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Chapter 9

Relation of Row-Column Determinants with Quasideterminants of Matrices over a Quaternion Algebra

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Abstract

Since product of quaternions is noncommutative, there is a problem how to determine a determinant of a matrix with noncommutative elements (it's called a noncommutative determinant). We consider two approaches to define a noncommutative determinant. Primarily, there are row – column determinants that are an extension of the classical definition of the determinant; however we assume predetermined order of elements in each of the terms of the determinant. In the chapter we extend the concept of an immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

Properties of the determinant of a Hermitian matrix are established. Based on these properties, analogs of the classical adjont matrix over a quaternion skew field have been obtained. As a result we have a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule by using row–column determinants.

Quasideterminants appeared from the analysis of the procedure of a matrix inversion. By using quasideterminants, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

The common feature in definition of row and column determinants and quasideterminants is that we have not one determinant of a quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminants, n row determinants, and n column determinants). We have obtained a relation of row-column determinants with quasideterminants of a matrix over a quaternion division algebra.

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1. Introduction

Linear algebra is a powerful tool that we use in different areas of mathematics, including the calculus, the analytic and differential geometry, the theory of differential equations, and the optimal control theory. Linear algebra has accumulated a rich set of different methods. Since some methods have a common final result, this gives us the opportunity to choose the most effective method, depending on the nature of calculations.

At transition from linear algebra over a field to linear algebra over a division ring, we want to save as much as possible tools that we regularly use. Already in the early XX century, shortly after Hamilton created a quaternion algebra, mathematicians began to search the answer how looks like the algebra with noncommutative multiplication. In particular, there is a problem how to determine a determinant of a matrix with elements belonging to a noncommutative ring. Such determinant is also called a noncommutative determinant.

There were a lot of approaches to the definition of the noncommutative determinant. However none of the introduced noncommutative determinants maintained all those properties that determinant possessed for matrices over a field. Moreover, in paper [1], J. Fan proved that there is no unique definition of determinant which would expands the definition of determinant of real matrices for matrices over the division ring of quaternions. Therefore, search for a solution of the problem to define a noncommutative determinant is still going on.

In this chapter, we consider two approaches to define noncommutative determinant. Namely, we explore row-column determinants and quasideterminant.

Row-column determinants are an extension of the classical definition of the determinant, however we assume predetermined order of elements in each of the terms of the determinant. Using row-column determinants, we obtain a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule.

Quasideterminant appeared from the analysis of the procedure of a matrix inversion. Using quasideterminant, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

There is common in definition of row and column determinants and quasideterminant. In both cases, we have not one determinant in correspondence to quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminant, n row determinants, and n column determinants).

Today there is wide application of quasideterminants in linear algebra ([2, 3]), and in physics ([4, 5, 6]). Row and column determinants ([7, 8]) introduced relatively recently are less well known. Purpose of the chapter is establishment of a relation of row-column determinants with quasideterminants of a matrix over a quaternion algebra. The authors are hopeful that the establishment of this relation can provide mutual development of both the theory of quasideterminants and the theory of row-column determinants.

1.1. Convention about Notations

There are different forms to write elements of a matrix. In this paper, we denote a_{ij} an element of the matrix $\mathbf{A}$. The index i labels rows, and the index j labels columns.

We use the following notation for different minors of the matrix $\mathbf{A}$.

$\mathbf{a}_{i\,\cdot}$ the i-th row

$\mathbf{A}_{S\,\cdot}$ the minor obtained from $\mathbf{A}$ by selecting rows with index from the set S

$\mathbf{A}^{i\,\cdot}$ the minor obtained from $\mathbf{A}$ by deleting row $\mathbf{a}_{i\,\cdot}$

$\mathbf{A}^{S\,\cdot}$ the minor obtained from $\mathbf{A}$ by deleting rows with index from the set S

$\mathbf{a}_{\cdot\,j}$ the j-th column

$\mathbf{A}_{\cdot\,T}$ the minor obtained from $\mathbf{A}$ by selecting columns with index from the set T

$\mathbf{A}^{\cdot\,j}$ the minor obtained from $\mathbf{A}$ by deleting column $\mathbf{a}_{\cdot\,j}$

$\mathbf{A}^{\cdot\,T}$ the minor obtained from $\mathbf{A}$ by deleting columns with index from the set T

$\mathbf{A}_{.j}\left(\mathbf{b}\right)$ the matrix obtained from $\mathbf{A}$ by replacing its j-th column by the column $\mathbf{b}$

$\mathbf{A}_{i.}\left(\mathbf{b}\right)$ the matrix obtained from $\mathbf{A}$ by replacing its i-th row by the row $\mathbf{b}$

Considered notations can be combined. For instance, the record

$$\mathbf{A}^{ii}_{k.}(\mathbf{b})$$

means replacing of the k-th row by the vector $\mathbf{b}$ followed by removal of both the i-th row and the i-th column.

As was noted in section 2.2 of the paper [9], we can define two types of matrix products: either product of rows of first matrix over columns of second one, or product of columns of first matrix over rows of second one. However, according to the theorem 2.2.5 in the paper [9], this product is symmetric relative operation of transposition. Hence in the chapter, we will restrict ourselves by traditional product of rows of first matrix over columns of second one; and we do not indicate clearly the operation like it was done in [9].

1.2. Preliminaries. A Brief Overview of the Theory of Noncommutative Determinants

Theory of determinants of matrices with noncommutative elements can be divided into three groups regarding their methods of definition. Denote $\mathrm{M}(n, \mathbf{K})$ the ring of matrices with elements from the ring $\mathbf{K}$. One of the ways to determine determinant of a matrix of $\mathrm{M}\left(n, \mathbf{K}\right)$ is following ([11, 12, 13]).

Definition 1.1. *Let the functional*

$$\mathrm{d} : \mathrm{M}\left(n, \mathbf{K}\right) \to \mathbf{K}$$

satisfy the following axioms.

Axiom 1. $\mathrm{d}(\mathbf{A}) = 0$ *iff* $\mathbf{A}$ *is singular (irreversible).*

Axiom 2. $\forall \mathbf{A}, \mathbf{B} \in \mathrm{M}(n, \mathbf{K})$, $\mathrm{d}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{d}(\mathbf{A}) \cdot \mathrm{d}(\mathbf{B})$.

Axiom 3. *If we obtain a matrix* $\mathbf{A}'$ *from matrix* $\mathbf{A}$ *either by adding of an arbitrary row multiplied on the left with its another row or by adding of an arbitrary column multiplied on the right with its another column, then*

$$\mathrm{d}\left(\mathbf{A}'\right) = \mathrm{d}(\mathbf{A})$$

Then the value of the functional d *is called determinant of* $\mathbf{A} \in \mathrm{M}(n, \mathbf{K})$. □

The known determinants of Dieudonné and Study are examples of such functionals. Aslaksen [11] proved that determinants which satisfy Axioms 1, 2 and 3 take their value in some commutative subset of the ring. It makes no sense for them such property of conventional determinants as the expansion along an arbitrary row or column. Therefore a determinantal representation of an inverse matrix using only these determinants is impossible. This is the reason that causes to introduce determinant functionals that do not satisfy all Axioms. Dyson [13] considers Axiom 1 as necessary to determine a determinant.

In another approach, a determinant of a square matrix over a noncommutative ring is considered as a rational function of entries of a matrix. The greatest success is achieved by Gelfand and Retakh [14, 15, 16, 17] in the theory of quasideterminants. We present introduction to the theory of quasideterminants in the section 5.

In third approach, a determinant of a square matrix over a noncommutative ring is considered as an alternating sum of $n!$ products of entries of a matrix. However, it assumed certain fixed order of factors in each term. E. H. Moore was first who achieved implementation of the key Axiom 1 using such definition of a noncommutative determinant. Moore had done this not for all square matrices, but only for Hermitian. He defined the determinant of a Hermitian matrix[1] $\mathbf{A} = (a_{ij})_{n\times n}$ over a division ring with involution by induction over n following way (see [13])

$$\mathrm{Mdet}\mathbf{A} = \begin{cases} a_{11}, & n = 1 \\ \sum\limits_{j=1}^{n} \varepsilon_{ij} a_{ij} \mathrm{Mdet}\left(\mathbf{A}(i \to j)\right), & n > 1 \end{cases} \tag{1.1}$$

Here $\varepsilon_{kj} = \begin{cases} 1, & i = j \\ -1, & i \neq j \end{cases}$, and $\mathbf{A}(i \to j)$ denotes the matrix obtained from $\mathbf{A}$ by replacing its j-th column with the i-th column and then by deleting both the i-th row and column. Another definition of this determinant is presented in [11] by using permutations,

$$\mathrm{Mdet}\,\mathbf{A} = \sum_{\sigma\in S_n} |\sigma| a_{n_{11}n_{12}} \cdot \ldots \cdot a_{n_{1l_1}n_{11}} \cdot a_{n_{21}n_{22}} \cdot \ldots \cdot a_{n_{rl_1}n_{r1}}.$$

Here S_n is symmetric group of n elements. A cycle decomposition of a permutation σ has form,

$$\sigma = (n_{11} \ldots n_{1l_1})(n_{21} \ldots n_{2l_2}) \ldots (n_{r1} \ldots n_{rl_r}).$$

[1]Hermitian matrix is such matrix $\mathbf{A} = (a_{ij})$ that $a_{ij} = \overline{a_{ji}}$.

