{4.44}$$

Denote by

$$d_{it}^{\mathbf{A}}:=$$

$$\sum_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\widetilde{\mathbf{d}}_{.t}\right)\,_{\beta}^{\beta}\right|=\sum_{l=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}}\left|\mathbf{A}_{.i}^{k_1+1}\left(\mathbf{e}_{.l}\right)\,_{\beta}^{\beta}\right|\widetilde{d}_{lt}$$

the t-th component of a row-vector $\mathbf{d}_{i\,.}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, ..., d_{im}^{\mathbf{A}})$ for all $t = \overline{1, m}$. Substituting it in (4.44), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{m} d_{it}^{\mathbf{A}} \sum\limits_{\alpha \in I_{r_2,m}\{j\}} \left|\mathbf{B}_{j\,.}^{k_2+1}(\mathbf{e}_{t.})\,{}_{\alpha}^{\alpha}\right|}{\sum\limits_{\beta \in J_{r_1,\,n}} \left|(\mathbf{A}^{k_1+1})\,{}_{\beta}^{\beta}\right| \sum\limits_{\alpha \in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})\,{}_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{m} d_{it}^{\mathbf{A}} \mathbf{e}_{t.} = \mathbf{d}_{i\,.}^{\mathbf{A}}$, then it follows (4.39).

If we denote by

$$d_{lj}^{\mathbf{B}} := \sum_{t=1}^{m} \widetilde{d}_{lt} \sum_{\alpha \in I_{r_2,m}\{j\}} \left|\mathbf{B}_{j\,.}^{k_2+1}(\mathbf{e}_{t.})\,{}_{\alpha}^{\alpha}\right| = \sum_{\alpha \in I_{r_2,m}\{j\}} \left|\mathbf{B}_{j\,.}^{k_2+1}(\widetilde{\mathbf{d}}_{l.})\,{}_{\alpha}^{\alpha}\right|$$

the l-th component of a column-vector $\mathbf{d}_{.\,j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, ..., d_{jn}^{\mathbf{B}})^T$ for all $l = \overline{1, n}$ and substitute it in (4.44), we obtain

$$x_{ij} = \frac{\sum\limits_{l=1}^{n} \sum\limits_{\beta \in J_{r_1,\,n}\{i\}} \left|\mathbf{A}_{.\,i}^{k_1+1}\,(\mathbf{e}_{.\,l})\,{}_{\beta}^{\beta}\right| d_{lj}^{\mathbf{B}}}{\sum\limits_{\beta \in J_{r_1,\,n}} \left|(\mathbf{A}^{k_1+1})\,{}_{\beta}^{\beta}\right| \sum\limits_{\alpha \in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})\,{}_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{l=1}^{n} \mathbf{e}_{.l} d_{lj}^{\mathbf{B}} = \mathbf{d}_{.\,j}^{\mathbf{B}}$, then it follows (4.38). ■

4.3. Examples

In this subsection, we give an example to illustrate results obtained in the section.

1. Let us consider the matrix equation

$$\mathbf{AXB} = \mathbf{D}, \tag{4.45}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & i & i \\ i & -1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -i \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} i & 1 & -i \\ -1 & i & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \\ 0 & 1 & i \end{pmatrix}.$$

Since $\operatorname{rank} \mathbf{A} = 2$ and $\operatorname{rank} \mathbf{B} = 1$, then we have the case (ii) of Theorem 4.9. We shall find the least squares solution of (4.45) by (4.11). Then we have

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} 3 & 2i & 3i \\ -2i & 3 & 2 \\ -3i & 2 & 3 \end{pmatrix}, \ \mathbf{BB}^* = \begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix}, \ \widetilde{\mathbf{D}} = \mathbf{A}^*\mathbf{DB}^* = \begin{pmatrix} 1 & -i \\ -i & -1 \\ -i & -1 \end{pmatrix},$$

and $\sum\limits_{\alpha\in I_{1,\,2}} |(\mathbf{BB}^*)\,{}^{\alpha}_{\alpha}| = 3+3=6,$

$$\sum_{\beta\in J_{2,\,3}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right| = \det\begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix} + \det\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} + \det\begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix} = 10.$$

By (4.17), we can get

$$\mathbf{d}^{\mathbf{B}}_{.1} = \begin{pmatrix} 1 \\ -i \\ -i \end{pmatrix}, \quad \mathbf{d}^{\mathbf{B}}_{.2} = \begin{pmatrix} -i \\ -1 \\ -1 \end{pmatrix}.$$

Since $(\mathbf{A}^*\mathbf{A})_{.\,1}\left(\mathbf{d}^{\,\mathbf{B}}_{.1}\right) = \begin{pmatrix} 1 & 2i & 3i \\ -i & 3 & 2 \\ -i & 2 & 3 \end{pmatrix}$, then finally we obtain

$$x_{11} = \frac{\sum\limits_{\beta\in J_{2,\,3}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.\,1}\left(\mathbf{d}^{\,\mathbf{B}}_{.1}\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{2,3}} \left|(\mathbf{A}^*\mathbf{A})^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{1,2}} |(\mathbf{BB}^*)^{\alpha}_{\alpha}|} = \frac{\det\begin{pmatrix} 1 & 2i \\ -i & 3 \end{pmatrix} + \det\begin{pmatrix} 1 & 3i \\ -i & 3 \end{pmatrix}}{60} = -\frac{1}{60}.$$

Similarly,

$$x_{12} = \frac{\det\begin{pmatrix} -i & 2i \\ -1 & 3 \end{pmatrix} + \det\begin{pmatrix} -i & 3i \\ -1 & 3 \end{pmatrix}}{60} = -\frac{i}{60},$$

$$x_{21} = \frac{\det\begin{pmatrix} 3 & 1 \\ -2i & -i \end{pmatrix} + \det\begin{pmatrix} -i & 2 \\ -i & 3 \end{pmatrix}}{60} = -\frac{2i}{60},$$

$$x_{22} = \frac{\det\begin{pmatrix} 3 & -i \\ -2i & -1 \end{pmatrix} + \det\begin{pmatrix} -1 & 2 \\ -1 & 3 \end{pmatrix}}{60} = -\frac{2}{60},$$

$$x_{31} = \frac{\det\begin{pmatrix} 3 & 1 \\ -3i & -i \end{pmatrix} + \det\begin{pmatrix} 3 & -i \\ 2 & -i \end{pmatrix}}{60} = -\frac{i}{60},$$

$$x_{32} = \frac{\det\begin{pmatrix} 3 & -i \\ -3i & -1 \end{pmatrix} + \det\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}}{60} = -\frac{1}{60}.$$

2. Let us consider the matrix equation (4.45), where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ -i & i & i \\ -i & -i & -i \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}.$$

We shall find the Drazin inverse solution of (4.45) by (4.11). We obtain

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 2-2i & 0 & 0 \\ -2-2i & 0 & 0 \end{pmatrix}, \ \mathbf{A}^3 = \begin{pmatrix} 8 & 0 & 0 \\ 4-4i & 0 & 0 \\ -4-4i & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}^2 = \begin{pmatrix} -i & i & 3-i \\ 1 & -1 & 1+3i \\ -3+i & 3-i & 3+i \end{pmatrix}.$$

Since $\operatorname{rank}\mathbf{A} = 2$ and $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A}^2 = 1$, then $k_1 = \operatorname{Ind}\mathbf{A} = 2$ and $r_1 = 1$. Since $\operatorname{rank}\mathbf{B} = \operatorname{rank}\mathbf{B}^2 = 2$, then $k_2 = \operatorname{Ind}\mathbf{B} = 1$ and $r_2 = 2$. Then we have

$$\widetilde{\mathbf{D}} = \mathbf{A}^2\mathbf{D}\mathbf{B} = \begin{pmatrix} -4 & 4 & 8 \\ -2+2i & 2-2i & 4-4i \\ 2+2i & -2-2i & -4-4i \end{pmatrix},$$

and $\sum\limits_{\beta\in J_{1,3}} \left|\left(\mathbf{A}^3\right)\,{}^{\beta}_{\beta}\right| = 8+0+0 = 8,$

$$\sum_{\alpha\in I_{2,3}} \left|\left(\mathbf{B}^2\right)\,{}^{\alpha}_{\alpha}\right| =$$
$$\det\begin{pmatrix} -i & i \\ 1 & -1 \end{pmatrix} + \det\begin{pmatrix} -1 & 1+3i \\ 3-i & 3+i \end{pmatrix} + \det\begin{pmatrix} -i & 3-i \\ -3+i & 3+i \end{pmatrix} =$$
$$0 + (-9-9i) + (9-9i) = -18i.$$

By (4.13), we can get

$$\mathbf{d}^{\mathbf{B}}_{.1} = \begin{pmatrix} 12-12i \\ -12i \\ -12 \end{pmatrix},\ \mathbf{d}^{\mathbf{B}}_{.2} = \begin{pmatrix} -12+12i \\ 12i \\ 12 \end{pmatrix},\ \mathbf{d}^{\mathbf{B}}_{.3} = \begin{pmatrix} 8 \\ -12-12i \\ -12+12i \end{pmatrix}.$$

Since $\mathbf{A}^3_{.1}\left(\mathbf{d}^{\mathbf{B}}_{.1}\right) = \begin{pmatrix} 12-12i & 0 & 0 \\ -12i & 0 & 0 \\ -12 & 0 & 0 \end{pmatrix}$, then finally we obtain

$$x_{11} = \frac{\sum\limits_{\beta\in J_{1,3}\{1\}} \left|\mathbf{A}^3_{.1}\left(\mathbf{d}^{\mathbf{B}}_{.1}\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{1,3}} \left|(\mathbf{A}^3)^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{2,3}} |(\mathbf{B}^2)^{\alpha}_{\alpha}|} = \frac{12-12i}{8\cdot(-18i)} = \frac{1+i}{12}.$$

Similarly,

$$x_{12} = \frac{-12+12i}{8\cdot(-18i)} = \frac{-1-i}{12},\ x_{13} = \frac{8}{8\cdot(-18i)} = \frac{i}{18},$$

$$x_{21} = \frac{-12i}{8\cdot(-18i)} = \frac{1}{12},\ x_{22} = \frac{12i}{8\cdot(-18i)} = -\frac{1}{12},\ x_{23} = \frac{-12-12i}{8\cdot(-18i)} = \frac{1-i}{12},$$

$$x_{31} = \frac{12}{8\cdot(-18i)} = -\frac{i}{12},\ x_{32} = \frac{-12}{8\cdot(-18i)} = \frac{i}{12}.\ x_{33} = \frac{-12+12i}{8\cdot(-18i)} = \frac{-1-i}{12}.$$

Then

$$\mathbf{X} = \begin{pmatrix} \frac{1+i}{12} & \frac{-1-i}{12} & \frac{i}{18} \\ \frac{1}{12} & -\frac{1}{12} & \frac{1-i}{12} \\ -\frac{i}{12} & \frac{i}{12} & \frac{-1-i}{12} \end{pmatrix}$$

is the Drazin inverse solution of (4.45).

5. An Application of the Determinantal Representations of the Drazin Inverse to Some Differential Matrix Equations

In this section we demonstrate an application of the determinantal representations (2.20) and (2.21) of the Drazin inverse to solutions of the following differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where the matrix $\mathbf{A}$ is singular.

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{AX} = \mathbf{B} \tag{5.1}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{C}^{n\times n}$ is unknown. It's well-known that the general solution of (5.1) is found to be

$$\mathbf{X}(t) = \exp^{-\mathbf{A}t}\left(\int \exp^{\mathbf{A}t}\, dt\right)\mathbf{B}$$

If $\mathbf{A}$ is invertible, then

$$\int \exp^{\mathbf{A}t}\, dt = \mathbf{A}^{-1}\exp^{\mathbf{A}t} + \mathbf{G},$$

where $\mathbf{G}$ is an arbitrary $n \times n$ matrix. If $\mathbf{A}$ is singular, then the following theorem gives an answer.

