Mathematics Research Developments

Quaternions

Theory and Applications

Sandra Griffin
Editor

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SANDRA GRIFFIN
EDITOR

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NOTICE TO THE READER

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PREFACE

This book focuses on the theory and applications of quaternions. Chapter One collects some old problems on lattice orders and directed partial orders on complex numbers and quaternions, and summarizes recent development in answering those questions. Chapter Two discusses spin 1 particles with anomalous magnetic moments in the external uniform electric field. Chapter Three examines techniques of projective operators used to construct solutions for a spin 1 particle with anomalous magnetic moment in the external uniform magnetic field. Chapter Four analyzes the implementation of a cheap Micro AHRS (Attitude and Heading Reference System) using low-cost inertial sensors. Chapter Five reviews the basic concepts of quaternion and reduced biquaternions algebra. It introduces the 2D Hermite-Gaussian functions (2D-HGF) as the eigenfunction of discrete quaternion Fourier transform (DQFT) and discrete reduced biquaternion Fourier transform (DRBQFT), and the eigenvalues of two dimensional Hermite-Gaussian functions for three types of DQFT and two types of DRBQFT. Chapter Six investigates a leader-follower formation control problem of quadrotors. Chapter Seven considers determinantal representations the Drazin and weighted Drazin inverses over the quaternion skew field.

Chapter 1 collects some old problems on lattice orders and directed partial orders on complex numbers and quaternions, and summarizes recent development in answering those questions. Within the matrix 10-dimensional Duffin-Kemmer-Petiau formalism applied to the Shamaly-Capri field, Chapter 2 studies the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components Ψ_0, Ψ_+, Ψ_+. It is enough to solve the equation for the main component Φ_0, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein-Fock-Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

Chapter 2 - Within *the matrix 10-dimensional Duffin-Kemmer-Petiau formalism* applied to *the Shamaly-Capri field*, Chapter 3 studies the behavior of a vector particle with anomalous magnetic moment in presence of an *external uniform magnetic field*. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components $\Psi_0, \Psi_{+}, \Psi_{+}$. It is enough to solve the equation for the main component Φ_0, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein--Fock--Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

Chapter 3 - The separation of variables in the wave equation is performed using projective operator techniques and the theory of DKP-algebras. The problem is reduced to a system of 2-nd order differential equations for three independent functions, which is solved in terms of confluent hypergeometric functions. Three series of energy levels are found, of which two substantially differ from those for spin 1 particles without anomalous magnetic moment. For assigning to them physical sense for all the values of the main quantum number $n = 0,1,2, \ldots,$ one must impose special restrictions on a parameter related to the anomalous moment. Otherwise, only some part of the energy levels corresponds to bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in the occurrence of a space scaling of the arguments of the wave functions, compared to a particle without such a moment. Traditionally, the automotive industry has been the largest employer of robots, but their control is inline and programmed to follow planning trajectories. As shown in Chapter 4, in this case, in the department motor’s test of Volkswagen Mexico a semi-autonomous robot is developed. To date, some critical technical problems must be solved in a number of areas, including in dynamics control. Generally, the attitude estimation and the measurement of the angular velocity are a requirements for the attitude control. As a result, the computational cost and the complexity of the control loop is relatively high.

Chapter 4 deals with the implementation of a cheap Micro AHRS (Attitude and Heading Reference System) using low-cost inertial sensors. In Chapter 4, the technique proposed is designed with attitude estimation and the prediction movement via the kinematic of a 4GDL robot. With this approach, only the measurements of at least two non-collinear directional sensors are needed. Since the control laws are highly simple and a model-based observer for angular velocity reconstruction is not needed, the proposed new strategy is very suitable for embedded implementations. The global convergence of the estimation and prediction techniques is proved. Simulation with some robustness tests is performed.

Chapter 5 - The quaternions, reduced biquaternions (RBs) and their respective Fourier transformations, i.e., discrete quaternion Fourier transform (DQFT) and discrete reduced biquaternion Fourier transform (DRBQFT), are very useful for multi-dimensional signal processing and analysis. In Chapter 5, the basic concepts of quaternion and RB algebra are reviewed, and the author introduce the two dimensional Hermite-Gaussian functions (2D-HGF) as the eigenfunction of DQFT/DRBQFT, and the eigenvalues of 2D-HGF for three types of DQFT and two types of DRBQFT. After that, the relation between 2D-HGF and Gauss-Laguerre circular harmonic function (GLCHF) is given. From the aforementioned relation and some derivations, the GLCHF can be proved as the eigenfunction of DQFT/DRBQFT and its eigenvalues are summarized. These GLCHFs can be used as the basis to perform color image expansion. The expansion coefficients can be used to reconstruct the original color image and as a rotation invariant feature. The GLCHFs can also be applied to color matching applications.

Chapter 6 - The unit quaternion system was invented in 1843 by William Rowan Hamilton as an extension to the complex number to find an answer to the question (how to multiply triplets?). Yet, quaternions are extensively used to represent the attitude of a rigid body such as quadrotors, which is able to alleviate the singularity problem caused by the Euler angles representation. The singularity is in general a point at which a given mathematical object is not defined and it outcome of the so called gimbal lock. The singularity is occur when the pitch angles rotation is $\theta = \pm 90^\circ$. In Chapter 6, a leader-follower formation control problem of quadrotors is investigated. The quadrotor dynamic model is represented by unit quaternion with the consideration of external disturbance. Three different control techniques are proposed for both the leader and the follower robots. First, a nonlinear H_∞ design approach is derived by solving a Hamilton-Jacobi inequality following from a result for general nonlinear affine systems. Second, integral backstepping (IBS) controllers are also addressed for the leader-follower formation control problem. Then, an iterative Linear Quadratic Regulator (iLQR) is derived to solve the problem of leader-follower formation. The simulation results from all types of controllers are compared and robustness performance of the H_∞ controllers, fast convergence and small tracking errors of iLQR controllers over the IBS controllers are demonstrated.

Chapter 7 - A generalized inverse of a given quaternion matrix (similarly, as for complex matrices) exists for a larger class of matrices than the invertible matrices. It has some of the properties of the usual inverse, and agrees with the inverse when a given matrix happens to be invertible. There exist many different generalized inverses. The authors consider determinantal representations of the Drazin and weighted Drazin inverses over the quaternion skew field. Due to the theory of column-row determinants recently introduced by the author, the authors derive determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field. Using obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathrm{AXB} = \mathrm{D}$ and, consequently, $\mathrm{AX} = \mathrm{D}$, $\mathrm{XB} = \mathrm{D}$ in both cases when A and B are Hermitian and arbitrary, where A, B can be noninvertible matrices of appropriate sizes. The author obtain determinantal representations of solutions of the differential quaternionic matrix equations, $X' + AX = B$ and $X' + XA = B$, where A is noninvertible as well. Also, the authors obtains new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. The author give determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, the authors derive explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $WAWX = D$, $XWAW = D$, and $W_1AW_1XW_2BW_2 = D$.

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Chapter 1

DIRECTED PARTIAL ORDERS ON QUATERNIONS - A BRIEF SUMMARY

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Abstract

This paper collects some old problems on lattice orders and directed partial orders on complex numbers and quaternions, and summarizes recent development in answering those questions.

Keywords: directed partial order, directed algebra, lattice order, ℓ-algebra, complex number, quaternion

2010 AMS Subject Classification: 06F25

1. Introduction

We will introduce some definitions and terminologies in this section. The reader is referred to [2, 3, 5] for more information on partially ordered rings and lattice-ordered rings (ℓ-rings).

Let R be a partially ordered ring. The positive cone of R is defined as $R^+ = \{r \in R \mid r \geq 0\}$. The positive cone R^+ is closed under the addition

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and multiplication of R and satisfies $R^+ \cap -R^+ = \{0\}$. Let P be a subset of a ring S that is closed under the addition and multiplication of S and satisfies $P \cap -P = \{0\}$. Define the partial order $\leq$ by for any $a, b \in S$, $a \leq b$ if $b - a \in P$. Then S is a partially ordered ring with respect to the partial order $\leq$. We often use the positive cone to denote a partial order on a partially ordered ring. A partial order is called *directed* if each element is a difference of two positive elements. A partially ordered ring is called a *lattice-ordered ring* (ℓ-ring) if the partial order is a lattice order. Clearly a lattice order is directed, but the converse is not true. Let T be a commutative totally ordered ring with the identity and A be an algebra over T. If A is a partially ordered ring and $T^+A^+ \subseteq A^+$, then A is called a partially ordered algebra over T. If the partial order on A is directed, then A is called a *directed algebra*, and if the partial order on A is a lattice order, then A is called a *lattice-ordered algebra* (ℓ-algebra).

Let D be a totally ordered integral domain, that is, D is a commutative totally ordered ring with the identity and without nonzero zero divisors. For $x, y \in D$, $x \ll y$ denotes that $nx \leq y$ for all positive integers n. Let T be a commutative totally ordered ring with the identity element 1. The complex numbers over T is defined as

$$C_T = \{a + bi \mid a, b \in T, i^2 = -1\},$$

and the quaternions over T is defined as

$$H_T = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in T, i^2 = j^2 = k^2 = -1\}.$$

The multiplication of H_T is given as follows,

$$\begin{aligned}
&(a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) = \\
&(a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i + \\
&(a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) j + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) k.
\end{aligned}$$

The following questions have been left unanswered for some time now, however they have greatly motivated research activities in the area.

- **Problem 1. (G. Birkhoff and R. Pierce, 1956)** Can the field of complex numbers be made into a lattice-ordered ring?

- **Problem 2. (L. Fuchs, 1963)** Describe the directed orders of the fields of complex numbers and quaternions.

- **Problem 3. (G. Birkhoff, 1967)** In how many ways can the quaternions be made into an ℓ-ring? an ℓ-algebra? a directed algebra?

We will summarize below recent developments in finding lattice orders and directed partial orders on quaternion algebras. Since this activity is closely related to and motivated by the same research for complex numbers, results for complex numbers are also included.

2. Directed Partial Orders on C_F

In this section, we present directed partial orders on C_F, where F is a totally ordered field. We start with lattice orders first. Lattice-ordered rings (ℓ-rings) were first systematically studied by G. Birkhoff and R. Pierce in the paper "Lattice-ordered Rings" published in 1956 [2]. Problem 1 was asked in the paper. Despite many efforts made over years, this problem remains unsolved.

In the same paper, the authors proved that the complex field $\mathbb{C}$ cannot be made into a lattice-ordered algebra (ℓ-algebra) over the real field $\mathbb{R}$. About 40 years later, motivated by the work on lattice orders of matrix algebras over totally ordered fields, the present author further proved that for any totally ordered subfield F of $\mathbb{R}$ with the usual total order, $M_n(C_F)$ cannot be made into an ℓ-algebra over F for any $n \geq 1$ [6], where $M_n(C_F)$ is the $n \times n$ matrix algebra with entries from C_F.

More generally, we have following result.

Theorem 1. $[8, Theorem\ 6]$ *Let D be a totally ordered integral domain. Suppose that C_D is an ℓ-algebra over D. If $a + bi \geq 0$ in C_D, then $a \geq 0$ and $|b| \ll a$ in D.*

A direct consequence of Theorem 1 is that if D is an archimedean totally ordered integral domain, then C_D cannot be an ℓ-algebra over D.