However, there was no any generalization of the definition of Moore's determinant to arbitrary square matrices. Freeman J. Dyson [13] pointed out the importance of this problem.

L. Chen [18, 19] offered the following definition of determinant of a square matrix over the quaternion skew field $\mathbf{H}$, by putting for $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbf{H})$,

$$\det \mathbf{A} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{n_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_s n_1} \cdot \ldots \cdot a_{n_r k_2} \cdot \ldots \cdot a_{k_l n_r},$$
$$\sigma = (n_1 i_2 \ldots i_s) \ldots (n_r k_2 \ldots k_r),$$
$$n_1 > i_2, i_3, \ldots, i_s; \ldots, n_r > k_2, k_3, \ldots, k_l,$$
$$n = n_1 > n_2 > \ldots > n_r \geq 1.$$

Despite the fact that this determinant does not satisfy Axiom 1, L. Chen got a determinantal representation of an inverse matrix. However it can not been expanded along arbitrary rows and columns (except for n-th row). Therefore, L. Chen did not obtain a classical adjoint matrix as well. For $\mathbf{A} = (\alpha_1, \ldots, \alpha_m)$ over the quaternion skew field $\mathbf{H}$, if $\|\mathbf{A}\| := \det(\mathbf{A}^* \mathbf{A}) \neq 0$, then $\exists \mathbf{A}^{-1} = (b_{jk})$, where

$$\overline{b_{jk}} = \frac{1}{\|\mathbf{A}\|}\omega_{kj}, \quad \left(j, k = \overline{1, n}\right),$$

$$\omega_{kj} = \det\left(\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \delta_k\right)^* \left(\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \alpha_j\right).$$

Here α_i is the i-th column of $\mathbf{A}$, δ_k is the n-dimensional column with 1 in the k-th entry and 0 in other ones. L. Chen defined $\|\mathbf{A}\| := \det(\mathbf{A}^* \mathbf{A})$ as the double determinant. If $\|\mathbf{A}\| \neq 0$, then the solution of a right system of linear equations

$$\sum\nolimits_{j=1}^{n} \alpha_j x_j = \beta$$

over $\mathbf{H}$ is represented by the following formula, which the author calls Cramer's rule

$$x_j = \|\mathbf{A}\|^{-1} \overline{\mathbf{D}_j},$$

for all $j = \overline{1, n}$, where

$$\mathbf{D}_j = \det \begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_{j-1}^* \\ \alpha_n^* \\ \alpha_{j+1}^* \\ \vdots \\ \alpha_{n-1}^* \\ \beta^* \end{pmatrix} \begin{pmatrix} \alpha_1 & \ldots & \alpha_{j-1} & \alpha_n & \alpha_{j+1} & \ldots & \alpha_{n-1} & \alpha_j \end{pmatrix}.$$

Here α_i^* is the i-th row of $\mathbf{A}^*$ and β^* is the n-dimensional vector-row conjugated with β.

In this chapter we explore the theory of row and column determinants which develops the classical approach to the definition of determinant of a square matrix, as an alternating sum of products of entries of a matrix but with a predetermined order of factors in each of the terms of the determinant.

2. Quaternion Algebra

A quaternion algebra $\mathbb{H}(a,b)$ (we also use notation $\left(\frac{a,b}{\mathbb{F}}\right)$) is a four-dimensional vector space over a field $\mathbb{F}$ with basis $\{1,i,j,k\}$ and the following multiplication rules:

$$\begin{aligned} i^2 &= a, \\ j^2 &= b, \\ ij &= k, \\ ji &= -k. \end{aligned}$$

The field $\mathbb{F}$ is the center of the quaternion algebra $\mathbb{H}(a,b)$.

In the algebra $\mathbb{H}(a,b)$ there are following mappings.

- A quadratic form

  $$\mathrm{n}: x \in \mathbb{H} \to \mathrm{n}(x) \in \mathbb{F}$$

  such that

  $$\mathrm{n}(x \cdot y) = \mathrm{n}(x)\mathrm{n}(y) \quad x, y \in \mathbb{H}$$

  is called the norm on a quaternion algebra $\mathbb{H}$.

- The linear mapping

  $$\mathrm{t}: x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H} \to \mathrm{t}(x) = 2x^0 \in \mathbb{F}$$

  is called the trace of a quaternion. The trace satisfies permutability property of the trace,

  $$\mathrm{t}(q \cdot p) = \mathrm{t}(p \cdot q).$$

  From the theorem 10.3.3 in the paper [9], it follows

  $$\mathrm{t}(x) = \frac{1}{2}(x - ixi - jxj - kxk). \tag{2.1}$$

- A linear mapping

  $$x \to \overline{x} = \mathrm{t}(x) - x \tag{2.2}$$

  is an involution. The involution has following properties

  $$\begin{aligned} \overline{\overline{x}} &= x, \\ \overline{x+y} &= \overline{x} + \overline{y}, \\ \overline{x \cdot y} &= \overline{y} \cdot \overline{x}. \end{aligned}$$

  A quaternion $\overline{x}$ is called the conjugate of $x \in \mathbb{H}$. The norm and the involution satisfy the following condition:

  $$\mathrm{n}(\overline{q}) = \mathrm{n}(q).$$

  The trace and the involution satisfy the following condition,

  $$\mathrm{t}(\overline{x}) = \mathrm{t}(x).$$

From equations (2.1), (2.2), it follows that

$$\overline{x} = -\frac{1}{2}(x + ixi + jxj + kxk).$$

Depending on the choice of the field $\mathbb{F}$, a and b, on the set of quaternion algebras there are only two possibilities [20]:

1. $\left(\frac{a,b}{\mathbb{F}}\right)$ is a division algebra.
2. $\left(\frac{a,b}{\mathbb{F}}\right)$ is isomorphic to the algebra of all 2×2 matrices with entries from the field $\mathbb{F}$. In this case, quaternion algebra is splittable.

The most famous example of a non-split quaternion algebra is Hamilton's quaternions $\mathbf{H} = (\frac{-1,-1}{\mathbb{R}})$, where $\mathbb{R}$ is real field. The set of quaternions can be represented as

$$\mathbf{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Consider some non-isomorphic quaternion algebra with division.

1. $\left(\frac{a,b}{\mathbb{R}}\right)$ is isomorphic to the Hamilton quaternion skew field $\mathbf{H}$ whenever $a < 0$ and $b < 0$. Otherwise $\left(\frac{a,b}{\mathbb{R}}\right)$ is splittable.
2. If $\mathbb{F}$ is the rational field $\mathbb{Q}$, then there exist infinitely many nonisomorphic division quaternion algebras $\left(\frac{a,b}{\mathbb{Q}}\right)$ depending on choice of $a < 0$ and $b < 0$.
3. Let $\mathbb{Q}_p$ be the p-adic field where p is a prime number. For each prime number p there is a unique division quaternion algebra.

The famous example of a split quaternion algebra is split quaternions of James Cockle $\mathbf{H_S}(\frac{-1,1}{\mathbb{R}})$, which can be represented as

$$\mathbf{H_S} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $i^2 = -1$, $j^2 = k^2 = 1$ and $ijk = 1$. Unlike quaternion division algebra, the set of split quaternions is a noncommutative ring with zero divisors, nilpotent elements and nontrivial idempotents. Recently there was conducted a number of studies in split quaternion matrices (see, for ex. [21, 22, 23, 24]).

3. Introduction to the Theory of the Row and Column Determinants over a Quaternion Algebra

The theory of the row and column determinants was introduced [7, 8] for matrices over a quaternion division algebra. Now this theory is in development for matrices over a split quaternion algebra. In the following two subsections we extend the concept of immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

3.1. Definitions and Properties of the Column and Row Immanants

The immanant of a matrix is a generalization of the concepts of determinant and permanent. The immanant of a complex matrix was defined by Dudley E. Littlewood and Archibald Read Richardson [25] as follows.

Definition 3.1. *Let $\sigma \in S_n$ denote the symmetric group on n elements. Let $\chi : S_n \to \mathbb{C}$ be a complex character. For any $n \times n$ matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n\times n}$ define the immanant of $\mathbf{A}$ as*

$$\operatorname{Imm}_\chi(\mathbf{A}) = \sum_{\sigma\in S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

Special cases of immanants are determinants and permanents. In the case where χ is the constant character ($\chi(x) = 1$ for all $x \in S_n$), $\operatorname{Imm}_\chi(\mathbf{A})$ is the permanent of $\mathbf{A}$. In the case where χ is the sign of the permutation (which is the character of the permutation group associated to the (non-trivial) one-dimensional representation), $\operatorname{Imm}_\chi(\mathbf{A})$ is the determinant of $\mathbf{A}$.

Denote by $\mathbb{H}^{n\times m}$ a set of $n \times m$ matrices with entries in an arbitrary (split) quaternion algebra $\mathbb{H}$ and $\mathrm{M}(n, \mathbb{H})$ a ring of matrices with entries in $\mathbb{H}$. For $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ we define n row immanants as follows.