Theorem 5.1. *([63], Theorem 1) If $\mathbf{A}$ has index k, then*

$$\int \exp^{\mathbf{A}t}\, dt = \mathbf{A}^D\exp^{\mathbf{A}t} + (\mathbf{I} - \mathbf{AA}^D)t\left[\mathbf{I} + \frac{\mathbf{A}}{2}t + \frac{\mathbf{A}^2}{3!}t^2 + \ldots + \frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right] + \mathbf{G}.$$

Using Theorem 5.1 and the power series expansion of $\exp^{-\mathbf{A}t}$, we get an explicit form for a general solution of (5.1)

$$\mathbf{X}(t) = \\ \left\{\mathbf{A}^D + (\mathbf{I} - \mathbf{AA}^D)t\left(\mathbf{I} - \tfrac{\mathbf{A}}{2}t + \tfrac{\mathbf{A}^2}{3!}t^2 - \ldots(-1)^{k-1}\tfrac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right) + \mathbf{G}\right\}\mathbf{B}.$$

If we put $\mathbf{G} = \mathbf{0}$, then we obtain the following partial solution of (5.1),

$$\begin{array}{c}\mathbf{X}(t) = \mathbf{A}^D\mathbf{B} + (\mathbf{B} - \mathbf{A}^D\mathbf{AB})t - \frac{1}{2}(\mathbf{AB} - \mathbf{A}^D\mathbf{A}^2\mathbf{B})t^2 + \ldots \\ \frac{(-1)^{k-1}}{k!}(\mathbf{A}^{k-1}\mathbf{B} - \mathbf{A}^D\mathbf{A}^k\mathbf{B})t^k.\end{array} \tag{5.2}$$

Denote $\mathbf{A}^l\mathbf{B} =: \widehat{\mathbf{B}}^{(l)} = (\widehat{b}_{ij}^{(l)}) \in \mathbb{C}^{n\times n}$ for all $l = \overline{1, 2k}$.

Theorem 5.2. *The partial solution (5.2), $\mathbf{X}(t) = (x_{ij})$, possess the following determinantal representation,*

$$\begin{array}{l} x_{ij} = \dfrac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|} + \left(b_{ij} - \dfrac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+1)}_{.j}\right)\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t \\ -\frac{1}{2}\left(\widehat{b}_{ij}^{(1)} - \dfrac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+2)}_{.j}\right)\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^2 + \ldots \\ \frac{(-1)^k}{k!}\left(\widehat{b}_{ij}^{(k-1)} - \dfrac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\left|\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(2k)}_{.j}\right)\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^k \end{array} \tag{5.3}$$

for all $i, j = \overline{1,n}$.

Proof. Using the determinantal representation of the identity $\mathbf{A}^D\mathbf{A}$ (2.27), we obtain the following determinantal representation of the matrix $\mathbf{A}^D\mathbf{A}^m\mathbf{B} := (y_{ij})$,

$$y_{ij} = \sum_{s=1}^{n} p_{is} \sum_{t=1}^{n} a_{st}^{(m-1)} b_{tj} = \sum_{\beta \in J_{r,n}\{i\}} \frac{\sum\limits_{s=1}^{n} \left| \left(\mathbf{A}_{.i}^{k+1} \left(\mathbf{a}_{.s}{}^{(k+1)} \right) \right)_{\beta}^{\beta} \right| \cdot \sum\limits_{t=1}^{n} a_{st}^{(m-1)} b_{tj}}{\sum\limits_{\beta \in J_{r,n}} \left| \left(\mathbf{A}^{k+1} \right)_{\beta}^{\beta} \right|} =$$

$$\sum_{\beta \in J_{r,n}\{i\}} \frac{\sum\limits_{t=1}^{n} \left| \left(\mathbf{A}_{.i}^{k+1} \left(\mathbf{a}_{.t}{}^{(k+m)} \right) \right)_{\beta}^{\beta} \right| \cdot b_{tj}}{\sum\limits_{\beta \in J_{r,n}} \left| \left(\mathbf{A}^{k+1} \right)_{\beta}^{\beta} \right|} = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left| \left(\mathbf{A}_{.\,i}^{k+1} \left(\widehat{\mathbf{b}}_{.j}^{(k+m)} \right) \right)_{\beta}^{\beta} \right|}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1} \right)_{\beta}^{\beta} \right|}$$

for all $i, j = \overline{1,n}$ and $m = \overline{1,k}$. From this and the determinantal representation of the Drazin inverse solution (4.34) and the identity (2.27) it follows (5.3). ■

Corollary 5.3. *If* $Ind\mathbf{A} = 1$*, then the partial solution of (5.1),*

$$\mathbf{X}(t) = (x_{ij}) = \mathbf{A}^g\mathbf{B} + (\mathbf{B} - \mathbf{A}^g\mathbf{A}\mathbf{B})t,$$

possess the following determinantal representation

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left| \left(\mathbf{A}^2_{.\,i} \left(\widehat{\mathbf{b}}_{.j}^{(1)} \right) \right)_{\beta}^{\beta} \right|}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^2 \right)_{\beta}^{\beta} \right|} + \left(b_{ij} - \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left| \left(\mathbf{A}^2_{.\,i} \left(\widehat{\mathbf{b}}_{.j}^{(2)} \right) \right)_{\beta}^{\beta} \right|}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^2 \right)_{\beta}^{\beta} \right|} \right) t. \tag{5.4}$$

for all $i, j = \overline{1,n}$.

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{X}\mathbf{A} = \mathbf{B} \tag{5.5}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{C}^{n\times n}$ is unknown. The general solution of (5.5) is found to be

$$\mathbf{X}(t) = \mathbf{B}\exp^{-\mathbf{A}t}\left(\int \exp^{\mathbf{A}t} dt\right)$$

If $\mathbf{A}$ is singular, then an explicit form for a general solution of (5.5) is

$$\mathbf{X}(t) =$$
$$\mathbf{B}\left\{\mathbf{A}^D + (\mathbf{I} - \mathbf{A}\mathbf{A}^D)t\left(\mathbf{I} - \tfrac{\mathbf{A}}{2}t + \tfrac{\mathbf{A}^2}{3!}t^2 + ...(-1)^{k-1}\tfrac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right) + \mathbf{G}\right\}.$$

If we put $\mathbf{G} = \mathbf{0}$, then we obtain the following partial solution of (5.5),

$$\begin{array}{c} \mathbf{X}(t) = \mathbf{B}\mathbf{A}^D + (\mathbf{B} - \mathbf{B}\mathbf{A}\mathbf{A}^D)t - \frac{1}{2}(\mathbf{B}\mathbf{A} - \mathbf{B}\mathbf{A}^2\mathbf{A}^D)t^2 + ... \\ \frac{(-1)^{k-1}}{k!}(\mathbf{B}\mathbf{A}^{k-1} - \mathbf{B}\mathbf{A}^k\mathbf{A}^D)t^k. \end{array} \tag{5.6}$$

Denote $\mathbf{B}\mathbf{A}^l =: \check{\mathbf{B}}^{(l)} = (\check{b}_{ij}^{(l)}) \in \mathbb{C}^{n\times n}$ for all $l = \overline{1,2k}$. Using the determinantal representation of the Drazin inverse solution (4.36), the group inverse (2.25) and the identity (2.26) we evidently obtain the following theorem.

Theorem 5.4. *The partial solution (5.6), $\mathbf{X}(t) = (x_{ij})$, possess the following determinantal representation,*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j\,.}\left(\check{\mathbf{b}}^{(k)}_{.\,i}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^{k+1}\right){}^{\alpha}_{\alpha}\right|} + \left(b_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j\,.}\left(\check{\mathbf{b}}^{(k+1)}_{i\,.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^{k+1}\right){}^{\alpha}_{\alpha}\right|}\right) t$$
$$-\frac{1}{2}\left(\check{b}^{(1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j\,.}\left(\check{\mathbf{b}}^{(k+2)}_{i\,.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^{k+1}\right){}^{\alpha}_{\alpha}\right|}\right) t^2 + \ldots$$
$$\frac{(-1)^k}{k!}\left(\check{b}^{(k-1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}_{j\,.}\left(\check{\mathbf{b}}^{(2k)}_{i\,.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^{k+1}\right){}^{\alpha}_{\alpha}\right|}\right) t^k$$

for all $i, j = \overline{1, n}$*.*

Corollary 5.5. *If* $Ind\mathbf{A} = 1$*, then the partial solution of (5.5),*

$$\mathbf{X}(t) = (x_{ij}) = \mathbf{B}\mathbf{A}^g + (\mathbf{B} - \mathbf{B}\mathbf{A}\mathbf{A}^g)t,$$

possess the following determinantal representation

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^2_{j\,.}\left(\widehat{\mathbf{b}}^{(1)}_{i\,.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^2\right){}^{\alpha}_{\alpha}\right|} + \left(b_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^2_{j\,.}\left(\widehat{\mathbf{b}}^{(2)}_{i\,.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,n}} \left|\left(\mathbf{A}^2\right){}^{\alpha}_{\alpha}\right|}\right) t.$$

for all $i, j = \overline{1, n}$*.*

5.1. Example

1. Let us consider the differential matrix equation

$$\mathbf{X}' + \mathbf{A}\mathbf{X} = \mathbf{B}, \tag{5.7}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}.$$

Since $\operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}^2 = 2$, then $k = \operatorname{Ind}\mathbf{A} = 1$ and $r = 2$. The matrix $\mathbf{A}$ is the group inverse. We shall find the partial solution of (5.7) by (5.4). We have

$$\mathbf{A}^2 = \begin{pmatrix} -i & i & 3-i \\ 1 & -1 & 1+3i \\ -3+i & 3-i & 3+i \end{pmatrix}, \ \widehat{\mathbf{B}}^{(1)} = \mathbf{A}\mathbf{B} = \begin{pmatrix} 2-i & 2i & 0 \\ 1+2i & -2 & 0 \\ 1+i & i & 0 \end{pmatrix},$$

$$\widehat{\mathbf{B}}^{(2)} = \mathbf{A}^2\mathbf{B} = \begin{pmatrix} 2-2i & 2+3i & 0 \\ 2+2i & -3+2i & 0 \\ 1+5i & -2 & 0 \end{pmatrix}.$$

and

$$\sum_{\alpha\in J_{2,\,3}}\left|\left(\mathbf{A}^2\right)_{\beta}^{\beta}\right|=$$
$$\det\begin{pmatrix}-i & i\\ 1 & -1\end{pmatrix}+\det\begin{pmatrix}-1 & 1+3i\\ 3-i & 3+i\end{pmatrix}+\det\begin{pmatrix}-i & 3-i\\ -3+i & 3+i\end{pmatrix}=$$
$$0+(-9-9i)+(9-9i)=-18i.$$

Since $\left(\mathbf{A}^2\right)_{.\,1}\left(\widehat{\mathbf{b}}_{.1}^{(1)}\right)=\begin{pmatrix}2-i & i & 3-i\\ 1+2i & -1 & 1+3i\\ 1+i & 3-i & 3+i\end{pmatrix}$ and

$$\left(\mathbf{A}^2\right)_{.\,1}\left(\widehat{\mathbf{b}}_{.1}^{(2)}\right)=\begin{pmatrix}2-2i & i & 3-i\\ 2+2i & -1 & 1+3i\\ 1+5i & 3-i & 3+i\end{pmatrix},$$

then finally we obtain

$$x_{11}=\frac{\sum\limits_{\beta\in J_{2,3}\{1\}}\left|\left(\mathbf{A}^2_{.\,1}\left(\widehat{\mathbf{b}}_{.1}^{(1)}\right)\right)_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{2,3}}\left|\left(\mathbf{A}^2\right)_{\beta}^{\beta}\right|}+\left(b_{11}-\frac{\sum\limits_{\beta\in J_{2,3}\{1\}}\left|\left(\mathbf{A}^2_{.\,1}\left(\widehat{\mathbf{b}}_{.1}^{(2)}\right)\right)_{\beta}^{\beta}\right|}{\sum\limits_{\beta\in J_{2,3}}\left|\left(\mathbf{A}^2\right)_{\beta}^{\beta}\right|}\right)t=$$
$$\tfrac{3-3i}{-18i}+\left(1-\tfrac{-18i}{-18i}\right)t=\tfrac{1+i}{6}.$$

Similarly,

$$x_{12}=\frac{-3+3i}{-18i}+\left(i-\frac{9+9i}{-18i}\right)t=\frac{-1-i}{6}+\frac{1+i}{2}t,\ x_{13}=0+(1-0)\,t=t,$$

$$x_{21}=\frac{3+3i}{-18i}+\left(i-\frac{-18}{-18i}\right)t=\frac{-1+i}{6},$$

$$x_{22}=\frac{-3-3i}{-18i}+\left(0-\frac{-9+9i}{-18i}\right)t=\frac{1-i}{6}+\frac{1+i}{2}t,\ x_{23}=0+(1-0)\,t=t,$$

$$x_{31}=\frac{-12i}{-18i}+\left(1-\frac{-18i}{-18i}\right)t=\frac{2}{3},$$

$$x_{32}=\frac{9+3i}{-18i}+\left(i-\frac{-18}{-18i}\right)t=\frac{-1+3i}{6},\ x_{33}=0+(0-0)\,t=0.$$

Then

$$\mathbf{X}=\frac{1}{6}\begin{pmatrix}1+i & -1-i+(3+3i)t & t\\ -1+i & 1-i+(3+3i)t & t\\ 4 & -1+3i & 0\end{pmatrix}$$

is the partial solution of (5.7) .