A natural question to ask is what happens in the non-archimedean case.

Theorem 2. $[8, Theorem\ 4]$ *Let F be a totally ordered field, archimedean or non-archimedean. C_F cannot be made into an ℓ-algebra over F.*

Now, let's consider directed partial orders on the set of complex numbers to make it into a directed algebra. Since it has been unsuccessful of finding lattice orders on complex numbers, researchers have tried to find directed partial orders on it. The first result states that there is no directed partial order on C_F when F is an archimedean totally ordered field.

Theorem 3. $[10, Corollary\ 2.2]$ *C_F cannot be made into a directed algebra over an archimedean totally ordered field F.*

In [14], Y. Yang showed that for some totally ordered field Q, C_Q admits directed partial orders to make it into a directed algebra with $1 > 0$, and hence i is an element with negative square, that is, $i^2 = -1 < 0$. Then in [13], N. Schwartz and Y. Yang proved that $\mathbb{C}$ can be made into a directed algebra over $\mathbb{R}$, and in [12], W. Rump and Y. Yang constructed directed partial orders on $K(i)$, where K could be any non-archimedean totally ordered field and $i^2 = -1$. Their method has used *multiplicative segment* that is a convex additive subgroup of F containing identity element 1.

Motivated by the above work, in [9], L. Wu, Y. Zhang and the present author have introduced a more general method to produce directed partial orders on C_F. Take an additive semigroup $S \subseteq F^+$ with $0, 1 \in S$, and take $x, y \in F^+$ with $0 < x, y \leq 1$. Define the positive cone $P_{x,y}(S)$ as follows.

$$P_{x,y}(S) = \{a + bi \in C_H \mid a \in F^+, -xa \leq sb \leq ya \text{ in } F \text{ for all } s \in S\}.$$

Then $(C_F, P_{x,y})$ is a partially ordered algebra over F and it is a directed algebra if there exists $z \in F^+$ such that $s \leq z$ for all $s \in S$ [9, Theorem 2.2].

For a non-archimedean totally ordered field F, take $S = \mathbb{Z}^+$ and $x = y = 1$, then $P_{1,1}(\mathbb{Z}^+)$ is a directed partial order on C_F that makes C_F into a directed algebra over F. We also observe that $P_{1,1}(\mathbb{Z}^+)$ is the largest directed partial order on C_F over a non-archimedean totally ordered field. Therefore $P_{1,1}(\mathbb{Z}^+)$ is *division closed* in the sense that for any $a, b \in C_F$, if $a, ab \in P_{1,1}(\mathbb{Z}^+)$, then $b \in P_{1,1}(\mathbb{Z}^+)$.

We notice that the partial orders defined above have positive identity element, that is, $1 \in P_{x,y}(S)$. This begs the question whether we can construct directed partial orders on C_F such that 1 is not positive?

Let S be an additive semigroup of F^+ containing 0, 1. Suppose that there exists $w \in F^+$ such that $s \leq w$ for all $s \in S$. Define

$$\begin{aligned} P(S)_> &= \{a + bi \mid a > 0, b > 0 \text{ in } F, sb \leq a, \forall s \in S\} \cup \{0\}, \\ P(S)_< &= \{a - bi \mid a > 0, b > 0 \text{ in } F, sb \leq a, \forall s \in S\} \cup \{0\}. \end{aligned}$$

Thus $P(S)_<$ is the conjugate of $P(S)_>$.

Theorem 4. *$P(S)_>$ and $P(S)_<$ are directed partial order on C_F with $1 \not> 0$.*

Proof. Let's just consider $P(S)_>$. We leave it to the reader to check that $P(S)_>$ is a partial order on C_F. Take $w \in F^+$ such that $s \leq w$ for all $s \in S$. For any $a + bi \in C_F$,

$$1 = (1 + w + i) - (w + i) \text{ and } (1 + w + i), (w + i) \in P(S)_>,$$

so 1 is not positive. We also have $i = (w+2i)-(w+i)$, and $(w+2i), (w+i) \in P(S)_>$. Thus $P(S)_>$ is a directed partial order.

The relation between $P_{1,1}(S)$ and $P(S)_>$ is given as follows.

$$P(S)_> = \{a + bi \in P_{1,1} \mid b > 0\} \cup \{0\},$$

and

$$P(S)_> + F^+ = \{a + bi \in P_{1,1} \mid b \geq 0\}.$$

The research in this direction continues. As a mater of fact, all directed partial orders with $1 > 0$ on C_F over a non-archidemean totally ordered field F have been described in [10] by using the similar positive cones as $P_{x,y}(S)$.

3. Directed Partial Orders on H_F

In this section, we present results on directed partial orders on H_F. First we consider lattice orders. In 1962, McHaffy showed that the division algebra of real quaternions cannot be an ℓ-algebra over $\mathbb{R}$ [11]; and much later in 2004, it was shown that $M_n(H_F)$ cannot be an ℓ-algebra over a archimedean totally ordered field F, for any $n \times n$ matrix algebra with entries from H_F [4]. In fact, the following more general result is true.

Theorem 5. $[8, Theorem\ 6]$ *Suppose that D is a totally ordered integral domain and H_D is a partially ordered algebra over D. If $a + bi + cj + dk \geq 0$ in H_D, then $a \geq 0$ and $|b| \ll a, |c| \ll a, |d| \ll a$ in D. In particular, If D is archimedean, then H_D cannot be an ℓ-algebra over D.*

How about H_F over a non-archimedean totally ordered field F? It was proved that for any totally ordered field F, H_F cannot be made into an ℓ-algebra over F with $1 > 0$ [8, Theorem 7]. Actually, now we can prove that H_F cannot be an ℓ-algebra over any totally ordered field F.

Theorem 6. *For a totally ordered field F, H_F cannot be an ℓ-algebra over F.*

Proof. Suppose that H_F is an ℓ-algebra over F and we derive a contradiction. Then we know that $1 \not> 0$. By [5, Corollary 1.3], H_F is the finite direct sum of convex totally ordered subspace of H_F over F. Since H_F cannot be a totally ordered algebra over F, there are at least two direct summands.

Let's assume first that $H_F = T_1 \oplus T_2$, where T_1, T_2 are totally ordered subspaces over F. Suppose $1 = q_1 + q_2$, where $q_i \in T_i$. Since $1 \not< 0$, one of q_1, q_2 must be positive. We may assume that $q_2 > 0$, and hence $q_1 < 0$. Let $q_1 = a_0 + a_1 i + a_2 j + a_3 k$. Then

$$\begin{aligned} & q_1^2 = 2a_0 q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2) > 0 \\ \Rightarrow\quad & 2a_0 q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2)(q_1 + q_2) > 0 \\ \Rightarrow\quad & (2a_0 - a_0^2 - a_1^2 - a_2^2 - a_3^2) q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2) q_2 > 0 \\ \Rightarrow\quad & (2a_0 - a_0^2 - a_1^2 - a_2^2 - a_3^2) q_1 > (a_0^2 + a_1^2 + a_2^2 + a_3^2) q_2 > 0. \end{aligned}$$

However, since $-q_1 \wedge q_2 = 0$, we must have $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 0$ [5, Theorem 1.13], so $a_0 = a_1 = a_2 = a_3 = 0$. Thus $q_1 = 0$, a contradiction.

Next, we assume $H_F = T_1 \oplus T_2 \oplus T_3$, where T_1, T_2, T_3 are convex totally ordered subspaces over F. Then $1 = q_1 + q_2 + q_3$, where $q_i \in T_i$. Similarly one of q_1, q_2, q_3 must be positive. Let $q_3 > 0$ and $q_1 = a_0 + a_1 i + a_2 j + a_3 k$. Then

$$\begin{aligned} & q_1^2 = 2a_0 q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2) > 0 \\ \Rightarrow\quad & 2a_0 q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2)(q_1 + q_2 + q_3) > 0 \\ \Rightarrow\quad & (2a_0 - a_0^2 - a_1^2 - a_2^2 - a_3^2) q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2) q_2 \\ & -(a_0^2 + a_1^2 + a_2^2 + a_3^2) q_3 > 0 \\ \Rightarrow\quad & (2a_0 - a_0^2 - a_1^2 - a_2^2 - a_3^2) q_1 - (a_0^2 + a_1^2 + a_2^2 + a_3^2) q_2 \\ & > (a_0^2 + a_1^2 + a_2^2 + a_3^2) q_3 > 0. \end{aligned}$$

Then since $|q_1| \wedge q_3 = |q_2| \wedge q_3 = 0$, we must have $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 0$, so $a_0 = a_1 = a_2 = a_3 = 0$ and $q_1 = 0$, a contradiction again.

Similar argument may be made to the case that H_F is a direct sum of four convex totally ordered subspaces over F. This completes the proof.

Theorem 6 answers the second question in Problem 3.

Now we consider directed partial orders on H_F, where F is a totally ordered field. By Theorem 5, H_F cannot be a directed algebra over F if F is an archimedean totally ordered field.

Motivated from the results obtained by W. Rump, N. Schwartz, and Y. Yang for complex numbers, we were able to make the real quaternions $\mathbb{H}$ into a directed algebra over $\mathbb{R}$ with a non-archemedean total order [6]. In fact, define the positive cone P on $\mathbb{H}$ as follows.

$$P = \{a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H} \mid a_0 \geq 0, |a_1| \ll a_0, |a_2| \ll a_0, |a_3| \ll a_0\}$$

Then P is a directed partial order on $\mathbb{H}$ that makes it into a directed algebra over $\mathbb{R}$ with $\mathbb{R} \cap P = \mathbb{R}^+$ [6, Theorem 1].

In [9], the authors proved a more general method to produce directed partial orders on H_F over a non-archimedean totally ordered field F. Take an additive semigroup $S \subseteq F^+$ with $0, 1 \in S$, and take $x \in F$ with $0 < x \leq 1$. Define the positive cone $P_x(S)$ as follows.

$$P_x(S) = \{a_0 + a_1 i + a_2 j + a_3 k \in H_F \mid a_0 \geq 0, |a_1| \ll_S xa_0, |a_2| \ll_S xa_0, |a_3| \ll_S xa_0\},$$

where $|a_1| \ll_S xa_0$ means $-xa_0 \leq sa_1 \leq xa_0$ for all $s \in S$. Similarly for $|a_2| \ll_S xa_0$ and $|a_3| \ll_S xa_0$. Then P_x is a partial order on H_F to make it into a partially ordered algebra over F, and if there exists an element $z \in F^+$ such that $s \leq z$ for all $s \in S$, then P_x is a directed partial order and H_F is a directed algebra [9, Theorem 3.2].

For instance, for a non-archimedean totally ordered field F, take $S = \mathbb{Z}^+$ and $x = 1$, then P_x is the positive cone P introduced in the previous paragraph, and $P_1(\mathbb{Z}^+)$ is the largest directed partial order on H_F.

Directed partial orders on H_F in which $1 \not> 0$ may be constructed similarly to the positive cone $P(S)_>$ and $P(S)_<$ on C_F. However, the last question in Problem 3 remains unsolved.