Definition 3.2. *The i-th row immanant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined by putting*

$$\operatorname{rImm}_i\mathbf{A} = \sum_{\sigma\in S_n} \chi(\sigma) a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ is written as follows

$$\sigma = (i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}). \tag{3.1}$$

Here the index i starts the first cycle from the left and other cycles satisfy the following conditions

$$i_{k_2} < i_{k_3} < \ldots < i_{k_r}, \quad i_{k_t} < i_{k_t+s}. \tag{3.2}$$

for all $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Consequently we have the following definitions.

Definition 3.3. *The i-th row permanent of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\operatorname{rper}_i\mathbf{A} = \sum_{\sigma\in S_n} a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ satisfies the conditions (3.1) and (3.2).

Definition 3.4. *The i-th row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\operatorname{rdet}_i\mathbf{A} = \sum_{\sigma\in S_n} (-1)^{n-r} a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation σ satisfies the conditions (3.1) and (3.2), (since $\operatorname{sign}(\sigma) = (-1)^{n-r}$).

For $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ we define n column immanants as well.

Definition 3.5. *The j-th column immanant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{cImm}_j \mathbf{A} = \sum_{\tau \in S_n} \chi(\tau) a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}} \ldots a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows

$$\tau = \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right) \left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right). \tag{3.3}$$

Here the first cycle from the right begins with the index j and other cycles satisfy the following conditions

$$j_{k_2} < j_{k_3} < \ldots < j_{k_r}, \quad j_{k_t} < j_{k_t+s}, \tag{3.4}$$

for all $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Consequently we have the following definitions as well.

Definition 3.6. *The j-th column permanent of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{rper}_j \mathbf{A} = \sum_{\tau \in S_n} a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}} \ldots a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation σ satisfies the conditions (3.3) and (3.4).

Definition 3.7. *The j-th column determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined as*

$$\mathrm{rdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}} \ldots a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation σ satisfies the conditions (3.3) and (3.4).

Consider the basic properties of the column and row immanants over $\mathbb{H}$.

Proposition 3.8. *(The first theorem about zero of an immanant) If one of the rows (columns) of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ consists of zeros only, then $\mathrm{rImm}_i \mathbf{A} = 0$ and $\mathrm{cImm}_i \mathbf{A} = 0$ for all $i = \overline{1, n}$.*

Proof. The proof immediately follows from the definitions. □

Denote by $\mathbb{H}a$ and $a\mathbb{H}$ left and right principal ideals of $\mathbb{H}$, respectively.

Proposition 3.9. *(The second theorem about zero of an row immanant) Let $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ and $a_{ki} \in \mathbb{H}a_i$ and $a_{ij} \in \overline{a_i}\mathbb{H}$, where $n(a_i) = 0$ for $k, j = \overline{1, n}$ and for all $i \neq k$. Let $a_{11} \in \mathbb{H}a_1$ and $a_{22} \in \overline{a_1}\mathbb{H}$ if $k = 1$, and $a_{kk} \in \mathbb{H}a_k$ and $a_{11} \in \overline{a_k}\mathbb{H}$ if $k = i > 1$, where $n(a_k) = 0$. Then $\mathrm{rImm}_k \mathbf{A} = 0$.*

Proof. Let $i \neq k$. Consider an arbitrary monomial of $\mathrm{rImm}_k\mathbf{A}$, if $i \neq k$,

$$d = \chi(\sigma)a_{ki}a_{ij}\dots a_{lm}$$

where $\{l, m\} \subset \{1, ..., n\}$. Since there exists $a_i \in \mathbb{H}$ such that $n(a_i) = 0$, and $a_{ki} \in \mathbb{H}a_i$, $a_{ij} \in \overline{a_i}\mathbb{H}$, than $a_{ki}a_{ij} = 0$ and $d = 0$.

Let $i = k = 1$. Then an arbitrary monomial of $\mathrm{rImm}_1\mathbf{A}$,

$$d = \chi(\sigma)a_{11}a_{22}\dots a_{lm}.$$

Since there exists $a_1 \in \mathbb{H}$ such that $n(a_1) = 0$, and $a_{11} \in \mathbb{H}a_1$, $a_{22} \in \overline{a_1}\mathbb{H}$, then $a_{11}a_{22} = 0$ and $d = 0$.

If $k = i > 1$, then an arbitrary monomial of $\mathrm{rImm}_k\mathbf{A}$,

$$d = \chi(\sigma)a_{kk}a_{11}\dots a_{lm}.$$

Since there exists $a_k \in \mathbb{H}$ such that $n(a_k) = 0$, and $a_{kk} \in \mathbb{H}a_k$, $a_{11} \in \overline{a_k}\mathbb{H}$, then $a_{kk}a_{11} = 0$ and $d = 0$. □

Proposition 3.10. *(The second theorem about zero of an column immanant) Let* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *and* $a_{ik} \in a_i\mathbb{H}$ *and* $a_{ji} \in \mathbb{H}\overline{a_i}$*, where* $n(a_i) = 0$ *for* $k, j = \overline{1, n}$ *and for all* $i \neq k$*. Let* $a_{11} \in a_1\mathbb{H}$ *and* $a_{22} \in \mathbb{H}\overline{a_1}$ *if* $k = 1$*, and* $a_{kk} \in a_k\mathbb{H}$ *and* $a_{11} \in \mathbb{H}\overline{a_k}$ *if* $k = i > 1$*, where* $n(a_k) = 0$*. Then* $\mathrm{cImm}_k\mathbf{A} = 0$*.*

Proof. The proof is similar to the proof of the Proposition 3.9. □

The proofs of the next theorems immediately follow from the definitions.

Proposition 3.11. *If the i-th row of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is left-multiplied by* $b \in \mathbb{H}$*, then* $\mathrm{rImm}_i\, \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rImm}_i\, \mathbf{A}$ *for all* $i = \overline{1, n}$*.*

Proposition 3.12. *If the j-th column of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is right-multiplied by* $b \in \mathbb{H}$*, then* $\mathrm{cImm}_j\, \mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cImm}_j\, \mathbf{A} \cdot b$ *for all* $j = \overline{1, n}$*.*

Proposition 3.13. *If for* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in \{1, ..., n\}$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = \overline{1, n}$*, then for all* $i = \overline{1, n}$

$$\begin{aligned} \mathrm{rImm}_i\, \mathbf{A} &= \mathrm{rImm}_i\, \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rImm}_i\, \mathbf{A}_{t.}(\mathbf{c}), \\ \mathrm{cImm}_i\, \mathbf{A} &= \mathrm{cImm}_i\, \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cImm}_i\, \mathbf{A}_{t.}(\mathbf{c}), \end{aligned}$$

where $\mathbf{b} = (b_1, \dots, b_n)$*,* $\mathbf{c} = (c_1, \dots, c_n)$*.*

Proposition 3.14. *If for* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in \{1, ..., n\}$ *such that* $a_{i\,t} = b_i + c_i$ *for all* $i = \overline{1, n}$*, then for all* $j = \overline{1, n}$

$$\begin{aligned} \mathrm{rImm}_j\, \mathbf{A} &= \mathrm{rImm}_j\, \mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{rImm}_j\, \mathbf{A}_{.\,t}(\mathbf{c}), \\ \mathrm{cImm}_j\, \mathbf{A} &= \mathrm{cImm}_j\, \mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{cImm}_j\mathbf{A}_{.\,t}(\mathbf{c}), \end{aligned}$$

where $\mathbf{b} = (b_1, \dots, b_n)^T$*,* $\mathbf{c} = (c_1, \dots, c_n)^T$*.*

Proposition 3.15. *If* $\mathbf{A}^*$ *is the Hermitian adjoint matrix (conjugate and transpose) of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, then* $\mathrm{rImm}_i\, \mathbf{A}^* = \overline{\mathrm{cImm}_i\, \mathbf{A}}$ *for all* $i = \overline{1, n}$*.*

Particular cases of these properties for the row-column determinants and permanents are evident.

Remark 3.16. *The peculiarity of the column immanant (permanent, determinant) is that, at the direct calculation, factors of each of the monomials are written from right to left.* □

In Lemmas 3.17 and 3.18, we consider the recursive definition of the column and row determinants. This definition is an analogue of the expansion of a determinant along a row and a column in commutative case.