6. Conclusion

From student years it is well known that Cramer's rule may only be used when the system is square and the coefficient matrix is invertible. In this chapter we are considered various cases of Cramer's rule for generalized inverse solutions of systems of linear equations and matrix equations when the coefficient matrix is not square or non-invertible. The results of this chapter have practical and theoretical importance because they give an explicit representation of an individual component of solutions independently of all other components. Also the results of this chapter can be extended to matrices over rings (and now this is done in the quaternion skew field), to polynomial matrices, etc.

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Chapter 9

Relation of Row-Column Determinants with Quasideterminants of Matrices over a Quaternion Algebra

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Abstract

Since product of quaternions is noncommutative, there is a problem how to determine a determinant of a matrix with noncommutative elements (it's called a noncommutative determinant). We consider two approaches to define a noncommutative determinant. Primarily, there are row – column determinants that are an extension of the classical definition of the determinant; however we assume predetermined order of elements in each of the terms of the determinant. In the chapter we extend the concept of an immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

Properties of the determinant of a Hermitian matrix are established. Based on these properties, analogs of the classical adjont matrix over a quaternion skew field have been obtained. As a result we have a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule by using row–column determinants.

Quasideterminants appeared from the analysis of the procedure of a matrix inversion. By using quasideterminants, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

The common feature in definition of row and column determinants and quasideterminants is that we have not one determinant of a quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminants, n row determinants, and n column determinants). We have obtained a relation of row-column determinants with quasideterminants of a matrix over a quaternion division algebra.

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1. Introduction

Linear algebra is a powerful tool that we use in different areas of mathematics, including the calculus, the analytic and differential geometry, the theory of differential equations, and the optimal control theory. Linear algebra has accumulated a rich set of different methods. Since some methods have a common final result, this gives us the opportunity to choose the most effective method, depending on the nature of calculations.

At transition from linear algebra over a field to linear algebra over a division ring, we want to save as much as possible tools that we regularly use. Already in the early XX century, shortly after Hamilton created a quaternion algebra, mathematicians began to search the answer how looks like the algebra with noncommutative multiplication. In particular, there is a problem how to determine a determinant of a matrix with elements belonging to a noncommutative ring. Such determinant is also called a noncommutative determinant.

There were a lot of approaches to the definition of the noncommutative determinant. However none of the introduced noncommutative determinants maintained all those properties that determinant possessed for matrices over a field. Moreover, in paper [1], J. Fan proved that there is no unique definition of determinant which would expands the definition of determinant of real matrices for matrices over the division ring of quaternions. Therefore, search for a solution of the problem to define a noncommutative determinant is still going on.

In this chapter, we consider two approaches to define noncommutative determinant. Namely, we explore row-column determinants and quasideterminant.

Row-column determinants are an extension of the classical definition of the determinant, however we assume predetermined order of elements in each of the terms of the determinant. Using row-column determinants, we obtain a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule.

Quasideterminant appeared from the analysis of the procedure of a matrix inversion. Using quasideterminant, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

There is common in definition of row and column determinants and quasideterminant. In both cases, we have not one determinant in correspondence to quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminant, n row determinants, and n column determinants).

Today there is wide application of quasideterminants in linear algebra ([2, 3]), and in physics ([4, 5, 6]). Row and column determinants ([7, 8]) introduced relatively recently are less well known. Purpose of the chapter is establishment of a relation of row-column determinants with quasideterminants of a matrix over a quaternion algebra. The authors are hopeful that the establishment of this relation can provide mutual development of both the theory of quasideterminants and the theory of row-column determinants.

1.1. Convention about Notations

There are different forms to write elements of a matrix. In this paper, we denote a_{ij} an element of the matrix $\mathbf{A}$. The index i labels rows, and the index j labels columns.

We use the following notation for different minors of the matrix $\mathbf{A}$.

$\mathbf{a}_{i\,.}$ the i-th row

$\mathbf{A}_{S\,.}$ the minor obtained from $\mathbf{A}$ by selecting rows with index from the set S

$\mathbf{A}^{i\,\cdot}$ the minor obtained from $\mathbf{A}$ by deleting row $\mathbf{a}_{i\,.}$

$\mathbf{A}^{S\,\cdot}$ the minor obtained from $\mathbf{A}$ by deleting rows with index from the set S

$\mathbf{a}_{.\,j}$ the j-th column

$\mathbf{A}_{.\,T}$ the minor obtained from $\mathbf{A}$ by selecting columns with index from the set T

$\mathbf{A}^{\cdot\,j}$ the minor obtained from $\mathbf{A}$ by deleting column $\mathbf{a}_{.\,j}$

$\mathbf{A}^{\cdot\,T}$ the minor obtained from $\mathbf{A}$ by deleting columns with index from the set T

$\mathbf{A}_{.j}\left(\mathbf{b}\right)$ the matrix obtained from $\mathbf{A}$ by replacing its j-th column by the column $\mathbf{b}$

$\mathbf{A}_{i.}\left(\mathbf{b}\right)$ the matrix obtained from $\mathbf{A}$ by replacing its i-th row by the row $\mathbf{b}$

Considered notations can be combined. For instance, the record

$$\mathbf{A}_{k.}^{ii}(\mathbf{b})$$

means replacing of the k-th row by the vector $\mathbf{b}$ followed by removal of both the i-th row and the i-th column.

As was noted in section 2.2 of the paper [9], we can define two types of matrix products: either product of rows of first matrix over columns of second one, or product of columns of first matrix over rows of second one. However, according to the theorem 2.2.5 in the paper [9], this product is symmetric relative operation of transposition. Hence in the chapter, we will restrict ourselves by traditional product of rows of first matrix over columns of second one; and we do not indicate clearly the operation like it was done in [9].

1.2. Preliminaries. A Brief Overview of the Theory of Noncommutative Determinants

Theory of determinants of matrices with noncommutative elements can be divided into three groups regarding their methods of definition. Denote $\mathrm{M}(n, \mathbf{K})$ the ring of matrices with elements from the ring $\mathbf{K}$. One of the ways to determine determinant of a matrix of $\mathrm{M}\left(n, \mathbf{K}\right)$ is following ([11, 12, 13]).

Definition 1.1. *Let the functional*

$$\mathrm{d} : \mathrm{M}\left(n, \mathbf{K}\right) \to \mathbf{K}$$

satisfy the following axioms.

Axiom 1. $\mathrm{d}(\mathbf{A}) = 0$ *iff* $\mathbf{A}$ *is singular (irreversible).*

Axiom 2. $\forall \mathbf{A}, \mathbf{B} \in \mathrm{M}(n, \mathbf{K})$, $\mathrm{d}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{d}(\mathbf{A}) \cdot \mathrm{d}(\mathbf{B})$.

Axiom 3. *If we obtain a matrix* $\mathbf{A}'$ *from matrix* $\mathbf{A}$ *either by adding of an arbitrary row multiplied on the left with its another row or by adding of an arbitrary column multiplied on the right with its another column, then*

$$\mathrm{d}\left(\mathbf{A}'\right) = \mathrm{d}(\mathbf{A})$$

Then the value of the functional d *is called determinant of* $\mathbf{A} \in \mathrm{M}(n, \mathbf{K})$. □

The known determinants of Dieudonné and Study are examples of such functionals. Aslaksen [11] proved that determinants which satisfy Axioms 1, 2 and 3 take their value in some commutative subset of the ring. It makes no sense for them such property of conventional determinants as the expansion along an arbitrary row or column. Therefore a determinantal representation of an inverse matrix using only these determinants is impossible. This is the reason that causes to introduce determinant functionals that do not satisfy all Axioms. Dyson [13] considers Axiom 1 as necessary to determine a determinant.

In another approach, a determinant of a square matrix over a noncommutative ring is considered as a rational function of entries of a matrix. The greatest success is achieved by Gelfand and Retakh [14, 15, 16, 17] in the theory of quasideterminants. We present introduction to the theory of quasideterminants in the section 5.

In third approach, a determinant of a square matrix over a noncommutative ring is considered as an alternating sum of $n!$ products of entries of a matrix. However, it assumed certain fixed order of factors in each term. E. H. Moore was first who achieved implementation of the key Axiom 1 using such definition of a noncommutative determinant. Moore had done this not for all square matrices, but only for Hermitian. He defined the determinant of a Hermitian matrix[1] $\mathbf{A} = (a_{ij})_{n\times n}$ over a division ring with involution by induction over n following way (see [13])

$$\mathrm{Mdet}\mathbf{A} = \begin{cases} a_{11}, & n = 1 \\ \sum\limits_{j=1}^{n} \varepsilon_{ij} a_{ij} \mathrm{Mdet}\left(\mathbf{A}(i \to j)\right), & n > 1 \end{cases} \tag{1.1}$$

Here $\varepsilon_{kj} = \begin{cases} 1, & i = j \\ -1, & i \neq j \end{cases}$, and $\mathbf{A}(i \to j)$ denotes the matrix obtained from $\mathbf{A}$ by replacing its j-th column with the i-th column and then by deleting both the i-th row and column. Another definition of this determinant is presented in [11] by using permutations,

$$\mathrm{Mdet}\,\mathbf{A} = \sum_{\sigma \in S_n} |\sigma| a_{n_{11} n_{12}} \cdot \ldots \cdot a_{n_{1l_1} n_{11}} \cdot a_{n_{21} n_{22}} \cdot \ldots \cdot a_{n_{rl_1} n_{r1}}.$$

Here S_n is symmetric group of n elements. A cycle decomposition of a permutation σ has form,

$$\sigma = (n_{11} \ldots n_{1l_1})(n_{21} \ldots n_{2l_2}) \ldots (n_{r1} \ldots n_{rl_r}).$$

[1]Hermitian matrix is such matrix $\mathbf{A} = (a_{ij})$ that $a_{ij} = \overline{a_{ji}}$.

However, there was no any generalization of the definition of Moore's determinant to arbitrary square matrices. Freeman J. Dyson [13] pointed out the importance of this problem.

L. Chen [18, 19] offered the following definition of determinant of a square matrix over the quaternion skew field $\mathbf{H}$, by putting for $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbf{H})$,

$$\det \mathbf{A} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{n_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_s n_1} \cdot \ldots \cdot a_{n_r k_2} \cdot \ldots \cdot a_{k_l n_r},$$
$$\sigma = (n_1 i_2 \ldots i_s) \ldots (n_r k_2 \ldots k_r),$$
$$n_1 > i_2, i_3, \ldots, i_s; \ldots, n_r > k_2, k_3, \ldots, k_l,$$
$$n = n_1 > n_2 > \ldots > n_r \geq 1.$$

Despite the fact that this determinant does not satisfy Axiom 1, L. Chen got a determinantal representation of an inverse matrix. However it can not been expanded along arbitrary rows and columns (except for n-th row). Therefore, L. Chen did not obtain a classical adjoint matrix as well. For $\mathbf{A} = (\alpha_1, \ldots, \alpha_m)$ over the quaternion skew field $\mathbf{H}$, if $\|\mathbf{A}\| := \det(\mathbf{A}^*\mathbf{A}) \neq 0$, then $\exists \mathbf{A}^{-1} = (b_{jk})$, where

$$\overline{b_{jk}} = \frac{1}{\|\mathbf{A}\|}\omega_{kj}, \quad \left(j, k = \overline{1, n}\right),$$

$$\omega_{kj} = \det(\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \delta_k)^* (\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \alpha_j).$$

Here α_i is the i-th column of $\mathbf{A}$, δ_k is the n-dimensional column with 1 in the k-th entry and 0 in other ones. L. Chen defined $\|\mathbf{A}\| := \det(\mathbf{A}^*\mathbf{A})$ as the double determinant. If $\|\mathbf{A}\| \neq 0$, then the solution of a right system of linear equations

$$\sum\nolimits_{j=1}^{n} \alpha_j x_j = \beta$$

over $\mathbf{H}$ is represented by the following formula, which the author calls Cramer's rule

$$x_j = \|\mathbf{A}\|^{-1} \overline{\mathbf{D}_j},$$

for all $j = \overline{1, n}$, where

$$\mathbf{D}_j = \det \begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_{j-1}^* \\ \alpha_n^* \\ \alpha_{j+1}^* \\ \vdots \\ \alpha_{n-1}^* \\ \beta^* \end{pmatrix} \begin{pmatrix} \alpha_1 & \ldots & \alpha_{j-1} & \alpha_n & \alpha_{j+1} & \ldots & \alpha_{n-1} & \alpha_j \end{pmatrix}.$$

Here α_i^* is the i-th row of $\mathbf{A}^*$ and β^* is the n-dimensional vector-row conjugated with β.