The directed partial orders constructed for complex numbers and quaternions over non-archimedean totally ordered fields have been generalized to complex numbers and quaternions over non-archimedean partially ordered fields that contain a totally ordered subfield [7, Theorems 1 and 2].

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References

[1] G. Birkhoff, *Lattice Theory*, vol. 25, 3rd ed. Colloquium Publications AMS, New York (1967).

[2] G. Birkhoff, R. S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil. Ci., **28** (1956), 41-69.

[3] L. Fuchs, *Partially ordered algebraic systems*, Dover Publications, Inc. (2011).

[4] J. Ma, Finite dimensional simple algebras that do not admit a lattice order, *Comm. Alg.* **32** (2004) 1615-1617.

[5] J. Ma, *Lecture Notes On Algebraic Structure Of Lattice-Ordered Rings*, World Scientific Publishing (2014).

[6] J. Ma, Directed partial orders on real quaternions, *Quaestiones Mathematicae* (2015), DOI: 10.2989/16073606.2015.1091044.

[7] J. Ma, Directed partial orders on complex number and real quaternions II, *Positivity* (2015), DOI: 10.1007/s11117-015-0388-7.

[8] J. Ma, Partial orders on $C = D + Di$ and $H = D + Di + Dj + Dk$, *International Journal of Advanced Mathematical Sciences*, **3** (2015) 156-160.

[9] J. Ma, L. Wu, Y. Zhang, *Directed partial orders on complex numbers and quaternions over non-archimedean linearly ordered fields*, Order (First Online: 05 March 2016, doi:10.1007/s11083-016-9387-y).

[10] J. Ma, L. Wu, Y. Zhang, *Describing directed partial orders on $F(i)$ with* $1 > 0$, (preprint).

[11] R. McHaffey, *A proof that the quaternions do not form a lattice-ordered algebra*, Proc. of Iraqi Scientific Societies, **5** (1962) 70-71.

[12] W. Rump, Y. Yang, Non-archimedean directed fields $K(i)$ with o-subfield and $i^2 = 1$, *J. Algebra* **400** (2014) 1-7.

[13] N. Schwartz, Y. Yang, Fields with directed partial orders, *J. Algebra* **336** (2011) 342-348.

[14] Y. Yang, On the existence of directed rings and algebras with negative squares, *J. Algebra* **295** (2006) 452-457.

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Chapter 2

SPIN 1 PARTICLE WITH ANOMALOUS MAGNETIC MOMENT IN THE EXTERNAL UNIFORM ELECTRIC FIELD

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Abstract

Within the matrix 10-dimensional Duffin-Kemmer-Petiau formalism applied to the Shamaly-Capri field, we study the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components Ψ_0, Ψ_+, Ψ_+. It is enough to solve the equation for the

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main component Φ_0, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein–Fock–Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

Keywords : Duffin–Kemmer–Petiau algebra, projective operators, spin 1 particle, anomalous magnetic moment, electric field, exact solutions

1. Intoduction

Commonly, we shall use only the simplest wave equations for fundamental particles of spin $0, 1/2, 1$. Meanwhile, it is known that other more complicated equations can be assigned to particles with such spins, which are based on the application of extended sets of Lorentz group representations (see [1]-[16]). Such generalized wave equations allow to describe more complicated objects, which have – besides mass, spin, and electric charge – other electromagnetic characteristics, like polarizability or anomalous magnetic moment. These additional characteristics manifest themselves explicitly in the presence of external electromagnetic fields.

In particular, within this approach, Petras [3] proposed a 20-component theory for spin 1/2 particle, which – after excluding 16 subsidiary components – turns to be equivalent to the Dirac particle theory modified by the presence of Pauli interaction term. In other words, this theory describes a spin 1/2 particle with anomalous magnetic moment.

A similar equation was proposed by Shamaly–Capri [6, 7] for spin 1 particles (also see [16, 17]). In the following, we investigate and solve this wave equation in the presence of the external uniform electric field.

The wave equation for spin 1 particle with anomalous magnetic moment

[6, 7] may be formulated as

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda F_{[\mu\nu]} P J_{[\mu\nu]} + M\right)\Psi = 0\,, \tag{1}$$

where the 10-dimensional wave function and the DKP-matrices are used[1]:

$$\Psi = \begin{pmatrix} \Psi_\mu \\ \Psi_{[\mu\nu]} \end{pmatrix}, \quad J_{[\mu\nu]} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu\,.$$

In tensor form, (1) rewrites as[2]:

$$D_\mu \Psi_\nu - D_\nu \Psi_\mu + M\Psi_{[\mu\nu]} = 0\,,$$

$$D_\nu \Psi_{[\mu\nu]} + 2\frac{ie}{M}\lambda F_{[\mu\nu]}\Psi_\nu + M\Psi_\mu = 0\,.$$

By using DKP-matrices, we apply the method [20] of generalized Kronecker's symbols [3]:

$$\beta_\mu = e^{\nu,[\nu\mu]} + e^{[\nu\mu],\nu}, \quad P = e^{\nu,\nu}\,,$$

$$(e^{A,B})_{CD} = \delta_{AC}\delta_{BD}\,, \quad e^{A,B} e^{C,D} \delta_{BC} e^{A,D},$$

$$\delta_{[\mu\nu],[\rho\sigma]} = \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\,,$$

and the main relationships in the DKP algebra:

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\rho + \delta_{\rho\nu}\beta_\mu\,, \quad [\beta_\lambda, J_{\rho\sigma}]_- = \delta_{\lambda\rho}\beta_\sigma - \delta_{\lambda\sigma}\beta_\rho\,.$$

We use the following representation for DKP-matrices

$$\beta_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1\\ 0&1&0&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0 \end{pmatrix}.$$

[1]Here P stands for a projective operator separating from Ψ its vector component Ψ_μ; $D_\mu = \partial_\mu - ieA_\mu$, abd λ_3 denotes an arbitrary real-valued number.

[2]In a Minkowski space, we use the metric with imaginary unit, since $x_4 = ict$.

[3]The indexes $A(B, C, D, ...)$ take the values 1, 2, 3, 4, [23], [31], [12], [14], [24], [34].

The uniform electric field is provided by the relations

$$(A_\mu) = (0, 0, 0, -iEx_3), \quad E = \text{const}\,,$$

$$F_{[\mu\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{[34]} = -F_{[43]} = -iE\,.$$

The non-minimal interaction through the anomalous magnetic moment is given by the term

$$\pm\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} P J_{[\mu\nu]} = \pm\frac{2eE}{M}\lambda_3\lambda_3^*\, P J_{[34]}\,.$$

Correspondingly, the main equation (1) is written as

$$\left[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} + \beta_3\frac{\partial}{\partial x^3} + \beta_4\left(\frac{\partial}{\partial x^4} - eEx_3\right) + \Gamma_0\, P J_{[34]} + M\right]\Psi = 0\,, \quad (2)$$

where $\Gamma_0 = \frac{2eE}{M}\lambda$.

2. Algebraic Transformation of the Wave Equation

Let us introduce the matrix $Y = iJ_{[34]} = i(\beta_3\beta_4 - \beta_4\beta_3)$, which satisfies the minimal polynomial equation $Y^3 = Y \Leftrightarrow Y(Y-1)(Y+1) = 0$, and allows us to define the tree projective operators:

$$P_0 = 1 - Y^2\,, \quad P_+ = \frac{1}{2}Y(Y+1)\,, \quad P_- = \frac{1}{2}Y(Y-1),$$

and solve the wave function in terms of the three components:

$$\Psi_0 = P_0\Psi, \quad \Psi_+ = P_+\Psi, \quad \Psi_- = P_-\Psi, \quad \Psi = \Psi_0 + \Psi_- + \Psi_+\,.$$

Acting on (2) by the operator P_0, and taking into account the algebraic identities

$$Y\beta_{1,2} = \beta_{1,2}Y, \quad P_0\beta_{1,2} = \beta_{1,2}P_0, \quad P_0 P J_{[34]} = -iP(1 - Y^2)Y \equiv 0\,,$$

we get

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0 + P_0\beta_3\,\partial_3\Psi + (\partial_4 - eEx_3)\,P_0\beta_4\Psi = 0\,. \quad (3)$$

Let us consider the operator $P_0\beta_3$ (we shall further use the computation rules within the DKP-algebra):

$$P_0\beta_3 = (1 - Y^2)\beta_3 = (1 + 2\beta_3\beta_3\beta_4\beta_4 - \beta_3\beta_3 - \beta_4\beta_4)\beta_3 =$$

$$= \beta_3 + 2\beta_3\beta_3\beta_4\beta_4\beta_3 - \beta_3 - \beta_4\beta_4\beta_3 =$$

$$= 2\beta_3\beta_3(\beta_3 - \beta_3\beta_4\beta_4) - (\beta_3 - \beta_3\beta_4\beta_4) = \beta_3 - \beta_3\beta_4\beta_4 \, .$$

Considering the identities

$$\beta_3(1 - P_0) = \beta_3 Y^2 = \beta_3[\beta_3\beta_3 + \beta_4\beta_4 - 2\beta_3\beta_3\beta_4\beta_4] =$$

$$= \beta_3 + \beta_3\beta_4\beta_4 - 2\beta_3\beta_4\beta_4 = \beta_3 - \beta_3\beta_4\beta_4 \, ,$$

the previous can be written in the form

$$P_0\beta_3 = \beta_3(1 - P_0) = \beta_3(P_+ + P_-) \, . \tag{4}$$

Similarly, one can obtain the identity

$$P_0\beta_4 = \beta_4(1 - P_0) = \beta_4(P_+ + P_-) \, . \tag{5}$$

Taking into account the relations (4)–(5), (3) reduces to the form

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0 +$$
$$+ [\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]\,\Psi_+ + [\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]\,\Psi_- = 0 \, . \tag{6}$$

Let us consider the operator

$$\beta_3 P_+ = \beta_3 \frac{1}{2}(Y + Y^2) =$$

$$= \beta_3 \frac{1}{2}[i(\beta_3\beta_4 - \beta_4\beta_3) - 2\beta_3\beta_3\beta_4\beta_4 + \beta_3\beta_3 + \beta_4\beta_4].$$

For $\beta_3^3 = \beta_3$ and $\beta_3\beta_4\beta_3 = 0$, it follows

$$\beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_3\beta_3\beta_4 - \beta_3\beta_4\beta_4) \, .$$

As well, for $\beta_4^3 = \beta_4$ and $\beta_4\beta_3\beta_4 = 0$, we infer

$$\beta_4 P_+ = \beta_4 \frac{1}{2}[i(\beta_3\beta_4 - \beta_4\beta_3) - 2\beta_3\beta_3\beta_4\beta_4 + \beta_3\beta_3 + \beta_4\beta_4] =$$

$$= \frac{1}{2}[-i\beta_4\beta_4\beta_3 - 2\beta_4\beta_3\beta_3\beta_4\beta_4 + \beta_4\beta_3\beta_3 + \beta_4] \, .$$