Lemma 3.17. *Let* $R_{i\,j}$ *be the right* ij*-th cofactor of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$*, namely*

$$\mathrm{rdet}_i\, \mathbf{A} = \sum_{j=1}^{n} a_{i\,j} \cdot R_{i\,j}$$

for all $i = \overline{1, n}$*. Then*

$$R_{i\,j} = \begin{cases} -\mathrm{rdet}_j\, (\mathbf{A}^{ii}_{.j}(\mathbf{a}_{.\,i})), & i \neq j \\ \mathrm{rdet}_k\, \mathbf{A}^{ii}, & i = j \end{cases}$$

$$k = \begin{cases} 2, & i = 1 \\ 1, & i > 1 \end{cases}$$

where the matrix $(\mathbf{A}^{ii}_{.j}(\mathbf{a}_{.\,i}))$ *is obtained from* $\mathbf{A}$ *by replacing its* j*-th column with the* i*-th column and then by deleting both the* i*-th row and column.* □

Lemma 3.18. *Let* $L_{i\,j}$ *be the left* ij*th cofactor of entry* $a_{i\,j}$ *of matrix* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$*, namely*

$$\mathrm{cdet}_j\, \mathbf{A} = \sum_{i=1}^{n} L_{i\,j} \cdot a_{i\,j}$$

for all $j = \overline{1, n}$*. Then*

$$L_{i\,j} = \begin{cases} -\mathrm{cdet}_i\, (\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j\,.})), & i \neq j \\ \mathrm{cdet}_k\, \mathbf{A}^{jj}, & i = j \end{cases}$$

$$k = \begin{cases} 2, & j = 1 \\ 1, & j > 1 \end{cases}$$

where the matrix $(\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j\,.}))$ *is obtained from* $\mathbf{A}$ *by replacing its* i*th row with the* j*th and then by deleting both the* j*th row and column.* □

Remark 3.19. *Clearly, an arbitrary monomial of each row or column determinant corresponds to a certain monomial of another row or column determinant such that both of them have the same sign, consist of the same factors and differ only in their ordering. If the entries of* $\mathbf{A}$ *are commutative, then* $\mathrm{rdet}_1\, \mathbf{A} = \ldots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1\, \mathbf{A} = \ldots = \mathrm{cdet}_n \mathbf{A}$. □

4. An Immanant of a Hermitian Matrix

If $\mathbf{A}^* = \mathbf{A}$ then $\mathbf{A} \in \mathbb{H}^{n\times n}$ is called a Hermitian matrix. In this section we consider the key theorem about row-column immanants of a Hermitian matrix.

The following lemma is needed for the sequel.

Lemma 4.1. *Let T_n be the sum of all possible products of n factors, each of their are either $h_i \in \mathbb{H}$ or $\overline{h_i}$ for all $i = \overline{1,n}$, by specifying the ordering in the terms, $T_n = h_1 \cdot h_2 \cdot \ldots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_n + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_n}$. Then T_n consists of the 2^n terms and $T_n = \mathrm{t}(h_1)\ \mathrm{t}(h_2)\ \ldots\ \mathrm{t}(h_n)$.*

Theorem 4.2. *If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is a Hermitian matrix, then*

$$\mathrm{rImm}_1\mathbf{A} = \ldots = \mathrm{rImm}_n\mathbf{A} = \mathrm{cImm}_1\mathbf{A} = \ldots = \mathrm{cImm}_n\mathbf{A} \in \mathbb{F}.$$

Proof. At first we note that if $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian, then we have $a_{ii} \in \mathbb{F}$ and $a_{ij} = \overline{a_{ji}}$ for all $i, j = \overline{1,n}$.

We divide the set of monomials of $\mathrm{rImm}_i\mathbf{A}$ for some $i \in \{1, ..., n\}$ into two subsets. If indices of coefficients of monomials form permutations as products of disjoint cycles of length 1 and 2, then we include these monomials to the first subset. Other monomials belong to the second subset. If indices of coefficients form a disjoint cycle of length 1, then these coefficients are a_{jj} for $j \in \{1, ..., n\}$ and $a_{jj} \in \mathbb{F}$.

If indices of coefficients form a disjoint cycle of length 2, then these entries are conjugated, $a_{i_k i_{k+1}} = \overline{a_{i_{k+1} i_k}}$, and

$$a_{i_k i_{k+1}} \cdot a_{i_{k+1} i_k} = \overline{a_{i_{k+1} i_k}} \cdot a_{i_{k+1} i_k} = \mathrm{n}(a_{i_{k+1} i_k}) \in \mathbb{F}.$$

So, all monomials of the first subset take on values in $\mathbb{F}$.

Now we consider some monomial d of the second subset. Assume that its index permutation σ forms a direct product of r disjoint cycles. Denote $i_{k_1} := i$, then

$$\begin{aligned} d = \chi(\sigma) a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i_{k_1}} a_{i_{k_2} i_{k_2+1}} \ldots a_{i_{k_2+l_2} i_{k_2}} \ldots a_{i_{k_m} i_{k_m+1}} \ldots \times \\ \times a_{i_{k_m+l_m} i_{k_m}} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}} = \chi(\sigma) h_1 h_2 \ldots h_m \ldots h_r, \end{aligned} \tag{4.1}$$

where $h_s = a_{i_{k_s} i_{k_s+1}} \cdot \ldots \cdot a_{i_{k_s+l_s} i_{k_s}}$ for all $s = \overline{1,r}$, and $m \in \{1, \ldots, r\}$. If $l_s = 1$, then $h_s = a_{i_{k_s} i_{k_s+1}} a_{i_{k_s+1} i_{k_s}} = \mathrm{n}(a_{i_{k_s} i_{k_s+1}}) \in \mathbb{F}$. If $l_s = 0$, then $h_s = a_{i_{k_s} i_{k_s}} \in \mathbb{F}$. If $l_s = 0$ or $l_s = 1$ for all $s = \overline{1,r}$ in (4.1), then d belongs to the first subset. Let there exists $s \in I_n$ such that $l_s \geq 2$. Then

$$\overline{h_s} = \overline{a_{i_{k_s} i_{k_s+1}} \ldots a_{i_{k_s+l_s} i_{k_s}}} = \overline{a_{i_{k_s+l_s} i_{k_s}}} \ldots \overline{a_{i_{k_s} i_{k_s+1}}} = a_{i_{k_s} i_{k_s+l_s}} \ldots a_{i_{k_s+1} i_{k_s}}.$$

Denote by $\sigma_s(i_{k_s}) := (i_{k_s} i_{k_s+1} \ldots i_{k_s+l_s})$ a disjoint cycle of indices of d for some $s \in \{1, ..., r\}$, then $\sigma = \sigma_1(i_{k_1})\, \sigma_2(i_{k_2}) \ldots \sigma_r(i_{k_r})$. The disjoint cycle $\sigma_s(i_{k_s})$ corresponds to the factor h_s. Then $\sigma_s^{-1}(i_{k_s}) = (i_{k_s} i_{k_s+l_s} i_{k_s+1} \ldots i_{k_s+1})$ is the inverse disjoint cycle and $\sigma_s^{-1}(i_{k_s})$ corresponds to the factor $\overline{h_s}$. By the Lemma 4.1, there exist another $2^p - 1$ monomials for d, (where $p = r - \rho$ and ρ is the number of disjoint cycles of length 1 and 2), such that their index permutations form the direct products of r disjoint cycles either $\sigma_s(i_{k_s})$

or $\sigma_s^{-1}(i_{k_s})$ by specifying their ordering by s from 1 to r. Their cycle notations are left-ordered according the to the Definition 3.2. These permutations are unique decomposition of the permutation σ including their ordering by s from 1 to r. Suppose C_1 is the sum of these $2^p - 1$ monomials and d, then, by the Lemma 4.1, we obtain

$$C_1 = \chi(\sigma)\alpha\, \mathrm{t}(h_{\nu_1})\ \ldots\ \mathrm{t}(h_{\nu_p}) \in \mathbb{F}.$$

Here $\alpha \in \mathbb{F}$ is the product of coefficients whose indices form disjoint cycles of length 1 and 2, $\nu_k \in \{1, \ldots, r\}$ for all $k = \overline{1, p}$.

Thus for an arbitrary monomial of the second subset of $\mathrm{rImm}_i\, \mathbf{A}$, we can find the 2^p monomials such that their sum takes on a value in $\mathbb{F}$. Therefore, $\mathrm{rImm}_i\, \mathbf{A} \in \mathbb{F}$.

Now we prove the equality of all row immanants of $\mathbf{A}$. Consider an arbitrary $\mathrm{rImm}_j\, \mathbf{A}$ such that $j \neq i$ for all $j = \overline{1, n}$. We divide the set of monomials of $\mathrm{rImm}_j\, \mathbf{A}$ into two subsets using the same rule as for $\mathrm{rImm}_i\, \mathbf{A}$. Monomials of the first subset are products of entries of the principal diagonal or norms of entries of $\mathbf{A}$. Therefore they take on a value in $\mathbb{F}$ and each monomial of the first subset of $\mathrm{rImm}_i\, \mathbf{A}$ is equal to a corresponding monomial of the first subset of $\mathrm{rImm}_j\, \mathbf{A}$.

Now consider the monomial d_1 of the second subset of monomials of $\mathrm{rImm}_j\, \mathbf{A}$ consisting of coefficients that are equal to the coefficients of d but they are in another order. Consider all possibilities of the arrangement of coefficients in d_1.