In this chapter we explore the theory of row and column determinants which develops the classical approach to the definition of determinant of a square matrix, as an alternating sum of products of entries of a matrix but with a predetermined order of factors in each of the terms of the determinant.

2. Quaternion Algebra

A quaternion algebra $\mathbb{H}(a,b)$ (we also use notation $\left(\frac{a,b}{\mathbb{F}}\right)$) is a four-dimensional vector space over a field $\mathbb{F}$ with basis $\{1,i,j,k\}$ and the following multiplication rules:

$$\begin{aligned} i^2 &= a, \\ j^2 &= b, \\ ij &= k, \\ ji &= -k. \end{aligned}$$

The field $\mathbb{F}$ is the center of the quaternion algebra $\mathbb{H}(a,b)$.

In the algebra $\mathbb{H}(a,b)$ there are following mappings.

- A quadratic form
$$\mathrm{n}: x \in \mathbb{H} \to \mathrm{n}(x) \in \mathbb{F}$$
such that
$$\mathrm{n}(x \cdot y) = \mathrm{n}(x)\mathrm{n}(y) \quad x, y \in \mathbb{H}$$
is called the norm on a quaternion algebra $\mathbb{H}$.

- The linear mapping
$$\mathrm{t}: x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H} \to \mathrm{t}(x) = 2x^0 \in \mathbb{F}$$
is called the trace of a quaternion. The trace satisfies permutability property of the trace,
$$\mathrm{t}\left(q \cdot p\right) = \mathrm{t}\left(p \cdot q\right).$$
From the theorem 10.3.3 in the paper [9], it follows
$$\mathrm{t}(x) = \frac{1}{2}(x - ixi - jxj - kxk). \tag{2.1}$$

- A linear mapping
$$x \to \overline{x} = \mathrm{t}(x) - x \tag{2.2}$$
is an involution. The involution has following properties
$$\begin{aligned} \overline{\overline{x}} &= x, \\ \overline{x+y} &= \overline{x} + \overline{y}, \\ \overline{x \cdot y} &= \overline{y} \cdot \overline{x}. \end{aligned}$$
A quaternion $\overline{x}$ is called the conjugate of $x \in \mathbb{H}$. The norm and the involution satisfy the following condition:
$$\mathrm{n}\left(\overline{q}\right) = \mathrm{n}(q).$$
The trace and the involution satisfy the following condition,
$$\mathrm{t}\left(\overline{x}\right) = \mathrm{t}(x).$$

From equations (2.1), (2.2), it follows that

$$\overline{x} = -\frac{1}{2}(x + ixi + jxj + kxk).$$

Depending on the choice of the field $\mathbb{F}$, a and b, on the set of quaternion algebras there are only two possibilities [20]:

1. $\left(\frac{a,b}{\mathbb{F}}\right)$ is a division algebra.
2. $\left(\frac{a,b}{\mathbb{F}}\right)$ is isomorphic to the algebra of all 2×2 matrices with entries from the field $\mathbb{F}$. In this case, quaternion algebra is splittable.

The most famous example of a non-split quaternion algebra is Hamilton's quaternions $\mathbf{H} = (\frac{-1,-1}{\mathbb{R}})$, where $\mathbb{R}$ is real field. The set of quaternions can be represented as

$$\mathbf{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Consider some non-isomorphic quaternion algebra with division.

1. $\left(\frac{a,b}{\mathbb{R}}\right)$ is isomorphic to the Hamilton quaternion skew field $\mathbf{H}$ whenever $a < 0$ and $b < 0$. Otherwise $\left(\frac{a,b}{\mathbb{R}}\right)$ is splittable.
2. If $\mathbb{F}$ is the rational field $\mathbb{Q}$, then there exist infinitely many nonisomorphic division quaternion algebras $\left(\frac{a,b}{\mathbb{Q}}\right)$ depending on choice of $a < 0$ and $b < 0$.
3. Let $\mathbb{Q}_p$ be the p-adic field where p is a prime number. For each prime number p there is a unique division quaternion algebra.

The famous example of a split quaternion algebra is split quaternions of James Cockle $\mathbf{H_S}(\frac{-1,1}{\mathbb{R}})$, which can be represented as

$$\mathbf{H_S} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $i^2 = -1$, $j^2 = k^2 = 1$ and $ijk = 1$. Unlike quaternion division algebra, the set of split quaternions is a noncommutative ring with zero divisors, nilpotent elements and nontrivial idempotents. Recently there was conducted a number of studies in split quaternion matrices (see, for ex. [21, 22, 23, 24]).

3. Introduction to the Theory of the Row and Column Determinants over a Quaternion Algebra

The theory of the row and column determinants was introduced [7, 8] for matrices over a quaternion division algebra. Now this theory is in development for matrices over a split quaternion algebra. In the following two subsections we extend the concept of immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

3.1. Definitions and Properties of the Column and Row Immanants

The immanant of a matrix is a generalization of the concepts of determinant and permanent. The immanant of a complex matrix was defined by Dudley E. Littlewood and Archibald Read Richardson [25] as follows.

Definition 3.1. *Let $\sigma \in S_n$ denote the symmetric group on n elements. Let $\chi : S_n \to \mathbb{C}$ be a complex character. For any $n \times n$ matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n\times n}$ define the immanent of $\mathbf{A}$ as*

$$\mathrm{Imm}_\chi(\mathbf{A}) = \sum_{\sigma\in S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

Special cases of immanants are determinants and permanents. In the case where χ is the constant character ($\chi(x) = 1$ for all $x \in S_n$), $\mathrm{Imm}_\chi(\mathbf{A})$ is the permanent of $\mathbf{A}$. In the case where χ is the sign of the permutation (which is the character of the permutation group associated to the (non-trivial) one-dimensional representation), $\mathrm{Imm}_\chi(\mathbf{A})$ is the determinant of $\mathbf{A}$.

Denote by $\mathbb{H}^{n\times m}$ a set of $n \times m$ matrices with entries in an arbitrary (split) quaternion algebra $\mathbb{H}$ and $\mathrm{M}\,(n, \mathbb{H})$ a ring of matrices with entries in $\mathbb{H}$. For $\mathbf{A} = (a_{ij}) \in \mathrm{M}\,(n, \mathbb{H})$ we define n row immanants as follows.

Definition 3.2. *The i-th row immanant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}\,(n, \mathbb{H})$ is defined by putting*

$$\mathrm{rImm}_i\mathbf{A} = \sum_{\sigma\in S_n} \chi(\sigma) a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ is written as follows

$$\sigma = (i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1})\,(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r})\,. \tag{3.1}$$

Here the index i starts the first cycle from the left and other cycles satisfy the following conditions

$$i_{k_2} < i_{k_3} < \ldots < i_{k_r}, \quad i_{k_t} < i_{k_t+s}. \tag{3.2}$$

for all $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Consequently we have the following definitions.

Definition 3.3. *The i-th row permanent of $\mathbf{A} = (a_{ij}) \in \mathrm{M}\,(n, \mathbb{H})$ is defined as*

$$\mathrm{rper}_i\mathbf{A} = \sum_{\sigma\in S_n} a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ satisfies the conditions (3.1) and (3.2).

Definition 3.4. *The i-th row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}\,(n, \mathbb{H})$ is defined as*

$$\mathrm{rdet}_i\mathbf{A} = \sum_{\sigma\in S_n} (-1)^{n-r}\, a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ satisfies the conditions (3.1) and (3.2), (since $\mathrm{sign}(\sigma) = (-1)^{n-r}$).

For $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ we define n column immanants as well.

Definition 3.5. *The j-th column immanant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{cImm}_j \mathbf{A} = \sum_{\tau \in S_n} \chi(\tau) a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} j_{k_r}} \dots a_{j\, j_{k_1+l_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows

$$\tau = \left(j_{k_r+l_r} \dots j_{k_r+1} j_{k_r}\right) \dots \left(j_{k_2+l_2} \dots j_{k_2+1} j_{k_2}\right) \left(j_{k_1+l_1} \dots j_{k_1+1} j_{k_1} j\right). \tag{3.3}$$

Here the first cycle from the right begins with the index j and other cycles satisfy the following conditions

$$j_{k_2} < j_{k_3} < \dots < j_{k_r}, \quad j_{k_t} < j_{k_t+s}, \tag{3.4}$$

for all $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Consequently we have the following definitions as well.

Definition 3.6. *The j-th column permanent of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{rper}_j \mathbf{A} = \sum_{\tau \in S_n} a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} j_{k_r}} \dots a_{j\, j_{k_1+l_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation σ satisfies the conditions (3.3) and (3.4).

Definition 3.7. *The j-th column determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{rdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} j_{k_r}} \dots a_{j\, j_{k_1+l_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation σ satisfies the conditions (3.3) and (3.4).

Consider the basic properties of the column and row immanants over $\mathbb{H}$.

Proposition 3.8. *(The first theorem about zero of an immanant) If one of the rows (columns) of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ consists of zeros only, then $\mathrm{rImm}_i \mathbf{A} = 0$ and $\mathrm{cImm}_i \mathbf{A} = 0$ for all $i = \overline{1, n}$.*

Proof. The proof immediately follows from the definitions. □

Denote by $\mathbb{H}a$ and $a\mathbb{H}$ left and right principal ideals of $\mathbb{H}$, respectively.

Proposition 3.9. *(The second theorem about zero of an row immanant) Let $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ and $a_{ki} \in \mathbb{H}a_i$ and $a_{ij} \in \overline{a_i}\mathbb{H}$, where $n(a_i) = 0$ for $k, j = \overline{1, n}$ and for all $i \neq k$. Let $a_{11} \in \mathbb{H}a_1$ and $a_{22} \in \overline{a_1}\mathbb{H}$ if $k = 1$, and $a_{kk} \in \mathbb{H}a_k$ and $a_{11} \in \overline{a_k}\mathbb{H}$ if $k = i > 1$, where $n(a_k) = 0$. Then $\mathrm{rImm}_k \mathbf{A} = 0$.*

Proof. Let $i \neq k$. Consider an arbitrary monomial of $\mathrm{rImm}_k\mathbf{A}$, if $i \neq k$,

$$d = \chi(\sigma) a_{ki} a_{ij} \dots a_{lm}$$

where $\{l, m\} \subset \{1, ..., n\}$. Since there exists $a_i \in \mathbb{H}$ such that $n(a_i) = 0$, and $a_{ki} \in \mathbb{H}a_i$, $a_{ij} \in \overline{a_i}\mathbb{H}$, than $a_{ki}a_{ij} = 0$ and $d = 0$.

Let $i = k = 1$. Then an arbitrary monomial of $\mathrm{rImm}_1\mathbf{A}$,

$$d = \chi(\sigma) a_{11} a_{22} \dots a_{lm}.$$

Since there exists $a_1 \in \mathbb{H}$ such that $n(a_1) = 0$, and $a_{11} \in \mathbb{H}a_1$, $a_{22} \in \overline{a_1}\mathbb{H}$, then $a_{11}a_{22} = 0$ and $d = 0$.