Further, by using the identities

$$\beta_4\beta_4\beta_3 = \beta_3 - \beta_3\beta_4\beta_4 \,, \quad \beta_4\beta_3\beta_3 = \beta_4 - \beta_3\beta_3\beta_4,$$

we get

$$\beta_4 P_+ = \frac{1}{2}[-i(\beta_3 - \beta_3\beta_4\beta_4) - 2(\beta_4 - \beta_3\beta_3\beta_4)\beta_4\beta_4 + (\beta_4 - \beta_3\beta_3\beta_4) + \beta_4] =$$

$$= -\frac{i}{2}[\beta_3 - \beta_3\beta_4\beta_4 + i\beta_3\beta_3\beta_4] \,.$$

Hence, we obtain the algebraic relation

$$\beta_3 P_+ = i\beta_4 P_+ \quad \Longrightarrow \quad (\beta_3 - i\beta_4)P_+ = 0 \,. \tag{7}$$

By combining the relations

$$i\beta_4 P_+ = \frac{1}{2}[\beta_3 - \beta_3\beta_4\beta_4 + i\beta_3\beta_3\beta_4], \quad \beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_3\beta_3\beta_4 - \beta_3\beta_4\beta_4) \,,$$

we easily derive

$$\beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_4)P_+ \,. \tag{8}$$

As well, by combining (7)–(8), we get

$$\beta_4 P_+ = -\frac{i}{2}(\beta_3 + i\beta_4)P_+ \,.$$

In the same manner, we get the following three identities

$$(\beta_3 + i\beta_4)P_- = 0 \,, \quad \beta_3 P_- = \frac{1}{2}(\beta_3 - i\beta_4)P_-, \quad \beta_4 P_- = \frac{i}{2}(\beta_3 - i\beta_4)P_- \,. \tag{9}$$

We further turn back to (6), which can be written as

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0 +$$

$$+(\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]P_+\Psi_+ + (\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]P_-\Psi_- = 0 \,.$$

With the help of above identities, (9) can be rewritten in the form[4]

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0 +$$

[4]We take into account that $P_\pm^2 = P_\pm$.

$$+\frac{1}{2}(\beta_3 + i\beta_4)\,[\partial_3 - i(\partial_4 - eEx_3)]\,\Psi_+ + \frac{1}{2}(\beta_3 - i\beta_4)\,[\partial_3 + i(\partial_4 - eEx_3)]\,\Psi_- = 0$$

or

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0 + \tfrac{1}{2}(\beta_3 + i\beta_4)\,[(\partial_3 + iEx_3) - i\partial_4]\,\Psi_+ + \\ + \tfrac{1}{2}(\beta_3 - i\beta_4)\,[(\partial_3 - eEx_3) + i\partial_4]\,\Psi_- = 0\,.$$

Now, let us consider the relation (2)

$$\left[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} + \beta_3\frac{\partial}{\partial x^3} + \beta_4(\frac{\partial}{\partial x^4} - eEx_3) - i\Gamma_0 PY + M\right]\Psi = 0\,,$$

and act on it by the operator $1 - P_0 = P_+ + P_-$; this yields

$$\left[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} - i\Gamma_0 PY + M\right](\Psi_+ + \Psi_-) + \\ +(1 - P_0)\beta_3\frac{\partial}{\partial x^3} + (1 - P_0)\beta_4(\frac{\partial}{\partial x^4} - eEx_3)\Psi = 0\,. \tag{10}$$

By using the easy-to-check identity

$$1 - P_0 = Y^2 = \beta_3\beta_3 + \beta_4\beta_4 - 2\beta_3\beta_3\beta_4\beta_4\,,$$

we get

$$(1 - P_0)\beta_3 = \beta_3 + (\beta_3 - \beta_3\beta_4\beta_4) - 2\beta_3\beta_3(\beta_3 - \beta_3\beta_4\beta_4) = +\beta_3\beta_4\beta_4\,.$$

Similarly, we derive

$$\beta_3 P_0 = \beta_3(1 - Y^2) = \beta_3(1 - \beta_3\beta_3 - \beta_4\beta_4 + 2\beta_3\beta_3\beta_4\beta_4) = +\beta_3\beta_4\beta_4\,.$$

By combining the two last relations, we obtain the commutation rule

$$(1 - P_0)\beta_3 = \beta_3 P_0\,.$$

In the same manner, we derive the following three similar relations

$$\beta_4 - \beta_3\beta_3\beta_4 = (1 - P_0)\beta_4, \quad \beta_4 - \beta_3\beta_3\beta_4 = \beta_4 P_0, \qquad (1 - P_0)\beta_4 = \beta_4 P_0\,,$$

which lead to the rewriting of (10) as

$$\left[\,\beta_1\tfrac{\partial}{\partial x^1} + \beta_2\tfrac{\partial}{\partial x^2} - i\Gamma_0 PY + M\,\right](\Psi_+ + \Psi_-) + \\ +\beta_3\tfrac{\partial}{\partial x^3}\Psi_0 + \beta_4(\tfrac{\partial}{\partial x^4} - eEx_3)\Psi_0 = 0\,, \tag{11}$$

By acting on (11) by the operator $\frac{1}{2}(1+Y)$ and with the help of the easy to check identities

$$\frac{1}{2}(1+Y)P_+ = \frac{1}{2}(1+Y)\frac{1}{2}Y(1+Y) = \frac{1}{2}Y(1+Y) = P_+\,,$$

$$\frac{1}{2}(1+Y)P_- = \frac{1}{4}(Y+Y^2)(Y-1) = 0\,,$$

we derive

$$\left[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} - i\Gamma_0 PY + M\right]\Psi_+ +$$

$$+\frac{1}{2}(1+Y)\beta_3\frac{\partial}{\partial x^3}\Psi_0 + \frac{1}{2}(1+Y)\beta_4(\frac{\partial}{\partial x^4} - eEx_3)\Psi_0 = 0\,, \tag{12}$$

We need three auxiliary relations. From the known formula

$$\beta_\lambda J_{[\rho\sigma]} - J_{[\rho\sigma]}\beta_\lambda = \delta_{\rho\sigma}\beta_\lambda - \delta_{\lambda\sigma}\beta_\rho$$

it follows

$$\beta_3 Y - Y\beta_3 = +i\beta_4 \quad \Longrightarrow \quad Y\beta_3 = \beta_3 Y - i\beta_4\,,$$

$$\beta_4 Y - Y\beta_4 = -i\beta_3 \quad \Longrightarrow \quad Y\beta_4 = \beta_4 Y + i\beta_4\,.$$

Therefore, (12) can be written as

$$\left[\,\beta_1\tfrac{\partial}{\partial x^1} + \beta_2\tfrac{\partial}{\partial x^2} - i\Gamma_0 PY + M\,\right]\Psi_+ +$$

$$+\tfrac{1}{2}(\beta_3 + \beta_3 Y - i\beta_4)\tfrac{\partial}{\partial x^3}\Psi_0 + \tfrac{1}{2}(\beta_4 + \beta_4 Y + i\beta_4)(\tfrac{\partial}{\partial x^4} - eEx_3)\Psi_0 = 0\,,$$

From this, taking into account $YP_0 \equiv 0$, we obtain the more simple form

$$\left[\beta_1\tfrac{\partial}{\partial x^1} + \beta_2\tfrac{\partial}{\partial x^2} - i\Gamma_0 PY + M\right]\Psi_+ +$$

$$+\tfrac{1}{2}(\beta_3 - i\beta_4)\tfrac{\partial}{\partial x^3}\Psi_0 + \tfrac{1}{2}(\beta_4 + i\beta_3)(\tfrac{\partial}{\partial x^4} - eEx_3)\Psi_0 = 0\,,$$

or

$$\left[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} - i\Gamma_0 PY + M\right]\Psi_+ + \frac{1}{2}(\beta_3 - i\beta_4)\left[\frac{\partial}{\partial x^3}\Psi_0 + i\left(\frac{\partial}{\partial x^4} - eEx_3\right)\right]\Psi_{\,0} = 0.$$

Now, let us take into account an identity

$$YP_+ = Y\frac{1}{2}Y(1+Y) = \frac{1}{2}(Y^2 + Y^3 = P_+ \Longrightarrow Y\Psi_+ = \Psi_+\,.$$

Then the previous equation reads

$$\left(\beta_1 \frac{\partial}{\partial x^1} + \beta_2 \frac{\partial}{\partial x^2} - i\Gamma_0 P + M\right) \Psi_+ + \frac{1}{2}(\beta_3 - i\beta_4) \left(\frac{\partial}{\partial x^3} - ieEx_3 + i\frac{\partial}{\partial x^4}\right) \Psi_0 = 0\,.$$

As well, by acting on (11) by the operator $\frac{1}{2}(1-Y)$, after similar calculations we get the equation

$$\left(\beta_1 \frac{\partial}{\partial x^1} + \beta_2 \frac{\partial}{\partial x^2} - i\Gamma_0 P + M\right) \Psi_- + \frac{1}{2}(\beta_3 + i\beta_4) \left(\frac{\partial}{\partial x^3} + ieEx_3 - i\frac{\partial}{\partial x^4}\right) \Psi_0 = 0\,.$$

3. The Separation of Variables

We start with the three equations

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\Psi_0 +$$
$$+\tfrac{1}{\sqrt{2}}\beta_+[(\partial_3 + ieEx_3) - i\partial_4]\Psi_+ + \tfrac{1}{\sqrt{2}}\beta_-[(\partial_3 - ieEx_3) + i\partial_4]\Psi_- = 0\,,$$
$$(\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 P + M)\Psi_+ + \tfrac{1}{\sqrt{2}}\beta_-[(\partial_3 - ieEx_3) + i\partial_4]\Psi_0 = 0\,,$$
$$(\beta_1\partial_1 + \beta_2\partial_2 + i\Gamma_0 P + M)\Psi_- + \tfrac{1}{\sqrt{2}}\beta_+[(\partial_3 + ieEx_3) - i\partial_4]\Psi_0 = 0\,,$$

where

$$\beta_+ = \frac{1}{\sqrt{2}}(\beta_3 + i\beta_4), \quad \beta_- = \frac{1}{\sqrt{2}}(\beta_3 - i\beta_4)\,.$$

We look for solutions of the form:

$$\Psi_0 = e^{ip_4x_4} e^{ip_1x_1} e^{ip_2x_2} f_0(x_3)\,,$$
$$\Psi_+ = e^{ip_4x_4} e^{ip_1x_1} e^{ip_2x_2} f_+(x_3)\,,$$
$$\Psi_- = e^{ip_4x_4} e^{ip_1x_1} e^{ip_2x_2} f_-(x_3)\,.$$

So, we have the system of three equations in the variable x_3:

$$(ip_1\beta_1 + ip_2\beta_2 + M)\Psi_0 +$$
$$+\tfrac{1}{\sqrt{2}}\beta_+[(\tfrac{d}{dx_3} + ieEx_3) + p_4]\Psi_+ + \tfrac{1}{\sqrt{2}}\beta_-[(\tfrac{d}{dx_3} - ieEx_3) - p_4]\Psi_- = 0\,,$$
$$(ip_1\beta_1 + ip_2\beta_2 - i\Gamma_0 P + M)\Psi_+ + \tfrac{1}{\sqrt{2}}\beta_-[(\tfrac{d}{dx_3} - ieEx_3) - p_4]\Psi_0 = 0\,,$$
$$(ip_1\beta_1 + ip_2\beta_2 + i\Gamma_0 P + M)\Psi_- + \tfrac{1}{\sqrt{2}}\beta_+[(\tfrac{d}{dx_3} + ieEx_3) + p_4]\Psi_0 = 0\,.$$