(i) Suppose that the index permutation σ' of its coefficients form a direct product of r disjoint cycles and these cycles coincide with the r disjoint cycles of d but differ by their ordering. Then $\sigma' = \sigma$ and we have

$$d_1 = \chi(\sigma)\alpha h_\mu \ldots h_\lambda,$$

where $\{\mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}$. By the Lemma 4.1, there exist $2^p - 1$ monomials of the second subset of $\mathrm{rImm}_j\, \mathbf{A}$ such that each of them is equal to a product of p factors either h_s or $\overline{h_s}$ for all $s \in \{\mu, \ldots, \lambda\}$. Hence by the Lemma 4.1, we obtain

$$C_2 = \chi(\sigma)\alpha\, t(h_\mu)\ \ldots\ t(h_\lambda) = \chi(\sigma)\ \alpha\ t(h_{\nu_1}) \ldots\ t(h_{\nu_p}) = C_1.$$

(ii) Now suppose that in addition to the case (i) the index j is placed inside some disjoint cycle of the index permutation σ of d, e.g., $j \in \{i_{k_m+1}, \ldots, i_{k_m+l_m}\}$. Denote $j = i_{k_m+q}$. Considering the above said and $\sigma_{k_m+1}(i_{k_m+1}) = \sigma_{k_m+q}(i_{k_m+q})$, we have $\sigma' = \sigma$. Then d_1 is represented as follows:

$$\begin{array}{c} d_1 = \chi(\sigma) a_{i_{k_m+q} i_{k_m+q+1}} \cdots \quad a_{i_{k_m+l_m} i_{k_m}}\, a_{i_{k_m} i_{k_m+1}} \ldots \times \\ \times a_{i_{k_m+q-1} i_{k_m+q}} a_{i_{k_\mu} i_{k_\mu+1}} \ldots a_{i_{k_\mu+l_\mu} i_{k_\mu}} \ldots a_{i_{k_\lambda} i_{k_\lambda+1}} \ldots a_{i_{k_\lambda+l_\lambda} i_{k_\lambda}} = \\ = \chi(\sigma)\alpha \tilde{h}_m h_\mu \ldots h_\lambda, \end{array} \tag{4.2}$$

where $\{m, \mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}$. Except for $\tilde{h}_m$, each factor of d_1 in (4.2) corresponds to the equal factor of d in (4.1). By the rearrangement property of the trace, we have $t(\tilde{h}_m) = t(h_m)$. Hence by the Lemma 4.1 and by analogy to the previous case, we obtain,

$$\begin{array}{c} C_2 = \chi(\sigma)\alpha\, t(\tilde{h}_m)\, t(h_\mu)\ \ldots\ t(h_\lambda) = \\ = \chi(\sigma)\ \alpha\ t(h_{\nu_1})\ \ldots\ t(h_m)\ \ldots\ t(h_{\nu_p}) = C_1. \end{array}$$

(iii) If in addition to the case (i) the index i is placed inside some disjoint cycles of the index permutation of d_1, then we apply the rearrangement property of the trace to this cycle. As in the previous cases we find 2^p monomials of the second subset of $\mathrm{rImm}_j\,\mathbf{A}$ such that by Lemma 4.1 their sum is equal to the sum of the corresponding 2^p monomials of $\mathrm{rImm}_i\mathbf{A}$. Clearly, we obtain the same conclusion at association of all previous cases, then we apply twice the rearrangement property of the trace.

Thus, in any case each sum of 2^p corresponding monomials of the second subset of $\mathrm{rImm}_j\,\mathbf{A}$ is equal to the sum of 2^p monomials of $\mathrm{rImm}_i\,\mathbf{A}$. Here p is the number of disjoint cycles of length more than 2. Therefore, for all $i, j = \overline{1, n}$ we have

$$\mathrm{rImm}_i\,\mathbf{A} = \mathrm{rImm}_j\,\mathbf{A} \in \mathbb{F}.$$

The equality $\mathrm{cImm}_i\,\mathbf{A} = \mathrm{rImm}_i\,\mathbf{A}$ for all $i = \overline{1, n}$ is proved similarly. □

Remark 4.3. *If $\mathbf{A} \in \mathbb{H}^{n\times n}$ is skew-hermitian ($\mathbf{A} = -\mathbf{A}^*$), then the Theorem 4.2 is not meaningful. It follows from the next example.*

Example 4.4. *Consider the following skew-hermitian matrix over the split quaternions of James Cockle $\mathbf{H_S}(\frac{-1,1}{\mathbb{R}})$,*

$$\mathbf{A} = \begin{pmatrix} j & 2+i \\ -2+i & -k \end{pmatrix}.$$

Since

$$\begin{aligned} \mathrm{rImm}_1\,\mathbf{A} &= -jk - (2+i)(-2+i) = 5+i, \\ \mathrm{rImm}_2\,\mathbf{A} &= -(-2+i)(2+i) - kj = 5-i, \end{aligned}$$

then $\mathrm{rImm}_1\,\mathbf{A} \neq \mathrm{rImm}_2\,\mathbf{A}$.

Since the Theorem 4.2, we have the following definition.

Definition 4.5. *Since all column and row immanants of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the immanant (permanent, determinant) of a Hermitian matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$. By definition, we put for all $i = \overline{1, n}$*

$$\begin{aligned} \mathrm{Imm}\,\mathbf{A} &:= \mathrm{rImm}_i\,\mathbf{A} = \mathrm{cImm}_i\,\mathbf{A}, \\ \mathrm{per}\,\mathbf{A} &:= \mathrm{rper}_i\,\mathbf{A} = \mathrm{cper}_i\,\mathbf{A}, \\ \det\mathbf{A} &:= \mathrm{rdet}_i\,\mathbf{A} = \mathrm{cdet}_i\,\mathbf{A}. \end{aligned}$$

4.1. Cramer's Rule for System of Linear Equations over a Quaternion Division Algebra

In this subsection we shall be consider $\mathbb{H}$ as a quaternion division algebra especially since quasideterminants are defined over the skew field as well.

Properties of the determinant of a Hermitian matrix is completely explored in [7, 8] by its row and column determinants. Among all, consider the following.

Theorem 4.6. *If the i-th row of the Hermitian matrix $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is replaced with a left linear combination of its other rows*

$$\mathbf{a}_{i\,.} = c_1\mathbf{a}_{i_1\,.} + \ldots + c_k\mathbf{a}_{i_k\,.}$$

where $c_l \in \mathbb{H}$ *for all* $l = \overline{1,k}$ *and* $\{i, i_l\} \subset \{1,\ldots,n\}$*, then for all* $i = \overline{1,n}$

$$\mathrm{cdet}_i \mathbf{A}_{i\,.}\left(c_1 \cdot \mathbf{a}_{i_1\,.} + \ldots + c_k \cdot \mathbf{a}_{i_k\,.}\right) = \mathrm{rdet}_i \mathbf{A}_{i\,.}\left(c_1 \cdot \mathbf{a}_{i_1\,.} + \ldots + c_k \cdot \mathbf{a}_{i_k\,.}\right) = 0.$$

Theorem 4.7. *If the* j*-th column of a Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n,\mathbb{H})$ *is replaced with a right linear combination of its other columns*

$$\mathbf{a}_{.\,j} = \mathbf{a}_{.\,j_1} c_1 + \ldots + \mathbf{a}_{.\,j_k} c_k$$

where $c_l \in \mathbb{H}$ *for all* $l = \overline{1,k}$ *and* $\{j, j_l\} \subset \{1,\ldots,n\}$ *, then for all* $j = \overline{1,n}$

$$\mathrm{cdet}_j \mathbf{A}_{.\,j}\left(\mathbf{a}_{.\,j_1} \cdot c_1 + \ldots + \mathbf{a}_{.\,j_k} \cdot c_k\right) = \mathrm{rdet}_j \mathbf{A}_{.\,j}\left(\mathbf{a}_{.\,j_1} \cdot c_1 + \ldots + \mathbf{a}_{.\,j_k} \cdot c_k\right) = 0.$$

The following theorem on the determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 4.8. *There exist a unique right inverse matrix* $(R\mathbf{A})^{-1}$ *and a unique left inverse matrix* $(L\mathbf{A})^{-1}$ *of a nonsingular Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n,\mathbb{H})$*,* $(\det \mathbf{A} \neq 0)$*, where* $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$*. Right inverse and left inverse matrices has following determinantal representation*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix},$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix},$$

where R_{ij}*,* L_{ij} *are right and left* ij*-th cofactors of* $\mathbf{A}$*, respectively, for all* $i, j = \overline{1,n}$*.*

To obtain the determinantal representation for an arbitrary inverse matrix over a quaternion division algebra $\mathbb{H}$, we consider the right $\mathbf{A}\mathbf{A}^*$ and left $\mathbf{A}^*\mathbf{A}$ corresponding Hermitian matrices.

Theorem 4.9 ([7])**.** *If an arbitrary column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is a right linear combination of its other columns, or an arbitrary row of* $\mathbf{A}^*$ *is a left linear combination of its other rows, then* $\det \mathbf{A}^*\mathbf{A} = 0$*.*

Since principal submatrices of a Hermitian matrix are also Hermitian, then the basis principal minor may be defined in this noncommutative case as a principal nonzero minor of a maximal order. We also can introduce the notion of the rank of a Hermitian matrix by principal minors, as a maximal order of a principal nonzero minor. The following theorem establishes the correspondence between the rank by principal minors of a Hermitian matrix and the rank of the corresponding matrix that are defined as a maximum number of right-linearly independent columns or left-linearly independent rows, which form a basis.

Theorem 4.10 ([7])**.** *A rank by principal minors of a Hermitian matrix* $\mathbf{A}^*\mathbf{A}$ *is equal to its rank and a rank of* $\mathbf{A} \in \mathbb{H}^{m\times n}$*.*

Theorem 4.11 ([7]). *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$, *then an arbitrary column of* $\mathbf{A}$ *is a right linear combination of its basic columns or arbitrary row of* $\mathbf{A}$ *is a left linear combination of its basic rows.*

It implies a criterion for the singularity of a corresponding Hermitian matrix.