If $k = i > 1$, then an arbitrary monomial of $\mathrm{rImm}_k\mathbf{A}$,

$$d = \chi(\sigma) a_{kk} a_{11} \dots a_{lm}.$$

Since there exists $a_k \in \mathbb{H}$ such that $n(a_k) = 0$, and $a_{kk} \in \mathbb{H}a_k$, $a_{11} \in \overline{a_k}\mathbb{H}$, then $a_{kk}a_{11} = 0$ and $d = 0$. □

Proposition 3.10. *(The second theorem about zero of an column immanant) Let* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *and* $a_{ik} \in a_i\mathbb{H}$ *and* $a_{ji} \in \mathbb{H}\overline{a_i}$*, where* $n(a_i) = 0$ *for* $k, j = \overline{1, n}$ *and for all* $i \neq k$*. Let* $a_{11} \in a_1\mathbb{H}$ *and* $a_{22} \in \mathbb{H}\overline{a_1}$ *if* $k = 1$*, and* $a_{kk} \in a_k\mathbb{H}$ *and* $a_{11} \in \mathbb{H}\overline{a_k}$ *if* $k = i > 1$*, where* $n(a_k) = 0$*. Then* $\mathrm{cImm}_k\mathbf{A} = 0$*.*

Proof. The proof is similar to the proof of the Proposition 3.9. □

The proofs of the next theorems immediately follow from the definitions.

Proposition 3.11. *If the* i*-th row of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is left-multiplied by* $b \in \mathbb{H}$*, then* $\mathrm{rImm}_i\,\mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rImm}_i\,\mathbf{A}$ *for all* $i = \overline{1, n}$*.*

Proposition 3.12. *If the* j*-th column of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is right-multiplied by* $b \in \mathbb{H}$*, then* $\mathrm{cImm}_j\,\mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cImm}_j\,\mathbf{A} \cdot b$ *for all* $j = \overline{1, n}$*.*

Proposition 3.13. *If for* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in \{1, ..., n\}$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = \overline{1, n}$*, then for all* $i = \overline{1, n}$

$$\mathrm{rImm}_i\,\mathbf{A} = \mathrm{rImm}_i\,\mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rImm}_i\,\mathbf{A}_{t.}(\mathbf{c}),$$
$$\mathrm{cImm}_i\,\mathbf{A} = \mathrm{cImm}_i\,\mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cImm}_i\,\mathbf{A}_{t.}(\mathbf{c}),$$

where $\mathbf{b} = (b_1, \dots, b_n)$*,* $\mathbf{c} = (c_1, \dots, c_n)$*.*

Proposition 3.14. *If for* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in \{1, ..., n\}$ *such that* $a_{i\,t} = b_i + c_i$ *for all* $i = \overline{1, n}$*, then for all* $j = \overline{1, n}$

$$\mathrm{rImm}_j\,\mathbf{A} = \mathrm{rImm}_j\,\mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{rImm}_j\,\mathbf{A}_{.\,t}(\mathbf{c}),$$
$$\mathrm{cImm}_j\,\mathbf{A} = \mathrm{cImm}_j\,\mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{cImm}_j\mathbf{A}_{.\,t}(\mathbf{c}),$$

where $\mathbf{b} = (b_1, \dots, b_n)^T$*,* $\mathbf{c} = (c_1, \dots, c_n)^T$*.*

Proposition 3.15. *If* $\mathbf{A}^*$ *is the Hermitian adjoint matrix (conjugate and transpose) of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *then* $\mathrm{rImm}_i\, \mathbf{A}^* = \overline{\mathrm{cImm}_i\, \mathbf{A}}$ *for all* $i = \overline{1, n}$.

Particular cases of these properties for the row-column determinants and permanents are evident.

Remark 3.16. *The peculiarity of the column immanant (permanent, determinant) is that, at the direct calculation, factors of each of the monomials are written from right to left.* □

In Lemmas 3.17 and 3.18, we consider the recursive definition of the column and row determinants. This definition is an analogue of the expansion of a determinant along a row and a column in commutative case.

Lemma 3.17. *Let* R_{ij} *be the right* ij*-th cofactor of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$, *namely*

$$\mathrm{rdet}_i\, \mathbf{A} = \sum_{j=1}^{n} a_{ij} \cdot R_{ij}$$

for all $i = \overline{1, n}$. *Then*

$$R_{ij} = \begin{cases} -\mathrm{rdet}_j\, (\mathbf{A}^{ii}_{.j}(\mathbf{a}_{.i})), & i \neq j \\ \mathrm{rdet}_k\, \mathbf{A}^{ii}, & i = j \end{cases}$$

$$k = \begin{cases} 2, & i = 1 \\ 1, & i > 1 \end{cases}$$

where the matrix $(\mathbf{A}^{ii}_{.j}(\mathbf{a}_{.i}))$ *is obtained from* $\mathbf{A}$ *by replacing its* j*-th column with the* i*-th column and then by deleting both the* i*-th row and column.* □

Lemma 3.18. *Let* L_{ij} *be the left* ij*th cofactor of entry* a_{ij} *of matrix* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$, *namely*

$$\mathrm{cdet}_j\, \mathbf{A} = \sum_{i=1}^{n} L_{ij} \cdot a_{ij}$$

for all $j = \overline{1, n}$. *Then*

$$L_{ij} = \begin{cases} -\mathrm{cdet}_i\, (\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j.})), & i \neq j \\ \mathrm{cdet}_k\, \mathbf{A}^{jj}, & i = j \end{cases}$$

$$k = \begin{cases} 2, & j = 1 \\ 1, & j > 1 \end{cases}$$

where the matrix $(\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j.}))$ *is obtained from* $\mathbf{A}$ *by replacing its* i*th row with the* j*th and then by deleting both the* j*th row and column.* □

Remark 3.19. *Clearly, an arbitrary monomial of each row or column determinant corresponds to a certain monomial of another row or column determinant such that both of them have the same sign, consist of the same factors and differ only in their ordering. If the entries of* $\mathbf{A}$ *are commutative, then* $\mathrm{rdet}_1\, \mathbf{A} = \ldots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1\, \mathbf{A} = \ldots = \mathrm{cdet}_n \mathbf{A}$. □

4. An Immanant of a Hermitian Matrix

If $\mathbf{A}^* = \mathbf{A}$ then $\mathbf{A} \in \mathbb{H}^{n\times n}$ is called a Hermitian matrix. In this section we consider the key theorem about row-column immanants of a Hermitian matrix.

The following lemma is needed for the sequel.

Lemma 4.1. *Let T_n be the sum of all possible products of n factors, each of their are either $h_i \in \mathbb{H}$ or $\overline{h_i}$ for all $i = \overline{1,n}$, by specifying the ordering in the terms, $T_n = h_1 \cdot h_2 \cdot \ldots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_n + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_n}$. Then T_n consists of the 2^n terms and $T_n = \mathrm{t}(h_1)\ \mathrm{t}(h_2)\ \ldots\ \mathrm{t}(h_n)$.*

Theorem 4.2. *If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is a Hermitian matrix, then*

$$\mathrm{rImm}_1\mathbf{A} = \ldots = \mathrm{rImm}_n\mathbf{A} = \mathrm{cImm}_1\mathbf{A} = \ldots = \mathrm{cImm}_n\mathbf{A} \in \mathbb{F}.$$

Proof. At first we note that if $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian, then we have $a_{ii} \in \mathbb{F}$ and $a_{ij} = \overline{a_{ji}}$ for all $i, j = \overline{1,n}$.

We divide the set of monomials of $\mathrm{rImm}_i\mathbf{A}$ for some $i \in \{1, ..., n\}$ into two subsets. If indices of coefficients of monomials form permutations as products of disjoint cycles of length 1 and 2, then we include these monomials to the first subset. Other monomials belong to the second subset. If indices of coefficients form a disjoint cycle of length 1, then these coefficients are a_{jj} for $j \in \{1, ..., n\}$ and $a_{jj} \in \mathbb{F}$.

If indices of coefficients form a disjoint cycle of length 2, then these entries are conjugated, $a_{i_k i_{k+1}} = \overline{a_{i_{k+1} i_k}}$, and

$$a_{i_k i_{k+1}} \cdot a_{i_{k+1} i_k} = \overline{a_{i_{k+1} i_k}} \cdot a_{i_{k+1} i_k} = \mathrm{n}(a_{i_{k+1} i_k}) \in \mathbb{F}.$$

So, all monomials of the first subset take on values in $\mathbb{F}$.

Now we consider some monomial d of the second subset. Assume that its index permutation σ forms a direct product of r disjoint cycles. Denote $i_{k_1} := i$, then

$$\begin{array}{l} d = \chi(\sigma) a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i_{k_1}} a_{i_{k_2} i_{k_2+1}} \ldots a_{i_{k_2+l_2} i_{k_2}} \ldots a_{i_{k_m} i_{k_m+1}} \ldots \times \\ \times a_{i_{k_m+l_m} i_{k_m}} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}} = \chi(\sigma) h_1 h_2 \ldots h_m \ldots h_r, \end{array} \tag{4.1}$$

where $h_s = a_{i_{k_s} i_{k_s+1}} \cdot \ldots \cdot a_{i_{k_s+l_s} i_{k_s}}$ for all $s = \overline{1,r}$, and $m \in \{1, \ldots, r\}$. If $l_s = 1$, then $h_s = a_{i_{k_s} i_{k_s+1}} a_{i_{k_s+1} i_{k_s}} = \mathrm{n}(a_{i_{k_s} i_{k_s+1}}) \in \mathbb{F}$. If $l_s = 0$, then $h_s = a_{i_{k_s} i_{k_s}} \in \mathbb{F}$. If $l_s = 0$ or $l_s = 1$ for all $s = \overline{1,r}$ in (4.1), then d belongs to the first subset. Let there exists $s \in I_n$ such that $l_s \geq 2$. Then

$$\overline{h_s} = \overline{a_{i_{k_s} i_{k_s+1}} \ldots a_{i_{k_s+l_s} i_{k_s}}} = \overline{a_{i_{k_s+l_s} i_{k_s}}} \ldots \overline{a_{i_{k_s} i_{k_s+1}}} = a_{i_{k_s} i_{k_s+l_s}} \ldots a_{i_{k_s+1} i_{k_s}}.$$

Denote by $\sigma_s(i_{k_s}) := (i_{k_s} i_{k_s+1} \ldots i_{k_s+l_s})$ a disjoint cycle of indices of d for some $s \in \{1, ..., r\}$, then $\sigma = \sigma_1(i_{k_1})\, \sigma_2(i_{k_2}) \ldots \sigma_r(i_{k_r})$. The disjoint cycle $\sigma_s(i_{k_s})$ corresponds to the factor h_s. Then $\sigma_s^{-1}(i_{k_s}) = (i_{k_s} i_{k_s+l_s} i_{k_s+1} \ldots i_{k_s+1})$ is the inverse disjoint cycle and $\sigma_s^{-1}(i_{k_s})$ corresponds to the factor $\overline{h_s}$. By the Lemma 4.1, there exist another $2^p - 1$ monomials for d, (where $p = r - \rho$ and ρ is the number of disjoint cycles of length 1 and 2), such that their index permutations form the direct products of r disjoint cycles either $\sigma_s(i_{k_s})$

or $\sigma_s^{-1}(i_{k_s})$ by specifying their ordering by s from 1 to r. Their cycle notations are left-ordered according the to the Definition 3.2. These permutations are unique decomposition of the permutation σ including their ordering by s from 1 to r. Suppose C_1 is the sum of these $2^p - 1$ monomials and d, then, by the Lemma 4.1, we obtain

$$C_1 = \chi(\sigma)\alpha\, \mathrm{t}(h_{\nu_1})\ \ldots\ \mathrm{t}(h_{\nu_p}) \in \mathbb{F}.$$

Here $\alpha \in \mathbb{F}$ is the product of coefficients whose indices form disjoint cycles of length 1 and 2, $\nu_k \in \{1, \ldots, r\}$ for all $k = \overline{1,p}$.

Thus for an arbitrary monomial of the second subset of $\mathrm{rImm}_i\, \mathbf{A}$, we can find the 2^p monomials such that their sum takes on a value in $\mathbb{F}$. Therefore, $\mathrm{rImm}_i\, \mathbf{A} \in \mathbb{F}$.

Now we prove the equality of all row immanants of $\mathbf{A}$. Consider an arbitrary $\mathrm{rImm}_j\, \mathbf{A}$ such that $j \neq i$ for all $j = \overline{1,n}$. We divide the set of monomials of $\mathrm{rImm}_j\, \mathbf{A}$ into two subsets using the same rule as for $\mathrm{rImm}_i\, \mathbf{A}$. Monomials of the first subset are products of entries of the principal diagonal or norms of entries of $\mathbf{A}$. Therefore they take on a value in $\mathbb{F}$ and each monomial of the first subset of $\mathrm{rImm}_i\, \mathbf{A}$ is equal to a corresponding monomial of the first subset of $\mathrm{rImm}_j\, \mathbf{A}$.