With the shortening notation

$$\hat{a} = \frac{1}{\sqrt{2}} \left(+ \frac{d}{dx_3} + ieEx_3 + p_4 \right), \quad \hat{b} = \frac{1}{\sqrt{2}} \left(- \frac{d}{dx_3} + ieEx_3 + p_4 \right) ;$$
$$i\Gamma_0 = \Gamma, \qquad p_1\beta_1 + p_2\beta_2 = \hat{p} ;$$

the equations are written as

$$(i\hat{p} + M)\Psi_0 + \beta_+ \hat{a}\Psi_+ - \beta_- \hat{b}\Psi_- = 0 , \tag{13}$$
$$(i\hat{p} - \Gamma P + M)\Psi_+ - \beta_- \hat{b}\Psi_0 = 0 , \tag{14}$$
$$(i\hat{p} + \Gamma P + M)\Psi_- + \beta_+ \hat{a}\Psi_0 = 0 . \tag{15}$$

By acting (14) by the operator

$$\frac{M - \Gamma \bar{P}}{M - \Gamma},$$

we infer

$$\left(\frac{M - \Gamma \bar{P}}{M - \Gamma} i\hat{p} + \frac{M - \Gamma \bar{P}}{M - \Gamma} (M - \Gamma P) \right) \Psi_+ - \frac{M - \Gamma \bar{P}}{M - \Gamma} \beta_- \hat{b}\Psi_0 = 0 .$$

With the help of the identities

$$\frac{M - \Gamma \bar{P}}{M - \Gamma} (M - \Gamma P) = \frac{1}{M - \Gamma} (M^2 - M\Gamma P - M\Gamma \bar{P}) = M ,$$

it reads

$$\left(\frac{M - \Gamma \bar{P}}{M - \Gamma} i\hat{p} + M \right) \Psi_+ - \frac{M - \Gamma \bar{P}}{M - \Gamma} \beta_- \hat{b}\Psi_0 = 0 .$$

By using the notations

$$\frac{M - \Gamma \bar{P}}{M - \Gamma} i\hat{p} = A, \quad \frac{M - \Gamma \bar{P}}{M - \Gamma} \beta_- = \beta'_-,$$

the previous equation shortens to

$$(A + M)\Psi_+ - \beta'_- \hat{b}\Psi_0 = 0 .$$

Analogously, by acting on (15) by the operator

$$\frac{M + \Gamma \bar{P}}{M + \Gamma},$$

we get

$$\left(\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p}+\frac{M+\Gamma\bar{P}}{M-\Gamma}(M+\Gamma P)\right)\Psi_-+\frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+\hat{a}\Psi_0=0\,.$$

Taking into account the identities

$$\frac{M+\Gamma\bar{P}}{M+\Gamma}(M+\Gamma P)=\frac{1}{M+\Gamma}(M^2+M\Gamma P+M\Gamma\bar{P})=M\,;$$

we derive

$$\left(\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p}+M\right)\Psi_-+\frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+\hat{a}\Psi_0=0\,.$$

With the notations

$$\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p}=C,\quad \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+=\beta'_+,$$

the last equation reads

$$(C+M)\Psi_-+\beta'_+\hat{b}\Psi_0=0\,.$$

Let us consider the powers of A

$$A^2=\frac{1}{(M-\Gamma)^2}(iM\hat{p}-i\Gamma\bar{P}\hat{p})(iM\hat{p}-i\Gamma\bar{P}\hat{p})=$$

$$=\frac{1}{(M-\Gamma)^2}[-M^2\hat{p}^{\,2}+M\Gamma\hat{p}\bar{P}\hat{p}+M\Gamma\bar{P}\hat{p}^2-\Gamma^2\bar{P}\hat{p}\bar{P}\hat{p}]\,.$$

Because

$$\beta_\mu=P\beta_\mu+\beta_\mu P=\bar{P}\beta_\mu+\beta_\mu\bar{P},\beta_\mu P=P\beta_\mu,\bar{P}\beta_\mu=\beta_\mu\bar{P},$$

$$P\beta_\mu P=\bar{P}\beta_\mu\bar{P}=0,\beta_\mu\beta_\nu P=P\beta_\mu\beta_\nu,\beta_\mu\beta_\nu\bar{P}=\bar{P}\beta_\mu\beta_\nu,$$

$$P+\bar{P}=1,P\bar{P}=\bar{P}P=0,$$

we get

$$A^2=\frac{1}{(M-\Gamma)^2}(-M^2\hat{p}^2+M\Gamma\hat{p}^2)=-\frac{M\hat{p}^2}{M-\Gamma}\,.$$

We calculate A^3:

$$A^3 = -\frac{M}{(M-\Gamma)^2}(M-\Gamma\bar{P})(i\hat{p})\hat{p}^2 = -\frac{Mp^2}{(M-\Gamma)}\frac{(M-\Gamma\bar{P})}{M-\Gamma}(i\hat{p})\,,$$

so, the minimal polynomial of A (or the Cayley-Hamilton identity for A) has the form

$$A^3 + \frac{Mp^2}{M-\Gamma}A = 0\,.$$

Similar results are valid for the operator C:

$$C^3 + \frac{Mp^2}{M+\Gamma}C = 0\,.$$

The Cayley-Hamilton identity for $i\hat{p}$) has the form

$$i\hat{p}\,[(i\hat{p})^2 + p^2] = 0\,.$$

Thus, the complete set of equations in the variable x_3 is of the form

$$\begin{aligned}
&(i\hat{p} + M)\,f_0 + \beta_+\hat{a}f_+ - \beta_-\hat{b}f_- = 0\,,\\
&(A+M)f_+ - \beta'_-\hat{b}f_0 = 0\,,\\
&(C+M)f_- + \beta'_+\hat{a}f_0 = 0\,.
\end{aligned}$$

To proceed with these equations, we introduce the matrices[5] with the properties

$$\begin{aligned}
\overline{(i\hat{p}+M)}(i\hat{p}+M) &= p^2 + M^2,\\
\overline{(A+M)}(A+M) &= p^2 + M^2,\\
\overline{(C+M)}(C+M) &= p^2 + M^2\,.
\end{aligned} \qquad (16)$$

In fact these formulas determine the inverse matrices up to numerical factors $(p^2+M^2)^{-1}$. Then the system of radial equations can be rewritten alternatively

$$\begin{aligned}
(i\hat{p} + M)\,(p^2+M^2)f_0 + \beta_+\hat{a}(p^2+M^2)f_+ - \beta_-\hat{b}(p^2+M^2)f_- &= 0\,,\\
(p^2+M^2)f_+ - \overline{(A+M)}\beta'_-\hat{b}f_0 &= 0\,,\\
(p^2+M^2)f_- + \overline{(C+M)}\beta'_+\hat{a}f_0 &= 0\,.
\end{aligned} \qquad (17)$$

[5]We take in the account that $p^2 = p_1^2 + p_2^2$

The first equation in (17), with the help of the other two ones, transforms into an equation in the component $f_0(r)$:

$$(i\hat{p}+M)(p^2+M^2)^2 f_0 + \beta_+ \hat{a}\overline{(A+M)}\beta'_- \hat{b} f_0 + \beta_- \hat{b}\overline{(C+M)}\beta'_+ \hat{a}_m f_0 = 0\,, \quad (18)$$

while the two remaining ones do not change

$$(p^2+M^2)f_+ - \overline{(A+M)}\beta'_- \hat{b} f_0 = 0\,,$$
$$(p^2+M^2)f_- + \overline{(C+M)}\beta'_+ \hat{a} f_0 = 0\,. \quad (19)$$

In fact, the equations (19) mean that it suffices to solve (18) with respect to f_0; the two other components f_+ and f_- can be calculated by means of the equations (19).

To proceed further, we need to know the explicit form of the inverse operators (16). To solve this task, we first establish the minimal polynomials for the relevant matrices.

Therefore, the needed inverse operators must be quadratic with respect to the relevant matrices. They are given by the formulas:

$$\overline{(M+i\hat{p})} = \tfrac{1}{M}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right],$$
$$\overline{(A+M)} = \tfrac{p^2+M^2}{M}\left[1 - \tfrac{M-\Gamma}{p^2+M^2-M\Gamma}A + \tfrac{M-\Gamma}{M(p^2+M^2-M\Gamma)}A^2\right],$$
$$\overline{(C+M)} = \tfrac{p^2+M^2}{M}\left[1 - \tfrac{M+\Gamma}{p^2+M^2+M\Gamma}C + \tfrac{M+\Gamma}{M(p^2+M^2+M\Gamma)}C^2\right].$$

Taking into account the explicit form of the inverse operators, we get

$$(p^2+M^2)f_0 +$$
$$+\frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right]\beta_+ \times$$
$$\times\left[1 - \frac{M-\Gamma}{p^2+M^2-M\Gamma}A + \frac{M-\Gamma}{M(p^2+M^2-M\Gamma)}A^2\right]\beta'_-\,\hat{a}\hat{b} f_0 +$$
$$+\frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right]\beta_- \times$$
$$\times\left[1 - \frac{M+\Gamma}{p^2+M^2+M\Gamma}C + \frac{M+\Gamma}{M(p^2+M^2+M\Gamma)}C^2\right]\beta'_+\,\hat{b}\hat{a} f_0 = 0\,.$$

Now, by considering the formulas

$$A = \frac{M-\Gamma\bar{P}}{M-\Gamma}\,i\hat{p}\,, \quad A^2 = -\frac{M\hat{p}^2}{M-\Gamma}\,,$$

$$C = \frac{M+\Gamma\bar{P}}{M+\Gamma}\,i\hat{p}\,, \quad C^2 = -\frac{M\hat{p}^2}{M+\Gamma}\,,$$

$$\beta'_{-} = \frac{M - \Gamma\bar{P}}{M - \Gamma}\beta_{-}\,, \quad \beta'_{+} = \frac{M + \Gamma\bar{P}}{M + \Gamma}\beta_{+}\,,$$

we transform the above equation into

$$(p^2 + M^2) f_0 +$$
$$+ \frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{+}\times$$
$$\times\left[1 - \frac{M - \Gamma\bar{P}}{p^2 + M^2 - M\Gamma}\, i\hat{p} + \frac{(i\hat{p})^2}{p^2 + M^2 - M\Gamma}\right]\frac{M - \Gamma\bar{P}}{M - \Gamma}\beta_{-}\,\hat{a}\hat{b} f_0 +$$
$$+ \frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{-}\times$$
$$\times\left[1 - \frac{M + \Gamma\bar{P}}{p^2 + M^2 + M\Gamma}\, i\hat{p} + \frac{(i\hat{p})^2}{p^2 + M^2 + M\Gamma}\right]\frac{M + \Gamma\bar{P}}{M + \Gamma}\beta_{+}\,\hat{b}\hat{a} f_0 = 0\,.$$