Theorem 4.12 ([7]). *The right linearly independence of columns of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *or the left linearly independence of rows of* $\mathbf{A}^*$ *is the necessary and sufficient condition for*

$$\det \mathbf{A}^*\mathbf{A} \neq 0$$

Theorem 4.13 ([7]). *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *then* $\det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

In the following example, we shall prove the Theorem 4.13 for the case $n = 2$.

Example 4.14. *Consider the matrix* $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, *then* $\mathbf{A}^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$. *Respectively, we have*

$$\mathbf{A}\mathbf{A}^* = \begin{pmatrix} a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}} \end{pmatrix},$$

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} \overline{a_{11}}a_{11} + \overline{a_{21}}a_{21} & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{12}}a_{11} + \overline{a_{22}}a_{21} & \overline{a_{12}}a_{12} + \overline{a_{22}}a_{22} \end{pmatrix}.$$

According to thw Theorem 4.2 and the Definition 4.5, we have

$$\begin{aligned} \det \mathbf{A}\mathbf{A}^* &= \mathrm{rdet}_1 \mathbf{A}\mathbf{A}^*, \\ \det \mathbf{A}^*\mathbf{A} &= \mathrm{rdet}_1 \mathbf{A}^*\mathbf{A} \end{aligned}$$

According to the Lemma 3.17

$$\begin{aligned} \det \mathbf{A}\mathbf{A}^* &= (\mathbf{A}\mathbf{A}^*)_{11}(\mathbf{A}\mathbf{A}^*)_{22} - (\mathbf{A}\mathbf{A}^*)_{12}(\mathbf{A}\mathbf{A}^*)_{21} \\ &= (a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}})(a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}}) \\ &\quad -(a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}})(a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}}) \\ &= a_{11}\overline{a_{11}}a_{21}\overline{a_{21}} + a_{12}\overline{a_{12}}a_{21}\overline{a_{21}} \\ &\quad +a_{11}\overline{a_{11}}a_{22}\overline{a_{22}} + a_{12}\overline{a_{12}}a_{22}\overline{a_{22}} \\ &\quad -a_{11}\overline{a_{21}}a_{21}\overline{a_{11}} - a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} \\ &\quad -a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} - a_{12}\overline{a_{22}}a_{22}\overline{a_{12}} \\ &= a_{12}\overline{a_{12}}a_{21}\overline{a_{21}} + a_{11}\overline{a_{11}}a_{22}\overline{a_{22}} \\ &\quad -a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} - a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} \end{aligned}, \tag{4.3}$$

$$\begin{aligned} \det \mathbf{A}^*\mathbf{A} &= (\mathbf{A}^*\mathbf{A})_{11}(\mathbf{A}^*\mathbf{A})_{22} - (\mathbf{A}^*\mathbf{A})_{12}(\mathbf{A}^*\mathbf{A})_{21} \\ &= (\overline{a_{11}}a_{11} + \overline{a_{21}}a_{21})(\overline{a_{12}}a_{12} + \overline{a_{22}}a_{22}) \\ &\quad -(\overline{a_{11}}a_{12} + \overline{a_{21}}a_{22})(\overline{a_{12}}a_{11} + \overline{a_{22}}a_{21}) \\ &= \overline{a_{11}}a_{11}\overline{a_{12}}a_{12} + \overline{a_{21}}a_{21}\overline{a_{12}}a_{12} \\ &\quad +\overline{a_{11}}a_{11}\overline{a_{22}}a_{22} + \overline{a_{21}}a_{21}\overline{a_{22}}a_{22} \\ &\quad -\overline{a_{11}}a_{12}\overline{a_{12}}a_{11} - \overline{a_{21}}a_{22}\overline{a_{12}}a_{11} \\ &\quad -\overline{a_{11}}a_{12}\overline{a_{22}}a_{21} - \overline{a_{21}}a_{22}\overline{a_{22}}a_{21} \\ &= \overline{a_{21}}a_{21}\overline{a_{12}}a_{12} + \overline{a_{11}}a_{11}\overline{a_{22}}a_{22} \\ &\quad -\overline{a_{21}}a_{22}\overline{a_{12}}a_{11} - \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} \end{aligned}. \tag{4.4}$$

Positive terms in equations (4.3), (4.4) *are real numbers and they obviously coincide. To prove equation*

$$a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} = \overline{a_{21}}a_{22}\overline{a_{12}}a_{11} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} \tag{4.5}$$

we use the rearrangement property of the trace of elements of the quaternion algebra, $\mathrm{t}(pq) = \mathrm{t}(qp)$. *Indeed,*

$$a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + a_{11}\overline{a_{21}}a_{22}\overline{a_{12}} = a_{12}\overline{a_{22}}a_{21}\overline{a_{11}} + \overline{a_{12}\overline{a_{22}}a_{21}\overline{a_{11}}} = \mathrm{t}(a_{12}\overline{a_{22}}a_{21}\overline{a_{11}}),$$

$$\overline{a_{21}}a_{22}\overline{a_{12}}a_{11} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} = \overline{\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}} + \overline{a_{11}}a_{12}\overline{a_{22}}a_{21} = \mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21})$$

Then by the rearrangement property of the trace, we obtain (4.5).

According to the Theorem 4.13, we introduce the concept of double determinant. For the first time this concept was introduced by L. Chen ([18]).

Definition 4.15. *The determinant of corresponding Hermitian matrices is called the double determinant of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *i.e.,* $\mathrm{ddet}\mathbf{A} := \det(\mathbf{A}^*\mathbf{A}) = \det(\mathbf{A}\mathbf{A}^*)$.

If $\mathbb{H}$ is the Hamilton's quaternion skew field $\mathbf{H}$, then the following theorem establishes the validity of Axiom 1 for the double determinant.

Theorem 4.16. *If* $\{\mathbf{A}, \mathbf{B}\} \subset \mathrm{M}(n, \mathbf{H})$, *then* $\mathrm{ddet}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}$.

Unfortunately, if a non-Hermitian matrix is not full rank, then nothing can be said about singularity of its row and column determinant. We show it in the following example.

Example 4.17. *Consider the matrix*

$$\mathbf{A} = \begin{pmatrix} i & j \\ j & -i \end{pmatrix}.$$

Its second row is obtained from the first row by left-multiplying by k. *Then, by the Theorem 4.12,* $\mathrm{ddet}A = 0$. *Indeed,*

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} -i & -j \\ -j & i \end{pmatrix} \cdot \begin{pmatrix} i & j \\ j & -i \end{pmatrix} = \begin{pmatrix} 2 & -2k \\ 2k & 2 \end{pmatrix}.$$

Then $\mathrm{ddet}\mathbf{A} = 4 + 4k^2 = 0$. *However*

$$\mathrm{cdet}_1\mathbf{A} = \mathrm{cdet}_2\mathbf{A} = \mathrm{rdet}_1\mathbf{A} = \mathrm{rdet}_2\mathbf{A} = -i^2 - j^2 = 2.$$

At the same time $\mathrm{rank}\,\mathbf{A} = 1$, *that corresponds to the Theorem 4.10.*

The correspondence between the double determinant and the noncommutative determinants of Moore, Stady and Dieudonné are as follows,

$$\mathrm{ddet}\mathbf{A} = \mathrm{Mdet}(\mathbf{A}^*\mathbf{A}) = \mathrm{Sdet}\mathbf{A} = \mathrm{Ddet}^2\mathbf{A}.$$

Definition 4.18. *Let* $\mathrm{ddet}\mathbf{A} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A}) = \sum_i \mathbb{L}_{ij} \cdot a_{ij}$ *for* $j = \overline{1,n}$. *Then* $\mathbb{L}_{ij}$ *is called the left double* ij*-th cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$.

Definition 4.19. *Let* $\mathrm{ddet}\mathbf{A} = \mathrm{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right) = \sum_j a_{ij} \cdot \mathbb{R}_{ij}$ *for* $i = \overline{1,n}$. *Then* $\mathbb{R}_{ij}$ *is called the right double* ij*-th cofactor of* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$.

Theorem 4.20. *The necessary and sufficient condition of invertibility of a matrix* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is* $\mathrm{ddet}\mathbf{A} \neq 0$. *Then* $\exists \mathbf{A}^{-1} = (L\mathbf{A})^{-1} = (R\mathbf{A})^{-1}$, *where*

$$(L\mathbf{A})^{-1} = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^* = \frac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \ldots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \ldots & \mathbb{L}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \ldots & \mathbb{L}_{nn} \end{pmatrix} \tag{4.6}$$

$$(R\mathbf{A})^{-1} = \mathbf{A}^*\left(\mathbf{A}\mathbf{A}^*\right)^{-1} = \frac{1}{\mathrm{ddet}\mathbf{A}^*}\begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \ldots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \ldots & \mathbb{R}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \ldots & \mathbb{R}_{nn} \end{pmatrix} \tag{4.7}$$

and $\mathbb{L}_{ij} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{a}^*_{.i}\right)$, $\mathbb{R}_{ij} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\left(\mathbf{a}^*_{j.}\right)$ *for all* $i, j = \overline{1,n}$.