Now consider the monomial d_1 of the second subset of monomials of $\mathrm{rImm}_j\, \mathbf{A}$ consisting of coefficients that are equal to the coefficients of d but they are in another order. Consider all possibilities of the arrangement of coefficients in d_1.

(i) Suppose that the index permutation σ' of its coefficients form a direct product of r disjoint cycles and these cycles coincide with the r disjoint cycles of d but differ by their ordering. Then $\sigma' = \sigma$ and we have

$$d_1 = \chi(\sigma)\alpha h_\mu \ldots h_\lambda,$$

where $\{\mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}$. By the Lemma 4.1, there exist $2^p - 1$ monomials of the second subset of $\mathrm{rImm}_j\, \mathbf{A}$ such that each of them is equal to a product of p factors either h_s or $\overline{h_s}$ for all $s \in \{\mu, \ldots, \lambda\}$. Hence by the Lemma 4.1, we obtain

$$C_2 = \chi(\sigma)\alpha\, t(h_\mu)\ \ldots\ t(h_\lambda) = \chi(\sigma)\ \alpha\, t(h_{\nu_1}) \ldots\ t(h_{\nu_p}) = C_1.$$

(ii) Now suppose that in addition to the case (i) the index j is placed inside some disjoint cycle of the index permutation σ of d, e.g., $j \in \{i_{k_m+1}, ..., i_{k_m+l_m}\}$. Denote $j = i_{k_m+q}$. Considering the above said and $\sigma_{k_m+1}(i_{k_m+1}) = \sigma_{k_m+q}(i_{k_m+q})$, we have $\sigma' = \sigma$. Then d_1 is represented as follows:

$$\begin{gathered} d_1 = \chi(\sigma) a_{i_{k_m+q} i_{k_m+q+1}} \cdots \quad a_{i_{k_m+l_m} i_{k_m}}\, a_{i_{k_m} i_{k_m+1}} \cdots \times \\ \times a_{i_{k_m+q-1} i_{k_m+q}} a_{i_{k_\mu} i_{k_\mu+1}} \cdots a_{i_{k_\mu+l_\mu} i_{k_\mu}} \cdots a_{i_{k_\lambda} i_{k_\lambda+1}} \cdots a_{i_{k_\lambda+l_\lambda} i_{k_\lambda}} = \\ = \chi(\sigma)\alpha \tilde{h}_m h_\mu \ldots h_\lambda, \end{gathered} \tag{4.2}$$

where $\{m, \mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}$. Except for $\tilde{h}_m$, each factor of d_1 in (4.2) corresponds to the equal factor of d in (4.1). By the rearrangement property of the trace, we have $t(\tilde{h}_m) = t(h_m)$. Hence by the Lemma 4.1 and by analogy to the previous case, we obtain,

$$\begin{gathered} C_2 = \chi(\sigma)\alpha\, t(\tilde{h}_m)\, t(h_\mu)\ \ldots\ t(h_\lambda) = \\ = \chi(\sigma)\ \alpha\, t(h_{\nu_1})\ \ldots\ t(h_m)\ \ldots\ t(h_{\nu_p}) = C_1. \end{gathered}$$

(iii) If in addition to the case (i) the index i is placed inside some disjoint cycles of the index permutation of d_1, then we apply the rearrangement property of the trace to this cycle. As in the previous cases we find 2^p monomials of the second subset of $\mathrm{rImm}_j\, \mathbf{A}$ such that by Lemma 4.1 their sum is equal to the sum of the corresponding 2^p monomials of $\mathrm{rImm}_i \mathbf{A}$. Clearly, we obtain the same conclusion at association of all previous cases, then we apply twice the rearrangement property of the trace.

Thus, in any case each sum of 2^p corresponding monomials of the second subset of $\mathrm{rImm}_j\, \mathbf{A}$ is equal to the sum of 2^p monomials of $\mathrm{rImm}_i\, \mathbf{A}$. Here p is the number of disjoint cycles of length more than 2. Therefore, for all $i, j = \overline{1, n}$ we have

$$\mathrm{rImm}_i\, \mathbf{A} = \mathrm{rImm}_j\, \mathbf{A} \in \mathbb{F}.$$

The equality $\mathrm{cImm}_i\, \mathbf{A} = \mathrm{rImm}_i\, \mathbf{A}$ for all $i = \overline{1, n}$ is proved similarly. □

Remark 4.3. *If $\mathbf{A} \in \mathbb{H}^{n\times n}$ is skew-hermitian ($\mathbf{A} = -\mathbf{A}^*$), then the Theorem 4.2 is not meaningful. It follows from the next example.*

Example 4.4. *Consider the following skew-hermitian matrix over the split quaternions of James Cockle $\mathbf{H_S}(\frac{-1,1}{\mathbb{R}})$,*

$$\mathbf{A} = \begin{pmatrix} j & 2+i \\ -2+i & -k \end{pmatrix}.$$

Since

$$\begin{aligned} \mathrm{rImm}_1\, \mathbf{A} &= -jk - (2+i)(-2+i) = 5+i, \\ \mathrm{rImm}_2\, \mathbf{A} &= -(-2+i)(2+i) - kj = 5-i, \end{aligned}$$

then $\mathrm{rImm}_1\, \mathbf{A} \neq \mathrm{rImm}_2\, \mathbf{A}$.

Since the Theorem 4.2, we have the following definition.

Definition 4.5. *Since all column and row immanants of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the immanant (permanent, determinant) of a Hermitian matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$. By definition, we put for all $i = \overline{1, n}$*

$$\begin{aligned} \mathrm{Imm}\, \mathbf{A} &:= \mathrm{rImm}_i\, \mathbf{A} = \mathrm{cImm}_i\, \mathbf{A}, \\ \mathrm{per}\, \mathbf{A} &:= \mathrm{rper}_i\, \mathbf{A} = \mathrm{cper}_i\, \mathbf{A}, \\ \det \mathbf{A} &:= \mathrm{rdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}. \end{aligned}$$

4.1. Cramer's Rule for System of Linear Equations over a Quaternion Division Algebra

In this subsection we shall be consider $\mathbb{H}$ as a quaternion division algebra especially since quasideterminants are defined over the skew field as well.

Properties of the determinant of a Hermitian matrix is completely explored in [7, 8] by its row and column determinants. Among all, consider the following.

Theorem 4.6. *If the i-th row of the Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is replaced with a left linear combination of its other rows*

$$\mathbf{a}_{i\,.} = c_1 \mathbf{a}_{i_1\,.} + \ldots + c_k \mathbf{a}_{i_k\,.}$$

where $c_l \in \mathbb{H}$ *for all* $l = \overline{1,k}$ *and* $\{i, i_l\} \subset \{1, \ldots, n\}$*, then for all* $i = \overline{1,n}$

$$\mathrm{cdet}_i \mathbf{A}_{i\,.}\left(c_1 \cdot \mathbf{a}_{i_1\,.} + \ldots + c_k \cdot \mathbf{a}_{i_k\,.}\right) = \mathrm{rdet}_i \mathbf{A}_{i\,.}\left(c_1 \cdot \mathbf{a}_{i_1\,.} + \ldots + c_k \cdot \mathbf{a}_{i_k\,.}\right) = 0.$$

Theorem 4.7. *If the* j*-th column of a Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is replaced with a right linear combination of its other columns*

$$\mathbf{a}_{.\,j} = \mathbf{a}_{.\,j_1} c_1 + \ldots + \mathbf{a}_{.\,j_k} c_k$$

where $c_l \in \mathbb{H}$ *for all* $l = \overline{1,k}$ *and* $\{j, j_l\} \subset \{1, \ldots, n\}$ *, then for all* $j = \overline{1,n}$

$$\mathrm{cdet}_j \mathbf{A}_{.\,j}\left(\mathbf{a}_{.\,j_1} \cdot c_1 + \ldots + \mathbf{a}_{.\,j_k} \cdot c_k\right) = \mathrm{rdet}_j \mathbf{A}_{.\,j}\left(\mathbf{a}_{.\,j_1} \cdot c_1 + \ldots + \mathbf{a}_{.\,j_k} \cdot c_k\right) = 0.$$

The following theorem on the determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 4.8. *There exist a unique right inverse matrix* $(R\mathbf{A})^{-1}$ *and a unique left inverse matrix* $(L\mathbf{A})^{-1}$ *of a nonsingular Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, (*$\det \mathbf{A} \neq 0$*), where* $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$*. Right inverse and left inverse matrices has following determinantal representation*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix},$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix},$$

where R_{ij}*,* L_{ij} *are right and left* ij*-th cofactors of* $\mathbf{A}$*, respectively, for all* $i, j = \overline{1,n}$*.*

To obtain the determinantal representation for an arbitrary inverse matrix over a quaternion division algebra $\mathbb{H}$, we consider the right $\mathbf{A}\mathbf{A}^*$ and left $\mathbf{A}^*\mathbf{A}$ corresponding Hermitian matrices.

Theorem 4.9 ([7])**.** *If an arbitrary column of* $\mathbf{A} \in \mathbb{H}^{m \times n}$ *is a right linear combination of its other columns, or an arbitrary row of* $\mathbf{A}^*$ *is a left linear combination of its other rows, then* $\det \mathbf{A}^*\mathbf{A} = 0$.

Since principal submatrices of a Hermitian matrix are also Hermitian, then the basis principal minor may be defined in this noncommutative case as a principal nonzero minor of a maximal order. We also can introduce the notion of the rank of a Hermitian matrix by principal minors, as a maximal order of a principal nonzero minor. The following theorem establishes the correspondence between the rank by principal minors of a Hermitian matrix and the rank of the corresponding matrix that are defined as a maximum number of right-linearly independent columns or left-linearly independent rows, which form a basis.

Theorem 4.10 ([7])**.** *A rank by principal minors of a Hermitian matrix* $\mathbf{A}^*\mathbf{A}$ *is equal to its rank and a rank of* $\mathbf{A} \in \mathbb{H}^{m \times n}$*.*

Theorem 4.11 ([7]). *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, then an arbitrary column of* $\mathbf{A}$ *is a right linear combination of its basic columns or arbitrary row of* $\mathbf{A}$ *is a left linear combination of its basic rows.*

It implies a criterion for the singularity of a corresponding Hermitian matrix.

Theorem 4.12 ([7]). *The right linearly independence of columns of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *or the left linearly independence of rows of* $\mathbf{A}^*$ *is the necessary and sufficient condition for*

$$\det \mathbf{A}^*\mathbf{A} \neq 0$$

Theorem 4.13 ([7]). *If* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, then* $\det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

In the following example, we shall prove the Theorem 4.13 for the case $n = 2$.