After some manipulation, this becomes

$$\{\,(p^2 + M^2) + \hat{a}\hat{b}_m\frac{1}{M^2(p^2 + M^2 - M\Gamma)}\frac{1}{M - \Gamma}\times$$
$$\times\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{+}\left[(p^2 + M^2 - M\Gamma) - (M - \Gamma\bar{P})\, i\hat{p} + (i\hat{p})^2\right](M - \Gamma\bar{P})\beta_{-} +$$
$$+ \hat{b}\hat{a}\frac{1}{M^2(p^2 + M^2 + M\Gamma)}\frac{1}{M + \Gamma}\times$$
$$\times\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{-}\left[(p^2 + M^2 + M\Gamma) - (M - \Gamma\bar{P})\, i\hat{p} + (i\hat{p})^2\right]$$
$$(M + \Gamma\bar{P})\beta_{+}\,\}\, f_0 = 0$$

or

$$\{\,(p^2 + M^2) + \hat{a}\hat{b}\frac{1}{M^2(p^2 + M^2 - M\Gamma)}\frac{1}{M - \Gamma}\times$$
$$\times\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{+}\left[(p^2 + M^2 - M\Gamma) - \, i\hat{p}(M - \Gamma P) + (i\hat{p})^2\right](M - \Gamma\bar{P})\beta_{-} +$$
$$+ \hat{b}\hat{a}\frac{1}{M^2(p^2 + M^2 + M\Gamma)}\frac{1}{M + \Gamma}\times$$
$$\times\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2)\right]\beta_{-}\left[(p^2 + M^2 + M\Gamma) - \, i\hat{p}(M + \Gamma P) + (i\hat{p})^2\right](M + \Gamma\bar{P})\beta_{+}\,\}\, f_0 = 0\,.$$

Due to the identity

$$\hat{p}\beta_{+}\hat{p} = \hat{p}\beta_{-}\hat{p} = 0,$$

this admits the simpler form:

$$\{\,(p^2 + M^2) + \hat{a}\hat{b}\tfrac{1}{M^2(p^2+M^2-M\Gamma)}\tfrac{1}{M-\Gamma}\times[\,(p^2 + M^2 - M\Gamma)(i\hat{p})^2\beta_{+} - M(p^2 + M^2 - M\Gamma)i\hat{p}\beta_{+} +$$
$$+ (p^2 + M^2)(p^2 + M^2 - M\Gamma)\beta_{+} - (p^2 + M^2)\beta_{+} i\hat{p}(M - \Gamma P) + (p^2 + M^2)\beta_{+}(i\hat{p})^2]\,(M - \Gamma\bar{P})\beta_{-} +$$
$$+ \hat{b}\hat{a}\tfrac{1}{M^2(p^2+M^2+M\Gamma)}\tfrac{1}{M+\Gamma}\times\times[\,(p^2 + M^2 + M\Gamma)(i\hat{p})^2\beta_{-} - M(p^2 + M^2 + M\Gamma)i\hat{p}\beta_{-} +$$
$$+ (p^2 + M^2)(p^2 + M^2 + M\Gamma)\beta_{-} - (p^2 + M^2)\beta_{-} i\hat{p}(M + \Gamma P) +$$
$$+ (p^2 + M^2)\beta_{-}(i\hat{p})^2]\,(M + \Gamma\bar{P})\beta_{+}\,\}\, f_0 = 0$$

Now we take into account the explicit form of f_0, $i\hat{p}$, and all the involved matrices:

$$
i\hat{p} = i \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -p_3 & 0 & p_4 & 0 & 0 \\
0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_3 \\
0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_4 & -p_3 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} .
$$

The explicit form of $\beta_{\pm}$ is:

$$
\beta_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & 0 & -1 & 0 & \pm i & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & \pm i & 0 \\
\cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 & \pm i \\
\cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 & -1 \\
[2mm]0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\pm i & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \pm i & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \pm i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix} .
$$

The explicit form of $f_{[34]}$ and T^2 is:

$$
f_{[34]} = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & -1 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & -1 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} , \quad Y = i f_{[34]} , \quad -Y^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & -1 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & -1 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & -1 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

$$
1 - Y^2 = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 1 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} , f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_{[23]} \\ f_{[31]} \\ f_{[12]} \\ f_{[14]} \\ f_{[24]} \\ f_{[34]} \end{pmatrix} , \quad F_0 = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{[12]} \\ 0 \\ 0 \\ f_{[34]} \end{pmatrix} .
$$

Then we obtain

$$
(p^2 + M^2) \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{[12]} \\ 0 \\ 0 \\ f_{[34]} \end{pmatrix} + \frac{\hat{a}\hat{b}}{M^2(M-\Gamma)(p^2+M^2-M\Gamma)} \left\{ (p^2+M^2)(p^2+M^2-M\Gamma) \begin{pmatrix} (M-\Gamma)f_1 \\ (M-\Gamma)f_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M f_{[34]} \end{pmatrix} - \right.
$$

$$
-iM(p^2+M^2-M\Gamma) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (M-\Gamma)(p_2 f_1 - p_1 f_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} - (p^2+M^2-M\Gamma) \begin{pmatrix} p_2(M-\Gamma)(p_2 f_1 - p_1 f_2) \\ -p_1(M-\Gamma)(p_2 f_1 - p_1 f_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} +
$$

$$
+(p^2+M^2)M(M-\Gamma)\begin{pmatrix} p_1 f_{[34]} \\ p_1 f_{[34]} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -(p_1 f_1 + p_2 f_2) \end{pmatrix} - (p^2+M^2)\begin{pmatrix} p_1(M-\Gamma)(p_1 f_1 + p_2 f_2) \\ p_2(M-\Gamma)(p_1 f_1 + p_2 f_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M p^2 f_{[34]} \end{pmatrix} \Bigg\} +
$$

$$
+\frac{\hat{b}\hat{a}}{M^2(M+\Gamma)(p^2+M^2+M\Gamma)}\Bigg\{(p^2+M^2)(p^2+M^2+M\Gamma)\begin{pmatrix} (M+\Gamma)f_1 \\ (M+\Gamma)f_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M f_{[34]} \end{pmatrix} -
$$

$$
-iM(p^2+M^2+M\Gamma)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (M+\Gamma)(p_2 f_1 - p_1 f_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} - (p^2+M^2+M\Gamma)\begin{pmatrix} p_2(M+\Gamma)(p_2 f_1 - p_1 f_2) \\ -p_1(M+\Gamma)(p_2 f_1 - p_1 f_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} +
$$

$$
+(p^2+M^2)M(M+\Gamma)\begin{pmatrix} p_1 f_{[34]} \\ p_1 f_{[34]} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -(p_1 f_1 + p_2 f_2) \end{pmatrix} - (p^2+M^2)\begin{pmatrix} p_1(M+\Gamma)(p_1 f_1 + p_2 f_2) \\ p_2(M+\Gamma)(p_1 f_1 + p_2 f_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M p^2 f_{[34]} \end{pmatrix} \Bigg\} = 0.
$$

from this there follow four equations for the constituents of f_0:

$$
(p^2+M^2)f_1 + \frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\times
$$

$$
\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_1 - p_2(p^2+M^2-M\Gamma)(p_2 f_1 - p_1 f_2)+
$$

$$
+Mp_1(p^2+M^2)f_{[34]} - (p^2+M^2)p_1(p_1 f_1 + p_2 f_2)\} + \frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\times
$$

$$
\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_1 - p_2(p^2+M^2+M\Gamma)(p_2 f_1 - p_1 f_2)-
$$

$$
-Mp_1(p^2+M^2)f_{[34]} - p_1(p^2+M^2)(p_1 f_1 + p_2 f_2)\} = 0\,, \tag{20}
$$

$$
(p^2+M^2)f_2 + \frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\times
$$

$$
\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_2 + p_1(p^2+M^2-M\Gamma)(p_2 f_1 - p_1 f_2)+
$$

$$
+M(p^2+M^2)p_2 f_{[34]} - p_2(p^2+M^2)(p_1 f_1 + p_2 f_2)\} + \frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\times
$$

$$\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_2+p_1(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)-$$

$$-Mp_2(p^2+M^2)f_{[34]}-p_2(p^2+M^2)(p_1f_1+p_2f_2)\}=0\,, \quad (21)$$

$$(p^2+M^2)f_{[12]}+\frac{\hat{a}\hat{b}}{M}\{-i(p_2f_1-p_1f_2)\}+\frac{\hat{b}\hat{a}}{M}\{-i(p_2f_1-p_1f_2)\}=0\,, \quad (22)$$

$$f_{[34]}+\frac{\hat{a}\hat{b}}{M(M-\Gamma)(p^2+M^2-M\Gamma)}\{(p^2+M^2-M\Gamma)f_{[34]}-(M-\Gamma)(p_1f_1+p_2f_2))-p^2f_{[34]}\}+$$

$$+\frac{\hat{b}\hat{a}}{M(M+\Gamma)(p^2+M^2+M\Gamma)}\{(p^2+M^2+M\Gamma)f_{[34]}+(M+\Gamma)(p_1f_1+p_2f_2))-p^2f_{[34]}\}=0\,.$$

$$(23)$$

From (22) we derive

$$(p^2+M^2)f_{[12]}-\frac{i}{M}(\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2)=0\,,$$

and from (23) it follows

$$f_{[34]}+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\{Mf_{[34]}-(p_1f_1+p_2f_2)\}+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{Mf_{[34]}+(p_1f_1+p_2f_2)\}=0\,.$$

Then, from (20)–(21), we get

$$[\text{EQ.I}] \qquad (p^2+M^2)f_1+\frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\times$$

$$\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_1-p_2(p^2+M^2-M\Gamma)(p_2f_1-p_1f_2)+$$

$$+Mp_1(p^2+M^2)f_{[34]}-(p^2+M^2)p_1(p_1f_1+p_2f_2)\}+$$

$$+\frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_1-p_2(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)-$$

$$-Mp_1(p^2+M^2)f_{[34]}-p_1(p^2+M^2)(p_1f_1+p_2f_2)\}=0\,,$$

$$[\text{EQ.II}] \qquad (p^2+M^2)f_2+\frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\times$$

$$\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_2+p_1(p^2+M^2-M\Gamma)(p_2f_1-p_1f_2)+$$
$$+M(p^2+M^2)p_2f_{[34]}-p_2(p^2+M^2)(p_1f_1+p_2f_2)\}+$$
$$+\frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_2+p_1(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)-$$
$$-Mp_2(p^2+M^2)f_{[34]}-p_2(p^2+M^2)(p_1f_1+p_2f_2)\}=0\,.$$

By combining these equations as follows:

$$p_1\cdot[\text{EQ.I}]+p_2\cdot[\text{EQ.II}],\qquad p_2\cdot[\text{EQ.I}]-p_1\cdot[\text{EQ.II}]\,,$$

we derive

$$(p_1f_1+p_2f_2)+\frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma)}\times$$
$$\times\{(p^2+M^2-M\Gamma)(p_1f_1+p_2f_2)+Mp^2f_{[34]}-p^2(p_1f_1+p_2f_2)\}+$$
$$+\frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma)}\times$$
$$\times\{(p^2+M^2+M\Gamma)(p_1f_1+p_2f_2)-Mp^2f_{[34]}-p^2(p_1f_1+p_2f_2)\}=0\,,$$

$$(p^2+M^2)(p_2f_1-p_1f_2)+$$
$$+\frac{\hat{a}\hat{b}}{M^2}\{(p^2+M^2)(p_2f_1-p_1f_2)-p^2(p_2f_1-p_1f_2)\}+$$
$$+\frac{\hat{b}\hat{a}}{M^2}\{(p^2+M^2)(p_2f_1-p_1f_2)-p^2(p_2f_1-p_1f_2)\}=0\,.$$