Remark 4.21. *In the Theorem 4.20, the inverse matrix* $\mathbf{A}^{-1}$ *of an arbitrary matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *under the assumption of* $\mathrm{ddet}\mathbf{A} \neq 0$ *is represented by the analog of the classical adjoint matrix. If we denote this analog of the adjoint matrix over* $\mathbb{H}$ *by* $\mathrm{Adj}[[\mathbf{A}]]$, *then the next formula is valid over* $\mathbb{H}$*:*

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}[[\mathbf{A}]]}{\mathrm{ddet}\mathbf{A}}.$$

An obvious consequence of a determinantal representation of the inverse matrix by the classical adjoint matrix is Cramer's rule.

Theorem 4.22. *Let*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{4.8}$$

be a right system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *a column of constants* $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{H}^{n\times 1}$, *and a column of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)^T$. *If* $\mathrm{ddet}\mathbf{A} \neq 0$, *then* (4.8) *has a unique solution that has represented as follows,*

$$x_j = \frac{\mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{f}\right)}{\mathrm{ddet}\mathbf{A}}, \qquad \forall j = \overline{1,n} \tag{4.9}$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{y}$.

Theorem 4.23. *Let*

$$\mathbf{x} \cdot \mathbf{A} = \mathbf{y} \tag{4.10}$$

be a left system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *a column of constants* $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{H}^{1\times n}$ *and a column of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)$. *If* $\mathrm{ddet}\mathbf{A} \neq 0$, *then* (4.10) *has a unique solution that has represented as follows,*

$$x_i = \frac{\mathrm{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right)_{i.}\left(\mathbf{z}\right)}{\mathrm{ddet}\mathbf{A}}, \qquad \forall i = \overline{1,n} \tag{4.11}$$

where $\mathbf{z} = \mathbf{y}\mathbf{A}^*$.

Equations (4.9) and (4.11) are the obvious and natural generalizations of Cramer's rule for systems of linear equations over a quaternion division algebra. As follows from the Theorem 4.8, the closer analog to Cramer's rule can be obtained in the following specific cases.

Theorem 4.24. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian in* (4.8)*. Then the solution of* (4.8) *has represented by the equation,*

$$x_j = \frac{\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{y})}{\det \mathbf{A}}, \quad \forall j = \overline{1, n}.$$

Theorem 4.25. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian in* (4.10)*. Then the solution of* (4.10) *has represented as follows,*

$$x_i = \frac{\mathrm{rdet}_i \mathbf{A}_{i.}(\mathbf{y})}{\det \mathbf{A}}, \quad \forall i = \overline{1, n}.$$

An application of the column-row determinants in the theory of generalized inverse matrices over the quaternion skew field recently has been received in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

5. Quasideterminants over a Quaternion Division Algebra

Theorem 5.1. *Suppose a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

with entries from a quaternion division algebra has an inverse $\mathbf{A}^{-1}$.[2] *Then a minor of the inverse matrix satisfies the following equation, provided that the inverse matrices exist*

$$((\mathbf{A}^{-1})_{IJ})^{-1} = \mathbf{A}_{JI} - \mathbf{A}_{J.}^{.I}(\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.} \tag{5.1}$$

Proof. Definition of an inverse matrix leads to the system of linear equations

$$\mathbf{A}^{JI}(\mathbf{A}^{-1})_{.J}^{I.} + \mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = 0 \tag{5.2}$$

$$\mathbf{A}_{J.}^{.I}(\mathbf{A}^{-1})_{.J}^{I.} + \mathbf{A}_{JI}(\mathbf{A}^{-1})_{IJ} = \mathbf{I} \tag{5.3}$$

where $\mathbf{I}$ is a unit matrix. We multiply (5.2) by $\left(\mathbf{A}^{JI}\right)^{-1}$

$$(\mathbf{A}^{-1})_{.J}^{I.} + (\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = 0 \tag{5.4}$$

Now we can substitute (5.4) into (5.3)

$$\mathbf{A}_{JI}(\mathbf{A}^{-1})_{IJ} - \mathbf{A}_{J.}^{.I}(\mathbf{A}^{JI})^{-1}\mathbf{A}_{.I}^{J.}(\mathbf{A}^{-1})_{IJ} = \mathbf{I} \tag{5.5}$$

(5.1) follows from (5.5). □

[2] This statement and its proof are based on statement 1.2.1 from [17] (page 8) for matrix over free division ring.

Corollary 5.2. *Suppose a matrix* $\mathbf{A}$ *has the inverse matrix. Then elements of the inverse matrix satisfy to the equation*

$$((\mathbf{A}^{-1})_{ij})^{-1} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}(\mathbf{A}^{ji})^{-1}\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.6}$$

Example 5.3. *Consider a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

According to (5.6)

$$(\mathbf{A}^{-1})_{11} = (a_{11} - a_{12}(a_{22})^{-1}\ a_{21})^{-1} \tag{5.7}$$
$$(\mathbf{A}^{-1})_{21} = (a_{21} - a_{22}(a_{12})^{-1}\ a_{11})^{-1} \tag{5.8}$$
$$(\mathbf{A}^{-1})_{12} = (a_{12} - a_{11}(a_{21})^{-1}\ a_{22})^{-1} \tag{5.9}$$
$$(\mathbf{A}^{-1})_{22} = (a_{22} - a_{21}(a_{11})^{-1}\ a_{12})^{-1} \tag{5.10}$$

We call a matrix

$$\mathcal{H}\mathbf{A} = ((\mathcal{H}\mathbf{A})_{ij}) = ((a_{ji})^{-1}) \tag{5.11}$$

a Hadamard inverse of[3] $\mathbf{A}$.

Definition 5.4. *The* (ji)*-quasideterminant of* $\mathbf{A}$ *is formal expression*

$$|\mathbf{A}|_{ji} = (\mathcal{H}\mathbf{A}^{-1})_{ji} = ((\mathbf{A}^{-1})_{ij})^{-1} \tag{5.12}$$

We consider the (ji)*-quasideterminant as an element of the matrix* $|\mathbf{A}|$*, which is called a quasideterminant.*

Theorem 5.5. *Expression for the* (ji)*-quasideterminant has form*

$$|\mathbf{A}|_{ji} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}(\mathbf{A}^{ji})^{-1}\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.13}$$
$$|\mathbf{A}|_{ji} = a_{ji} - \mathbf{A}_{j\cdot}^{\cdot i}\,\mathcal{H}|\mathbf{A}^{ji}|\,\mathbf{A}_{\cdot i}^{j\cdot} \tag{5.14}$$

Proof. The statement follows from (5.6) and (5.12). □

Example 5.6. *Let*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.15}$$

It is clear from (5.7) *and* (5.10) *that* $(\mathbf{A}^{-1})_{11} = 1$ *and* $(\mathbf{A}^{-1})_{22} = 1$. *However expression for* $(\mathbf{A}^{-1})_{21}$ *and* $(\mathbf{A}^{-1})_{12}$ *cannot be defined from* (5.8) *and* (5.9) *since* $(a_{21} - a_{22}(a_{12})^{-1}\ a_{11})^{-1} = (a_{12} - a_{11}(a_{21})^{-1}\ a_{22})^{-1} = 0$. *We can transform these expressions. For instance*

$$\begin{aligned}(\mathbf{A}^{-1})_{21} &= (a_{21} - a_{22}(a_{12})^{-1}\ a_{11})^{-1} \\ &= (a_{11}((a_{11})^{-1}\ a_{12} - (a_{21})^{-1}\ a_{22}))^{-1} \\ &= ((a_{21})^{-1}\ a_{11}(a_{21}(a_{11})^{-1}\ a_{12} - a_{22}))^{-1} \\ &= (a_{11}(a_{21}(a_{11})^{-1}\ a_{12} - a_{22}))^{-1}\ a_{21}\end{aligned}$$

[3]See also page 4 in paper [16].

It follows immediately that $(\mathbf{A}^{-1})_{21} = 0$. *In the same manner we can find that* $(\mathbf{A}^{-1})_{12} = 0$. *Therefore,*

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.16}$$

□

From the Example 5.6 we see that we cannot always use Equation (5.6) to find elements of the inverse matrix and we need more transformations to solve this problem. From the theorem 4.6.3 in the paper [9], it follows that if

$$\operatorname{rank} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \le n-2$$

then $|\mathbf{A}|_{ij}$ for all $i, j = \overline{1, n}$ is not defined. From this, it follows that although a quasideterminant is a powerful tool, use of a determinant is a major advantage.