Example 4.14. *Consider the matrix* $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$*, then* $\mathbf{A}^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$*. Respectively, we have*

$$\mathbf{A}\mathbf{A}^* = \begin{pmatrix} a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}} \end{pmatrix},$$

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} \overline{a_{11}}a_{11} + \overline{a_{21}}a_{21} & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{12}}a_{11} + \overline{a_{22}}a_{21} & \overline{a_{12}}a_{12} + \overline{a_{22}}a_{22} \end{pmatrix}.$$

According to thw Theorem 4.2 and the Definition 4.5, we have

$$\begin{aligned} \det \mathbf{A}\mathbf{A}^* &= \mathrm{rdet}_1 \mathbf{A}\mathbf{A}^*, \\ \det \mathbf{A}^*\mathbf{A} &= \mathrm{rdet}_1 \mathbf{A}^*\mathbf{A} \end{aligned}$$

According to the Lemma 3.17

$$\begin{aligned} \det \mathbf{A}\mathbf{A}^* &= (\mathbf{A}\mathbf{A}^*)_{11}(\mathbf{A}\mathbf{A}^*)_{22} - (\mathbf{A}\mathbf{A}^*)_{12}(\mathbf{A}\mathbf{A}^*)_{21} \\ &= (a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}})(a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}}) \\ &\quad -(a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}})(a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}}) \\ &= a_{11}\overline{a_{11}}a_{21}\overline{a_{21}} + a_{12}\overline{a_{12}}a_{21}\overline{a_{21}} \\ &\quad +a_{11}\overline{a_{11}}a_{22}\overline{a_{22}} + a_{12}\overline{a_{12}}a_{22}\overline{a_{22}} \\ &\quad -a_{11}\overline{a_{21}}a_{21}\overline{a_{11}} - a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} \\ &\quad -a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} - a_{12}\overline{a_{22}}a_{22}\overline{a_{12}} \\ &= a_{12}\overline{a_{12}}a_{21}\overline{a_{21}} + a_{11}\overline{a_{11}}a_{22}\overline{a_{22}} \\ &\quad -a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} - a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} \end{aligned} \quad , \tag{4.3}$$

$$\begin{aligned} \det \mathbf{A}^*\mathbf{A} &= (\mathbf{A}^*\mathbf{A})_{11}(\mathbf{A}^*\mathbf{A})_{22} - (\mathbf{A}^*\mathbf{A})_{12}(\mathbf{A}^*\mathbf{A})_{21} \\ &= (\overline{a_{11}}a_{11} + \overline{a_{21}}a_{21})(\overline{a_{12}}a_{12} + \overline{a_{22}}a_{22}) \\ &\quad -(\overline{a_{11}}a_{12} + \overline{a_{21}}a_{22})(\overline{a_{12}}a_{11} + \overline{a_{22}}a_{21}) \\ &= \overline{a_{11}}a_{11}\overline{a_{12}}a_{12} + \overline{a_{21}}a_{21}\overline{a_{12}}a_{12} \\ &\quad +\overline{a_{11}}a_{11}\overline{a_{22}}a_{22} + \overline{a_{21}}a_{21}\overline{a_{22}}a_{22} \\ &\quad -\overline{a_{11}}a_{12}\overline{a_{12}}a_{11} - \overline{a_{21}}a_{22}\overline{a_{12}}a_{11} \\ &\quad -\overline{a_{11}}a_{12}\overline{a_{22}}a_{21} - \overline{a_{21}}a_{22}\overline{a_{22}}a_{21} \\ &= \overline{a_{21}}a_{21}\overline{a_{12}}a_{12} + \overline{a_{11}}a_{11}\overline{a_{22}}a_{22} \\ &\quad -\overline{a_{21}}a_{22}\overline{a_{12}}a_{11} - \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} \end{aligned} \quad . \tag{4.4}$$

Positive terms in equations (4.3), (4.4) *are real numbers and they obviously coincide. To prove equation*

$$a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} = \overline{a_{21}}a_{22}\overline{a_{12}}a_{11} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} \tag{4.5}$$

we use the rearrangement property of the trace of elements of the quaternion algebra, $\mathrm{t}(pq) = \mathrm{t}(qp)$. *Indeed,*

$$a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} = a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + \overline{a_{12}\overline{a_{22}}a_{21}\overline{a_{11}}} = \mathrm{t}(a_{12}\overline{a_{22}}a_{21}\overline{a_{11}}),$$

$$\overline{a_{21}}a_{22}\overline{a_{12}}a_{11} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} = \overline{\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} = \mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21})$$

Then by the rearrangement property of the trace, we obtain (4.5).

According to the Theorem 4.13, we introduce the concept of double determinant. For the first time this concept was introduced by L. Chen ([18]).

Definition 4.15. *The determinant of corresponding Hermitian matrices is called the double determinant of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, i.e.,* $\mathrm{ddet}\mathbf{A} := \det(\mathbf{A}^*\mathbf{A}) = \det(\mathbf{A}\mathbf{A}^*)$.

If $\mathbb{H}$ is the Hamilton's quaternion skew field $\mathbf{H}$, then the following theorem establishes the validity of Axiom 1 for the double determinant.

Theorem 4.16. *If* $\{\mathbf{A}, \mathbf{B}\} \subset \mathrm{M}(n, \mathbf{H})$*, then* $\mathrm{ddet}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}$.

Unfortunately, if a non-Hermitian matrix is not full rank, then nothing can be said about singularity of its row and column determinant. We show it in the following example.

Example 4.17. *Consider the matrix*

$$\mathbf{A} = \begin{pmatrix} i & j \\ j & -i \end{pmatrix}.$$

Its second row is obtained from the first row by left-multiplying by k. *Then, by the Theorem 4.12,* $\mathrm{ddet}A = 0$. *Indeed,*

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} -i & -j \\ -j & i \end{pmatrix} \cdot \begin{pmatrix} i & j \\ j & -i \end{pmatrix} = \begin{pmatrix} 2 & -2k \\ 2k & 2 \end{pmatrix}.$$

Then $\mathrm{ddet}\mathbf{A} = 4 + 4k^2 = 0$. *However*

$$\mathrm{cdet}_1\mathbf{A} = \mathrm{cdet}_2\mathbf{A} = \mathrm{rdet}_1\mathbf{A} = \mathrm{rdet}_2\mathbf{A} = -i^2 - j^2 = 2.$$

At the same time $\mathrm{rank}\,\mathbf{A} = 1$*, that corresponds to the Theorem 4.10.*

The correspondence between the double determinant and the noncommutative determinants of Moore, Stady and Dieudonné are as follows,

$$\mathrm{ddet}\mathbf{A} = \mathrm{Mdet}(\mathbf{A}^*\mathbf{A}) = \mathrm{Sdet}\mathbf{A} = \mathrm{Ddet}^2\mathbf{A}.$$

Definition 4.18. *Let* $\mathrm{ddet}\mathbf{A} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A}) = \sum\limits_i \mathbb{L}_{ij} \cdot a_{ij}$ *for* $j = \overline{1,n}$. *Then* $\mathbb{L}_{ij}$ *is called the left double* ij*-th cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$.

Definition 4.19. *Let* $\mathrm{ddet}\mathbf{A} = \mathrm{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right) = \sum_j a_{ij} \cdot \mathbb{R}_{ij}$ *for* $i = \overline{1,n}$*. Then* $\mathbb{R}_{ij}$ *is called the right double* ij*-th cofactor of* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$.

Theorem 4.20. *The necessary and sufficient condition of invertibility of a matrix* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is* $\mathrm{ddet}\mathbf{A} \neq 0$*. Then* $\exists \mathbf{A}^{-1} = \left(L\mathbf{A}\right)^{-1} = \left(R\mathbf{A}\right)^{-1}$*, where*

$$\left(L\mathbf{A}\right)^{-1} = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^* = \frac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \dots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \dots & \mathbb{L}_{n2} \\ \dots & \dots & \dots & \dots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \dots & \mathbb{L}_{nn} \end{pmatrix} \tag{4.6}$$

$$\left(R\mathbf{A}\right)^{-1} = \mathbf{A}^*\left(\mathbf{A}\mathbf{A}^*\right)^{-1} = \frac{1}{\mathrm{ddet}\mathbf{A}^*}\begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \dots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \dots & \mathbb{R}_{n2} \\ \dots & \dots & \dots & \dots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \dots & \mathbb{R}_{nn} \end{pmatrix} \tag{4.7}$$

and $\mathbb{L}_{ij} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.\,j}\left(\mathbf{a}^*_{.i}\right)$, $\mathbb{R}_{ij} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\left(\mathbf{a}^*_{j\,.}\right)$ *for all* $i, j = \overline{1,n}$*.*

Remark 4.21. *In the Theorem 4.20, the inverse matrix* $\mathbf{A}^{-1}$ *of an arbitrary matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *under the assumption of* $\mathrm{ddet}\mathbf{A} \neq 0$ *is represented by the analog of the classical adjoint matrix. If we denote this analog of the adjoint matrix over* $\mathbb{H}$ *by* $\mathrm{Adj}[[\mathbf{A}]]$*, then the next formula is valid over* $\mathbb{H}$*:*

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}[[\mathbf{A}]]}{\mathrm{ddet}\mathbf{A}}.$$

An obvious consequence of a determinantal representation of the inverse matrix by the classical adjoint matrix is Cramer's rule.

Theorem 4.22. *Let*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{4.8}$$

be a right system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a column of constants* $\mathbf{y} = \left(y_1, \dots, y_n\right)^T \in \mathbb{H}^{n \times 1}$*, and a column of unknowns* $\mathbf{x} = \left(x_1, \dots, x_n\right)^T$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then* (4.8) *has a unique solution that has represented as follows,*

$$x_j = \frac{\mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{f}\right)}{\mathrm{ddet}\mathbf{A}}, \qquad \forall j = \overline{1,n} \tag{4.9}$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{y}$.

Theorem 4.23. *Let*

$$\mathbf{x} \cdot \mathbf{A} = \mathbf{y} \tag{4.10}$$

be a left system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a column of constants* $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{H}^{1 \times n}$ *and a column of unknowns* $\mathbf{x} = (x_1, \dots, x_n)$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then* (4.10) *has a unique solution that has represented as follows,*

$$x_i = \frac{\mathrm{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right)_{i.}\left(\mathbf{z}\right)}{\mathrm{ddet}\mathbf{A}}, \qquad \forall i = \overline{1,n} \tag{4.11}$$

where $\mathbf{z} = \mathbf{y}\mathbf{A}^*$.

Equations (4.9) and (4.11) are the obvious and natural generalizations of Cramer's rule for systems of linear equations over a quaternion division algebra. As follows from the Theorem 4.8, the closer analog to Cramer's rule can be obtained in the following specific cases.

Theorem 4.24. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian in* (4.8)*. Then the solution of* (4.8) *has represented by the equation,*

$$x_j = \frac{\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{y})}{\det \mathbf{A}}, \quad \forall j = \overline{1, n}.$$

Theorem 4.25. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian in* (4.10)*. Then the solution of* (4.10) *has represented as follows,*

$$x_i = \frac{\mathrm{rdet}_i \mathbf{A}_{i.}(\mathbf{y})}{\det \mathbf{A}}, \quad \forall i = \overline{1, n}.$$

An application of the column-row determinants in the theory of generalized inverse matrices over the quaternion skew field recently has been received in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

5. Quasideterminants over a Quaternion Division Algebra

Theorem 5.1. *Suppose a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

with entries from a quaternion division algebra has an inverse $\mathbf{A}^{-1}$.[2] *Then a minor of the inverse matrix satisfies the following equation, provided that the inverse matrices exist*

$$((\mathbf{A}^{-1})_{IJ})^{-1} = \mathbf{A}_{JI} - \mathbf{A}_{J.}^{.I}(\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.} \tag{5.1}$$

Proof. Definition of an inverse matrix leads to the system of linear equations

$$\mathbf{A}^{JI}(\mathbf{A}^{-1})_{.J}^{I.} + \mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = 0 \tag{5.2}$$

$$\mathbf{A}_{J.}^{.I}(\mathbf{A}^{-1})_{.J}^{I.} + \mathbf{A}_{JI}(\mathbf{A}^{-1})_{IJ} = \mathbf{I} \tag{5.3}$$

where **I** is a unit matrix. We multiply (5.2) by $\left(\mathbf{A}^{JI}\right)^{-1}$

$$(\mathbf{A}^{-1})_{.J}^{I.} + (\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = 0 \tag{5.4}$$

Now we can substitute (5.4) into (5.3)

$$\mathbf{A}_{JI}(\mathbf{A}^{-1})_{IJ} - \mathbf{A}_{J.}^{.I}(\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = \mathbf{I} \tag{5.5}$$

(5.1) follows from (5.5). □

[2]This statement and its proof are based on statement 1.2.1 from [17] (page 8) for matrix over free division ring.

Corollary 5.2. *Suppose a matrix $\mathbf{A}$ has the inverse matrix. Then elements of the inverse matrix satisfy to the equation*

$$((\mathbf{A}^{-1})_{ij})^{-1} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}(\mathbf{A}^{ji})^{-1}\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.6}$$

Example 5.3. *Consider a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

According to (5.6)

$$(\mathbf{A}^{-1})_{11} = (a_{11} - a_{12}(a_{22})^{-1}\, a_{21})^{-1} \tag{5.7}$$

$$(\mathbf{A}^{-1})_{21} = (a_{21} - a_{22}(a_{12})^{-1}\, a_{11})^{-1} \tag{5.8}$$

$$(\mathbf{A}^{-1})_{12} = (a_{12} - a_{11}(a_{21})^{-1}\, a_{22})^{-1} \tag{5.9}$$

$$(\mathbf{A}^{-1})_{22} = (a_{22} - a_{21}(a_{11})^{-1}\, a_{12})^{-1} \tag{5.10}$$

We call a matrix

$$\mathcal{H}\mathbf{A} = ((\mathcal{H}\mathbf{A})_{ij}) = ((a_{ji})^{-1}) \tag{5.11}$$

a Hadamard inverse of[3] $\mathbf{A}$.