After elementary manipulations, they read

$$(p_1f_1+p_2f_2)+$$
$$+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\,\{(M-\Gamma)(p_1f_1+p_2f_2)+p^2f_{[34]}\}+$$
$$+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{(M+\Gamma)(p_1f_1+p_2f_2)-p^2f_{[34]}\}=0\,,\qquad(24)$$

$$(p^2+M^2)(p_2f_1-p_1f_2)+(\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2)=0\,.\qquad(25)$$

Let us write down here the remaining two equations (see (22)–(23)) as well:

$$(p^2+M^2)f_{[12]}-\frac{i}{M}(\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2)=0\,.\qquad(26)$$

$$f_{[34]} + \frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\{Mf_{[34]} - (p_1 f_1 + p_2 f_2)\} +$$
$$+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{Mf_{[34]} + (p_1 f_1 + p_2 f_2)\} = 0 \qquad (27)$$

From (25)– 26), we easily derive

$$[(\hat{a}\hat{b} + \hat{b}\hat{a}) + (p^2+M^2)](p_2 f_1 - p_1 f_2) = 0\,,$$
$$f_{[12]} = \tfrac{1}{iM}(p_2 f_1 - p_1 f_2)\,.$$

Thus, we need to investigate only the two remaining equations. We introduce the shortening notation:

$$F = f_{[34]}, \qquad G = p_1 f_1 + p_2 f_2\,,$$

and then the equations get the form

$$F + \frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}(MF-G) + \frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}(MF+G) = 0|, \qquad (28)$$

$$G + \frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\,[(M-\Gamma)G + p^2 F] +$$
$$+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}[(M+\Gamma)G - p^2 F] = 0\,. \qquad (29)$$

Let us transform the first equation (28) to the form

$$F + \tfrac{1}{M(p^2+M^2-M\Gamma)(p^2+M^2+M\Gamma)} \times \{(p^2+M^2+M\Gamma)\,\hat{a}\hat{b}\,(MF-G))+$$
$$+(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,(MF+G)\} = 0\,,$$

which after elementary manipulation yields

$$F + \tfrac{1}{M[(p^2+M^2)^2 - M^2\Gamma^2]}\times$$
$$\times\{M(p^2+M^2)\,\hat{a}\hat{b}\,F + M^2\Gamma\,\hat{a}\hat{b}\,F - (p^2+M^2)\,\hat{a}\hat{b}\,G - M\Gamma\,\hat{a}\,\hat{b}G+$$
$$+M(p^2+M^2)\,\hat{b}\hat{a}\,MF - M^2\Gamma\,\hat{b}\hat{a}\,F + (p^2+M^2)\,\hat{b}\hat{a}\,G - M\Gamma\,\hat{b}\hat{a}\,G\} = 0,$$

or,

$$\text{[EQ.I]} \qquad F + \tfrac{1}{M[(p^2+M^2)^2 - M^2\Gamma^2]} \times \{ M(p^2+M^2)\,(\hat{a}\hat{b}+\hat{b}\hat{a})\,F + M^2\Gamma\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,F -$$

$$- M\Gamma\,(\hat{a}\,\hat{b}+\hat{b}\hat{a})G - (p^2+M^2)\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,G\} = 0\,.$$

We further rewrite (29) as

$$G + \tfrac{1}{M[(p^2+M^2)^2 - M^2\Gamma^2)}\{\,(p^2+M^2+M\Gamma)(M-\Gamma)\,\hat{a}\hat{b}\,G + p^2(p^2+M^2+M\Gamma)\,\hat{a}\hat{b}\,F +$$

$$+ (M+\Gamma)(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,G - p^2(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,F\} = 0\,,$$

or,

$$G + \tfrac{1}{M[(p^2+M^2)^2 - M^2\Gamma^2)} \times$$

$$\times\{\,(p^2+M^2)(M-\Gamma)\,\hat{a}\hat{b}\,G + (M^2\Gamma - M\Gamma^2)\,\hat{a}\hat{b}\,G + p^2(p^2+M^2)\,\hat{a}\hat{b}\,F + p^2M\Gamma\,\hat{a}\hat{b}\,F +$$

$$+ (M+\Gamma)(p^2+M^2)\,\hat{b}\hat{a}\,G - (M^2\Gamma + M\Gamma^2)\,\hat{b}\hat{a}\,G - p^2(p^2+M^2)\,\hat{b}\hat{a}\,F + p^2M\Gamma\,\hat{b}\hat{a}\,F\} = 0\,,$$

so we infer

$$\text{[EQ.II]} \qquad G + \tfrac{1}{M[(p^2+M^2)^2 - M^2\Gamma^2)} \times \{\,M(p^2+M^2)(\hat{a}\hat{b}+\hat{b}\hat{a})\,G - \Gamma(p^2+M^2)(\hat{a}\hat{b}-\hat{b}\hat{a})G -$$

$$- M\Gamma^2(\,\hat{a}\hat{b}+\hat{b}\hat{a})\,G + \Gamma M^2(\hat{a}\hat{b}-\hat{b}\hat{a})G + M\Gamma p^2\,(\hat{a}\hat{b}+\hat{b}\hat{a})\,F + p^2(p^2+M^2)\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,F\} = 0\,,$$

Let us combine the equations [EQ.I] and [EQ.II] as follows:

$$-\Gamma p^2\cdot\text{[EQ.I]} + (p^2+M^2)\cdot\text{[EQ.II]}, \qquad (p^2+M^2-\Gamma^2)\cdot\text{[EQ.I]} + \Gamma\cdot\cdot\text{[EQ.II]} = \ldots;$$

this leads to

$$[(\hat{a}\hat{b}+\hat{b}\hat{a}) + p^2 + M^2]G - \Gamma p^2 F + \frac{p^2}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})F = 0\,,$$

$$[(\hat{a}\hat{b}+\hat{b}\hat{a}) + p^2 + M^2]F + \Gamma G - \frac{1}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})G - \Gamma^2 F + \frac{\Gamma}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})F = 0\,.$$

Taking into account the explicit form of the operators $\hat{a}$, $\hat{b}$, and by considering

$$\hat{a} = \frac{1}{\sqrt{2}}\left(+\frac{+d}{dx_3} + ieEx_3 + i\epsilon\right), \quad \hat{b} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx_3} + ieEx_3 + i\epsilon\right),$$

we get

$$(\hat{a}\hat{b}-\hat{b}\hat{a}) = ieE.$$

Then the last equations rewrite in the simpler form

$$[(\hat{a}\hat{b}+\hat{b}\hat{a}) + p^2 + M^2]G = p^2(\Gamma - \tfrac{ieE}{M})F\,,$$

$$[(\hat{a}\hat{b}+\hat{b}\hat{a}) + p^2 + M^2]F = -\left(\Gamma - \tfrac{ieE}{M}\right)G + \Gamma\left(\Gamma - \tfrac{ieE}{M}\right)F\,.$$

Let us introduce the notation $ieE = E_0$; then the system is written as

$$\left(\Gamma - \tfrac{E_0}{M}\right)^{-1} \left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2\right] G = p^2 F \,,$$
$$\left(\Gamma - \tfrac{E_0}{M}\right)^{-1} \left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2\right] F = -G + \Gamma F \,.$$

This sub-system is solved by diagonalizing the mixing matrix. To this aim, let us introduce the new functions

$$\Phi_1 = G - \lambda_1 F \,, \quad \Phi_2 = G - \lambda_2 F \,,$$

where

$$\lambda_1 = \frac{\Gamma + \sqrt{\Gamma^2 - 4p^2}}{2} \,, \quad \lambda_2 = \frac{\Gamma - \sqrt{\Gamma^2 - 4p^2}}{2} \,.$$

So we get two independent equations:

$$\left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2] - \lambda'_1\right] \Phi_1 = 0 \,, \quad \lambda'_1 = \lambda_1 \left(\Gamma - \tfrac{E_0}{M}\right) \;;$$
$$\left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2] - \lambda'_2\right] \Phi_2 = 0 \,, \quad \lambda'_2 = \lambda_2 \left(\Gamma - \tfrac{E_0}{M}\right) \,.$$

For the second order operator, we have the explicit for, e.g., $x_3 = z$:

$$(\hat{a}\hat{b} + \hat{b}\hat{a}) = -\frac{d^2}{dz^2} - e^2E^2z^2 - 2eE\epsilon z - \epsilon^2 \,.$$

Thus, we get the equation[6]:

$$\left[\frac{d^2}{dz^2} + (e^2E^2z^2 + 2eE\epsilon z + \epsilon^2) - \mu^2 p^2 - M^2 + \lambda'_{1,2}\right] \Phi_{1,2}(z) = 0 \,,$$

where $\mu^2 = p^2 - M^2 + \lambda'_{1,2}$.

This task coincides with that which arises for the scalar Klein–Fock–Gordon particle in external uniform electric field modified by an anomalous magnetic moment.

[6]We consider both variants.

4. Restrictions on the Values of Anomalous Magnetic Moment

On physical grounds, the above parameter μ^2 must be positive, for both cases $\lambda = \lambda'_{1,2}$:

$$\mu^2 = M^2 + p^2 - \left(\Gamma - \frac{ieE}{M}\right) \frac{\Gamma \pm \sqrt{\Gamma^2 - 4p^2}}{2} > 0\,.$$

We take into account that $\Gamma = i\Gamma_0$[7]:

$$\mu^2 = M^2 + p^2 + \left(\Gamma_0 - \frac{eE}{M}\right) \frac{\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}}{2} > 0\,.$$

Clearly, the region for Γ_0, given by[8]:

$$\Gamma_0 - \frac{eE}{M} > 0 \quad (eE > 0),$$

has no physical sense, because it does not contain the vicinity of the point $\Gamma_0 = 0$. So, in the following we assume that

$$\Gamma_0 - y < 0, \quad y = \frac{eE}{M} > 0\,.$$

Then, the main inequality takes the form

$$\frac{2(M^2 + p^2)}{(y - \Gamma_0)} > \Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}\,.$$

Let us study the variant $\Gamma_0 < 0$, $(-)$ – lower sign :

$$\frac{2(M^2 + p^2)}{(y - \Gamma_0)} > \Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2}\,,$$

which is valid without any additional restrictions.