Theorem 5.7. *Let a matrix* $\mathbf{A}$ *have an inverse. Then for any matrices* $\mathbf{B}$ *and* $\mathbf{C}$ *equation*

$$\mathbf{B} = \mathbf{C} \tag{5.17}$$

follows from the equation

$$\mathbf{BA} = \mathbf{CA} \tag{5.18}$$

Proof. Equation (5.17) follows from (5.18) if we multiply both parts of (5.18) over $\mathbf{A}^{-1}$. □

Theorem 5.8. *The solution of a nonsingular system of linear equations*

$$\mathbf{A}x = b \tag{5.19}$$

is determined uniquely and can be presented in either form[4]

$$x = \mathbf{A}^{-1} b \tag{5.20}$$

$$x = \mathcal{H}|\mathbf{A}|\, b \tag{5.21}$$

Proof. Multiplying both sides of (5.19) from left by $\mathbf{A}^{-1}$ we get (5.20). Using the Definition 5.4, we get (5.21). Since the Theorem 5.7, the solution is unique. □

6. Relation of Row-Column Determinants with Quasideterminants

Theorem 6.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is an invertible matrix, then, for arbitrary* $p, q = \overline{1, n}$, *we have the following representation of the pq-quasideterminant*

$$|\ \mathbf{A}\ |_{pq} = \frac{\operatorname{ddet}\mathbf{A} \cdot \overline{\operatorname{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right)}}{\operatorname{n}(\operatorname{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right))}, \tag{6.1}$$

[4]See similar statement in the theorem 1.6.1 in the paper [17] on pagen 19.

$$| \mathbf{A} |_{pq} = \frac{\mathrm{ddet}\mathbf{A} \cdot \overline{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{a}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{a}^*_{q.}\right))}. \tag{6.2}$$

Proof. Let $\mathbf{A}^{-1} = (b_{ij})$ to $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. Equation (5.12) reveals the relationship between a quasideterminant $| \mathbf{A} |_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ and elements of the inverse matrix $\mathbf{A}^{-1} = (b_{ij})$, namely

$$| \mathbf{A} |_{pq} = b_{qp}^{-1}$$

for all $p, q = \overline{1, n}$. At the same time, the theory of row and column determinants (the theorem 4.20) gives us representation of the inverse matrix through its left (4.6) and right (4.7) double cofactors. Thus, accordingly, we obtain

$$| \mathbf{A} |_{pq} = b_{qp}^{-1} = \left(\frac{\mathbb{L}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}, \tag{6.3}$$

$$| \mathbf{A} |_{pq} = b_{qp}^{-1} = \left(\frac{\mathbb{R}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}. \tag{6.4}$$

Since $\mathrm{ddet}\mathbf{A} \neq 0 \in \mathbb{F}$, then $\exists (\mathrm{ddet}\mathbf{A})^{-1} \in \mathbb{F}$. It follows that

$$\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)^{-1} = \frac{\overline{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}}{\mathrm{n}(\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right))}, \tag{6.5}$$

$$\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)^{-1} = \frac{\overline{\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{AA}^*)_{p.}\left(\mathbf{A}^*_{q.}\right))}. \tag{6.6}$$

Substituting (6.5) into (6.3), and (6.6) into (6.4), we accordingly obtain (6.1) and (6.2).

We proved the theorem. □

Equation (6.1) gives an explicit representation of a quasideterminant $| \mathbf{A} |_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ for all $p, q = \overline{1, n}$ by the column determinant of its corresponding left Hermitian matrix $\mathbf{A}^*\mathbf{A}$, and (6.2) does by the row determinant of its corresponding right Hermitian matrix $\mathbf{AA}^*$.

Example 6.2. *Consider a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

According to (5.13)

$$|\mathbf{A}| = \begin{pmatrix} a_{11} - a_{12}(a_{22})^{-1}\, a_{21} & a_{12} - a_{11}(a_{21})^{-1}\, a_{22} \\ a_{21} - a_{22}(a_{12})^{-1}\, a_{11} & a_{22} - a_{21}(a_{11})^{-1}\, a_{12} \end{pmatrix} \tag{6.7}$$

Our goal is to find this quasideterminant using the Theorem 6.1. It is evident that

$$\mathbf{A}^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} \quad \mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathrm{n}(a_{11}) + \mathrm{n}(a_{21}) & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{12}}a_{11} + \overline{a_{22}}a_{21} & \mathrm{n}(a_{12}) + \mathrm{n}(a_{22}) \end{pmatrix}.$$

Calculate the necessary determinants

$$\begin{aligned}
\mathrm{ddet}\mathbf{A} &= \mathrm{rdet}_1(\mathbf{A}^*\mathbf{A})\\
&= (\mathrm{n}(a_{11})+\mathrm{n}(a_{21}))\cdot(\mathrm{n}(a_{12})+\mathrm{n}(a_{22}))\\
&-(\overline{a_{11}}a_{12}+\overline{a_{21}}a_{22})\cdot(\overline{a_{12}}a_{11}+\overline{a_{22}}a_{21})\\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{12})+\mathrm{n}(a_{11})\mathrm{n}(a_{22})+\mathrm{n}(a_{21})\mathrm{n}(a_{12})+\mathrm{n}(a_{21})\mathrm{n}(a_{22})\\
&-\overline{a_{11}}a_{12}\overline{a_{12}}a_{11}-\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}-\overline{a_{21}}a_{22}\overline{a_{12}}a_{11}-\overline{a_{21}}a_{22}\overline{a_{22}}a_{21}\\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{22})+\mathrm{n}(a_{21})\mathrm{n}(a_{12})-(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}+\overline{\overline{a_{11}}a_{12}\overline{a_{22}}a_{21}})\\
&= \mathrm{n}(a_{11})\mathrm{n}(a_{22})+\mathrm{n}(a_{21})\mathrm{n}(a_{12})-\mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21})
\end{aligned}$$

$$\begin{aligned}
\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2}) &= \mathrm{cdet}_1\begin{pmatrix}\overline{a_{21}} & \overline{a_{11}}a_{12}+\overline{a_{21}}a_{22}\\ \overline{a_{22}} & \mathrm{n}(a_{12})+\mathrm{n}(a_{22})\end{pmatrix}\\
&= \mathrm{n}(a_{12})\overline{a_{21}}+\mathrm{n}(a_{22})\overline{a_{21}}-\overline{a_{11}}a_{12}\overline{a_{22}}-\overline{a_{21}}a_{22}\overline{a_{22}}\\
&= \mathrm{n}(a_{12})\overline{a_{21}}-\overline{a_{11}}a_{12}\overline{a_{22}}.
\end{aligned}$$

Then

$$\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})} = \mathrm{n}(a_{12})a_{21}-a_{22}\overline{a_{12}}a_{11},$$

$$\begin{aligned}
\mathrm{n}(\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})) &= \overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})}\cdot\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})\\
&= (\mathrm{n}(a_{12})a_{21}-a_{22}\overline{a_{12}}a_{11})\cdot(\mathrm{n}(a_{12})\overline{a_{21}}-\overline{a_{11}}a_{12}\overline{a_{22}})\\
&= \mathrm{n}^2(a_{12})\mathrm{n}(a_{21})-\mathrm{n}(a_{12})a_{21}\overline{a_{11}}a_{12}\overline{a_{22}}\\
&-\mathrm{n}(a_{12})a_{22}\overline{a_{12}}a_{11}\overline{a_{21}}+a_{22}\overline{a_{12}}a_{11}\overline{a_{11}}a_{12}\overline{a_{22}}\\
&= \mathrm{n}(a_{12})(\mathrm{n}(a_{12})\mathrm{n}(a_{21})-\mathrm{t}(\overline{a_{11}}a_{12}\overline{a_{22}}a_{21})+\mathrm{n}(a_{21})\mathrm{n}(a_{12}))\\
&= \mathrm{n}(a_{12})\mathrm{ddet}\mathbf{A}.
\end{aligned}$$

Following (6.1)*, we obtain*

$$\begin{aligned}
|\mathbf{A}|_{21} &= \frac{\mathrm{ddet}\mathbf{A}}{\mathrm{n}(\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2}))}\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})}\\
&= \frac{\mathrm{ddet}\mathbf{A}}{\mathrm{n}(a_{12})\mathrm{ddet}\mathbf{A}}\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})}\\
&= \frac{1}{\mathrm{n}(a_{12})}\cdot\overline{\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.2})}\\
&= \frac{1}{\mathrm{n}(a_{12})}\cdot(\mathrm{n}(a_{12})a_{21}-a_{22}\overline{a_{12}}a_{11})\\
&= a_{21}-a_{22}(a_{12})^{-1}a_{11}.
\end{aligned} \tag{6.8}$$

The last expression in (6.8) *coincides with the expression* $|\mathbf{A}|_{21}$ *in* (6.7). □

7. Conclusion

In the chapter we consider two approaches to define a noncommutative determinant, row-column determinants and quasideterminants. These approaches of studying of a matrix with entryes from non commutative division ring have their own field of applications.

The theory of the row and column determinants as an extension of the classical definition of determinant has been elaborated for matrices over a quaternion division algebra. It has applications in the theories of matrix equations and of generalized inverse matrices over the quaternion skew field. Now it is in development for matrices over a split quaternion algebra. In the chapter we have extended the concepts of an immanant, a permanent and a determinant to a split quaternion algebra and have established their basic properties.

Quasideterminants of Gelfand-Retax are rational matrix functions that requires the invertibility of certain submatrices. Now they are widely used. Though we can use quasideterminant in any division ring,[5] row-column determinant is more attractive to find solution of system of linear equations when division ring has conjugation.

In the chapter we have derived relations of row-column determinants with quasideterminants of a matrix over a quaternion division algebra. The use of equations (6.1) and (6.2) allows us direct calculation of quasideterminants. It already gives significance in establishing these relations.

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