Definition 5.4. *The (ji)-quasideterminant of $\mathbf{A}$ is formal expression*

$$|\mathbf{A}|_{ji} = (\mathcal{H}\mathbf{A}^{-1})_{ji} = ((\mathbf{A}^{-1})_{ij})^{-1} \tag{5.12}$$

We consider the (ji)-quasideterminant as an element of the matrix $|\mathbf{A}|$, which is called a quasideterminant.

Theorem 5.5. *Expression for the (ji)-quasideterminant has form*

$$|\mathbf{A}|_{ji} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}(\mathbf{A}^{ji})^{-1}\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.13}$$

$$|\mathbf{A}|_{ji} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}\,\mathcal{H}|\mathbf{A}^{ji}|\,\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.14}$$

Proof. The statement follows from (5.6) and (5.12). □

Example 5.6. *Let*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.15}$$

It is clear from (5.7) *and* (5.10) *that* $(\mathbf{A}^{-1})_{11} = 1$ *and* $(\mathbf{A}^{-1})_{22} = 1$. *However expression for* $(\mathbf{A}^{-1})_{21}$ *and* $(\mathbf{A}^{-1})_{12}$ *cannot be defined from* (5.8) *and* (5.9) *since* $(a_{21} - a_{22}(a_{12})^{-1}\, a_{11})^{-1} = (a_{12} - a_{11}(a_{21})^{-1}\, a_{22})^{-1} = 0$. *We can transform these expressions. For instance*

$$\begin{aligned}(\mathbf{A}^{-1})_{21} &= (a_{21} - a_{22}(a_{12})^{-1}\, a_{11})^{-1} \\ &= (a_{11}((a_{11})^{-1}\, a_{12} - (a_{21})^{-1}\, a_{22}))^{-1} \\ &= ((a_{21})^{-1}\, a_{11}(a_{21}(a_{11})^{-1}\, a_{12} - a_{22}))^{-1} \\ &= (a_{11}(a_{21}(a_{11})^{-1}\, a_{12} - a_{22}))^{-1}\, a_{21}\end{aligned}$$

[3]See also page 4 in paper [16].

It follows immediately that $(\mathbf{A}^{-1})_{21} = 0$. *In the same manner we can find that* $(\mathbf{A}^{-1})_{12} = 0$. *Therefore,*

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.16}$$

□

From the Example 5.6 we see that we cannot always use Equation (5.6) to find elements of the inverse matrix and we need more transformations to solve this problem. From the theorem 4.6.3 in the paper [9], it follows that if

$$\operatorname{rank} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \le n-2$$

then $|\mathbf{A}|_{ij}$ for all $i, j = \overline{1,n}$ is not defined. From this, it follows that although a quasideterminant is a powerful tool, use of a determinant is a major advantage.

Theorem 5.7. *Let a matrix* $\mathbf{A}$ *have an inverse. Then for any matrices* $\mathbf{B}$ *and* $\mathbf{C}$ *equation*

$$\mathbf{B} = \mathbf{C} \tag{5.17}$$

follows from the equation

$$\mathbf{BA} = \mathbf{CA} \tag{5.18}$$

Proof. Equation (5.17) follows from (5.18) if we multiply both parts of (5.18) over $\mathbf{A}^{-1}$. □

Theorem 5.8. *The solution of a nonsingular system of linear equations*

$$\mathbf{A}x = b \tag{5.19}$$

is determined uniquely and can be presented in either form[4]

$$x = \mathbf{A}^{-1} b \tag{5.20}$$

$$x = \mathcal{H}|\mathbf{A}|\, b \tag{5.21}$$

Proof. Multiplying both sides of (5.19) from left by $\mathbf{A}^{-1}$ we get (5.20). Using the Definition 5.4, we get (5.21). Since the Theorem 5.7, the solution is unique. □

6. Relation of Row-Column Determinants with Quasideterminants

Theorem 6.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is an invertible matrix, then, for arbitrary* $p, q = \overline{1,n}$, *we have the following representation of the pq-quasideterminant*

$$|\ \mathbf{A}\ |_{pq} = \frac{\mathrm{ddet}\mathbf{A} \cdot \overline{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right)}}{\mathrm{n}(\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right))}, \tag{6.1}$$

[4]See similar statement in the theorem 1.6.1 in the paper [17] on pagen 19.

$$| \mathbf{A} |_{pq} = \frac{\mathrm{ddet}\mathbf{A} \cdot \overline{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{a}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{a}^*_{q.}\right))}. \tag{6.2}$$

Proof. Let $\mathbf{A}^{-1} = (b_{ij})$ to $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. Equation (5.12) reveals the relationship between a quasideterminant $| \mathbf{A} |_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ and elements of the inverse matrix $\mathbf{A}^{-1} = (b_{ij})$, namely

$$| \mathbf{A} |_{pq} = b_{qp}^{-1}$$

for all $p, q = \overline{1, n}$. At the same time, the theory of row and column determinants (the theorem 4.20) gives us representation of the inverse matrix through its left (4.6) and right (4.7) double cofactors. Thus, accordingly, we obtain

$$| \mathbf{A} |_{pq} = b_{qp}^{-1} = \left(\frac{\mathbb{L}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}, \tag{6.3}$$

$$| \mathbf{A} |_{pq} = b_{qp}^{-1} = \left(\frac{\mathbb{R}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}. \tag{6.4}$$

Since $\mathrm{ddet}\mathbf{A} \neq 0 \in \mathbb{F}$, then $\exists(\mathrm{ddet}\mathbf{A})^{-1} \in \mathbb{F}$. It follows that

$$\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)^{-1} = \frac{\overline{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}}{\mathrm{n}(\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right))}, \tag{6.5}$$

$$\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)^{-1} = \frac{\overline{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right))}. \tag{6.6}$$

Substituting (6.5) into (6.3), and (6.6) into (6.4), we accordingly obtain (6.1) and (6.2).

We proved the theorem. □

Equation (6.1) gives an explicit representation of a quasideterminant $| \mathbf{A} |_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ for all $p, q = \overline{1, n}$ by the column determinant of its corresponding left Hermitian matrix $\mathbf{A}^*\mathbf{A}$, and (6.2) does by the row determinant of its corresponding right Hermitian matrix $\mathbf{AA}^*$.

Example 6.2. *Consider a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

According to (5.13)

$$|\mathbf{A}| = \begin{pmatrix} a_{11} - a_{12}(a_{22})^{-1}\, a_{21} & a_{12} - a_{11}(a_{21})^{-1}\, a_{22} \\ a_{21} - a_{22}(a_{12})^{-1}\, a_{11} & a_{22} - a_{21}(a_{11})^{-1}\, a_{12} \end{pmatrix} \tag{6.7}$$

Our goal is to find this quasideterminant using the Theorem 6.1. It is evident that

$$\mathbf{A}^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} \quad \mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathrm{n}(a_{11}) + \mathrm{n}(a_{21}) & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{12}}a_{11} + \overline{a_{22}}a_{21} & \mathrm{n}(a_{12}) + \mathrm{n}(a_{22}) \end{pmatrix}.$$

Calculate the necessary determinants

$$\begin{aligned}
\mathrm{ddet}\mathbf{A} &= \mathrm{rdet}_1(\mathbf{A}^*\mathbf{A}) \\
&= (\mathrm{n}(a_{11}) + \mathrm{n}(a_{21})) \cdot (\mathrm{n}(a_{12}) + \mathrm{n}(a_{22})) \\
&-(\overline{a_{11}}a_{12} + \overline{a_{21}}a_{22}) \cdot (\overline{a_{12}}a_{11} + \overline{a_{22}}a_{21}) \\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{12}) + \mathrm{n}(a_{11})\mathrm{n}(a_{22}) + \mathrm{n}(a_{21})\mathrm{n}(a_{12}) + \mathrm{n}(a_{21})\mathrm{n}(a_{22}) \\
&-\overline{a_{11}}a_{12}\overline{a_{12}}a_{11} - \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} - \overline{a_{21}}a_{22}\overline{a_{12}}a_{11} - \overline{a_{21}}a_{22}\overline{a_{22}}a_{21} \\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{22}) + \mathrm{n}(a_{21})\mathrm{n}(a_{12}) - (\overline{a_{11}}a_{12}\overline{a_{22}}a_{21} + \overline{\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}}) \\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{22}) + \mathrm{n}(a_{21})\mathrm{n}(a_{12}) - \mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21})
\end{aligned}$$

$$\begin{aligned}
\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2}) &= \mathrm{cdet}_1\begin{pmatrix} \overline{a_{21}} & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{22}} & \mathrm{n}(a_{12}) + \mathrm{n}(a_{22}) \end{pmatrix} \\
&= \mathrm{n}(a_{12})\overline{a_{21}} + \mathrm{n}(a_{22})\overline{a_{21}} - \overline{a_{11}}a_{12}\overline{a_{22}} - \overline{a_{21}}a_{22}\overline{a_{22}} \\
&= \mathrm{n}(a_{12})\overline{a_{21}} - \overline{a_{11}}a_{12}\overline{a_{22}}.
\end{aligned}$$

Then

$$\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} = \mathrm{n}(a_{12})a_{21} - a_{22}\overline{a_{12}}a_{11},$$

$$\begin{aligned}
\mathrm{n}(\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})) &= \overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} \cdot \mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2}) \\
&= (\mathrm{n}(a_{12})a_{21} - a_{22}\overline{a_{12}}a_{11}) \cdot (\mathrm{n}(a_{12})\overline{a_{21}} - \overline{a_{11}}a_{12}\overline{a_{22}}) \\
&= \mathrm{n}^2(a_{12})\mathrm{n}(a_{21}) - \mathrm{n}(a_{12})a_{21}\overline{a_{11}}a_{12}\overline{a_{22}} \\
&-\mathrm{n}(a_{12})a_{22}\overline{a_{12}}a_{11}\overline{a_{21}} + a_{22}\overline{a_{12}}a_{11}\overline{a_{11}}a_{12}\overline{a_{22}} \\
&= \mathrm{n}(a_{12})(\mathrm{n}(a_{12})\mathrm{n}(a_{21}) - \mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}) + \mathrm{n}(a_{21})\mathrm{n}(a_{12})) \\
&= \mathrm{n}(a_{12})\mathrm{ddet}\mathbf{A}.
\end{aligned}$$

Following (6.1)*, we obtain*

$$\begin{aligned}
|\mathbf{A}|_{21} &= \frac{\mathrm{ddet}\mathbf{A}}{\mathrm{n}(\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2}))}\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} \\
&= \frac{\mathrm{ddet}\mathbf{A}}{\mathrm{n}(a_{12})\mathrm{ddet}\mathbf{A}}\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} \\
&= \frac{1}{\mathrm{n}(a_{12})} \cdot \overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} \\
&= \frac{1}{\mathrm{n}(a_{12})} \cdot (\mathrm{n}(a_{12})a_{21} - a_{22}\overline{a_{12}}a_{11}) \\
&= a_{21} - a_{22}(a_{12})^{-1}a_{11}.
\end{aligned} \tag{6.8}$$

The last expression in (6.8) *coincides with the expression* $|\mathbf{A}|_{21}$ *in* (6.7). □

7. Conclusion

In the chapter we consider two approaches to define a noncommutative determinant, row-column determinants and quasideterminants. These approaches of studying of a matrix with entryes from non commutative division ring have their own field of applications.

The theory of the row and column determinants as an extension of the classical definition of determinant has been elaborated for matrices over a quaternion division algebra. It has applications in the theories of matrix equations and of generalized inverse matrices over the quaternion skew field. Now it is in development for matrices over a split quaternion algebra. In the chapter we have extended the concepts of an immanant, a permanent and a determinant to a split quaternion algebra and have established their basic properties.

Quasideterminants of Gelfand-Retax are rational matrix functions that requires the invertibility of certain submatrices. Now they are widely used. Though we can use quasideterminant in any division ring,[5] row-column determinant is more attractive to find solution of system of linear equations when division ring has conjugation.

In the chapter we have derived relations of row-column determinants with quasideterminants of a matrix over a quaternion division algebra. The use of equations (6.1) and (6.2) allows us direct calculation of quasideterminants. It already gives significance in establishing these relations.

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