We first address the variant $\Gamma_0 < 0$, $(+)$ – upper sign:

$$\frac{2(M^2 + p^2)}{(y - \Gamma_0)} > \Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2}\,.$$

[7] In our considerations, Γ_0 is real-valued

[8] For definiteness, we assume that $eE > 0$

This yields

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} - \Gamma_0 > +\sqrt{\Gamma_0^2 + 4p^2}\ ;$$

which, after squaring, takes the form

$$\frac{4(M^2+p^2)^2}{(y-\Gamma_0)^2} - 2\Gamma_0 \frac{2(M^2+p^2)}{(y-\Gamma_0)} - 4p^2 > 0\ ,$$

or

$$4(M^2+p^2)^2 - 4\Gamma_0(M^2+p^2)(y-\Gamma_0) - 4p^2(y-\Gamma_0)^2 > 0\ .$$

It is convenient to use the variable x:

$$y - \Gamma_0 = x > 0\,,$$

which leads to

$$(M^2+p^2)^2 - (y-x)x(M^2+p^2) - p^2x^2 > 0\ ,$$

equivalent to

$$x^2 - 2x\frac{(M^2+p^2)y}{2M^2} + \frac{(M^2+p^2)^2}{M^2} > 0\ .$$

The roots of this quadratic equations are

$$x_{1,2} = \frac{(M^2+p^2)}{2M^2} \pm \frac{(M^2+p^2)y}{2M^2}\sqrt{y^2-4M^2},$$

and the whole parabola lays above the horizontal axes only if the discriminant is negative, and this yields

$$y^2 - 4M^2 < 0 \quad \Longrightarrow \quad \frac{eE}{M} < 2M\ .$$

Thus, we get the essential restriction on the magnitude of the electric field $\Gamma_0 < 0$.

We further address the variant $\Gamma_0 > 0, (-)$ – lower sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} > \Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2}\ ;$$

which is evidently valid.

Let us now consider the variant $\Gamma_0 > 0$, $(+)$ – upper sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} - \Gamma_0 > \sqrt{\Gamma_0^2 + 4p^2} \; .$$

After squaring this inequality, we obtain

$$4(M^2+p^2)^2 - 4\Gamma_0(M^2+p^2)(y-\Gamma_0) - 4p^2(y-\Gamma_0)^2 > 0 \; .$$

Now, we can repeat the previous analysis. With the help of the variable $y - \Gamma_0 = x$, we get

$$(M^2+p^2)^2 - (y-x)x(M^2+p^2) - p^2x^2 > 0$$

or

$$x^2 - 2x\frac{(M^2+p^2)y}{2M^2} + \frac{(M^2+p^2)^2}{M^2} > 0 \; ;$$

the corresponding roots are

$$x_{1,2} = \frac{(M^2+p^2)}{2M^2} \pm \frac{(M^2+p^2)y}{2M^2}\sqrt{y^2 - 4M^2}.$$

The whole parabola lays above the horizontal axis only if the discriminant is negative, which yields

$$y^2 - 4M^2 < 0 \quad \Longrightarrow \quad \frac{eE}{M} < 2M \; .$$

Considering the previous assumption $0 < \Gamma_0$, we derive $0 < \Gamma_0 < \frac{eE}{M}$.

All in all, we conclude that μ^2 is positive, if the following double inequality is valid

$$\Gamma_0 < \frac{eE}{M} < 2M \; .$$

5. Solving the Differential Equation

We start with the equation

$$\left(\frac{d^2}{dz^2} + (\epsilon + eEz)^2 - \mu^2 \right) \Phi(z) = 0 \; , \quad \mu^2 = M^2 + p^2 - \lambda'_{1,2} > 0 \; . \quad (30)$$

We remark that this equation, after its transforming to a new variable x is – from mathematical point of view – very similar to that arising for the non-relativistic quantum harmonic oscillator:

$$x = \frac{\epsilon + eEz}{eE}, \quad \left(\frac{d^2}{dx^2} - \mu^2 + (eE)^2 x^2\right)\Phi = 0\,, \left(\frac{d^2}{dx^2} + E - kx^2\right) f = 0\,.$$

Let us use in (30) a new variable Z

$$Z = i\frac{(\epsilon + eEz)^2}{eE} \quad (\text{let it be} \quad \sigma = \frac{\mu^2}{4eE},\ eE > 0)\,.$$

Then we obtain an equation of the form

$$\left(\frac{d^2}{dZ^2} + \frac{1/2}{Z}\frac{d}{dZ} - \frac{1}{4} + \frac{i\sigma}{Z}\right)\Phi(Z) = 0\,,$$

which has two singular points. The point $Z = 0$ is regular, and the behavior of the solutions in its neighborhood may be as follows

$$Z \to 0, \quad \Phi(Z) = Z^A, \qquad A \in \left\{0, \frac{1}{2}\right\}\,.$$

The point $Z = \infty$ is an irregular singularity of rank 2. Indeed, in terms of the variable $y = Z^{-1}$, the above equation reads

$$\left(\frac{d^2}{dy^2} + \frac{3/2}{y}\frac{d}{dy} - \frac{1}{4y^4} + \frac{i\sigma}{y^3}\right)\Phi = 0\,.$$

When $y \to 0$, the corresponding asymptotic structure is given by

$$y \to 0, \qquad \Phi = y^C e^{D/y}, \quad \Phi' = Cy^{C-1}e^{D/y} - Dy^{C-2}e^{D/y},$$

$$\Phi'' = C(C-1)y^{C-2}e^{D/y} - CDy^{C-3}e^{D/y} - D(C-2)y^{C-3}e^{D/y} + D^2y^{C-4}e^{D/y}.$$

Then, the above equation gives

$$\frac{C(C-1)}{y^2} - \frac{2CD - 2D}{y^3} + \frac{D^2}{y^4} + \frac{3}{2}\frac{C}{y^2} - \frac{3}{2}\frac{D}{y^3} - \frac{1}{4y^4} + \frac{i\sigma}{y^3} = 0\,.$$

We retain only the main terms proportional to y^{-3} and y^{-4}, and require that their coefficients vanish:

$$D^2 - \frac{1}{4} = 0, \qquad -2CD + 2D - \frac{3}{2}D + i\sigma = 0\,,$$

whence it follows

$$D_1 = +\frac{1}{2}, \quad C_1 = \frac{1}{4} + i\sigma \; ; \qquad D_2 = -\frac{1}{2}, \quad C_2 = \frac{1}{4} - i\sigma \; .$$

Thus, at infinity, two asymptotics are possible

$$Z \to \infty, \qquad \Phi = Z^{-C} e^{DZ} = \begin{cases} Z^{-C_1} e^{D_1 Z} = Z^{-1/4 - i\sigma} e^{+Z/2} \\ Z^{-C_2} e^{D_2 Z} = Z^{-1/4 + i\sigma} e^{-Z/2}, \end{cases}$$

where[9]

$$Z = i\tfrac{(\epsilon + eEz)^2}{eE} = iZ_0, \qquad Z_0 > 0 \; , \quad e^{\pm Z/2} = e^{\pm i Z_0/2},$$
$$Z^{-1/4 \mp i\sigma} = \left(e^{\ln iZ_0}\right)^{-1/4 \mp i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4 \mp i\sigma} \; .$$

We shall further construct a solution in the whole region of Z. We start with the substitution

$$\Phi(Z) = Z^A \, e^{BZ} \, f(Z) \; .$$

This leads to

$$\left(Z \frac{d^2}{dZ^2} + (2A + \frac{1}{2} + 2BZ) \frac{d}{dZ} + (B^2 - \frac{1}{4}) \, Z + \frac{A\,(2A-1)}{2Z} + 2AB + \frac{B}{2} + i\sigma \right) f(Z) = 0 \, .$$

We fix $A \in \{0 \, , 1/2\}$, $B = -1/2$; then the equation becomes simpler

$$\left(Z \frac{d^2}{dZ^2} + (2A + 1/2 - Z) \frac{d}{dy} - (A + 1/4 - i\sigma) \right) f(Z) = 0 \, ,$$

which coincides with the confluent hypergeometric equation with

$$a = A + 1/4 - i\sigma \, , \qquad c = 2A + 1/2 \, , \qquad f(Z) = Z^A \, e^{-Z/2} \, F(a, c; Z).$$

Without loss of generality, we may take the value $A = 0$:

$$A = 0, \quad a = 1/4 - i\sigma \, , \quad c = +1/2 \, , \quad \Phi(Z) = e^{-Z/2} \, f(Z) \; .$$

Let us consider two definite independent solutions of the confluent hypergeometric equation[10]:

$$Y_1(Z) = F(a, c; Z) = e^Z F(c - a, c; -Z)$$
$$Y_2(Z) = Z^{1-c} F(a - c + 1, 2 - c; Z) = Z^{1-c} e^Z F(1 - a, 2 - c; -Z).$$

[9] We use the main branch of the logarithmic function.
[10] Note the equivalent representations for each solution.

These two lead to the corresponding Φ's:

$$\Phi_1 = e^{-Z/2} F(a, c; Z) = e^{+Z/2} F(c-a, c; -Z) \; ;$$
$$\Phi_2 = e^{-Z/2} Z^{1-c} F(a-c+1, 2-c; Z) = Z^{1-c} e^{+Z/2} F(1-a, 2-c; -Z) \; .$$

By taking into account the identities

$$c = \frac{1}{2}, \qquad a = \frac{1}{4} - i\sigma, \quad c - a = \frac{1}{4} + i\sigma = a^*, \quad c = c^* = \frac{1}{2}, \qquad Z^* = -Z \; ,$$
$$a - c + 1 = \frac{3}{4} - i\sigma = (1-a)^*, \quad (2-c) = (2-c)^* = \frac{3}{2} \; ,$$

we conclude that the first solution $\Phi_1(Z)$ is given by a real-valued function, whereas the second one, $\Phi_2(Z)$, has the following property with respect to the complex conjugation

$$\Phi_1(Z) = +[\Phi_1(Z)]^*, \qquad \Phi_2(Z) = i[\Phi_2(Z)]^* \; .$$

This behavior of $\Phi_2(Z)$ can be presented as the property of real-valuedness, if one uses another normalizing factor

$$\bar{\Phi}_2(Z) = \frac{1-i}{\sqrt{2}} \Phi_2(Z) = \left(\frac{1-i}{\sqrt{2}} \Phi_2(Z) \right)^* = (\bar{\Phi}_2(Z))^* \; . \qquad (31)$$

For small values of Z, the solutions behave as follows

$$Y_1(Z) \approx 1, \qquad Y_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\tfrac{i}{eE}} \, (\epsilon + eEz) \; ;$$
$$\Phi_1(Z) \approx 1, \qquad \Phi_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\tfrac{i}{eE}} \; (\epsilon + eEz) \; .$$

For large values of $Z = iZ_0$, $Z_0 \to +\infty$, on can employ the known asymptotic formulas

$$F(a, c, Z) = \left(\frac{\Gamma(c)}{\Gamma(c-a)} (-Z)^{-a} + ... \right) + \left(\frac{\Gamma(c)}{\Gamma(a)} e^Z Z^{a-c} + ... \right) \; .$$

In this way, we derive[11]

$$(-Z)^{-a} = (-iZ_0)^{-1/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2} \right)^{-1/4+i\sigma} = e^{-(-1/4+i\sigma)i\pi/2} \; e^{(-1/4+i\sigma)\ln Z_0} \; ,$$

[11] We use (again) the main branch of the logarithmic function

$$Z^{a-c} = (iZ_0)^{-1/4-i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4-i\sigma} = e^{+(-1/4-i\sigma)i\pi/2}\, e^{(-1/4-i\sigma)\ln Z_0}\,,$$

$$\frac{\Gamma(c)}{\Gamma(c-a)} = \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}, \qquad \frac{\Gamma(c)}{\Gamma(a)} = \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)},$$

so we get

$$Y_1(Z) = F(a,c,Z) = e^{iZ_0/2}\times$$

$$\times\left\{\frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}\; e^{-(-1/4+i\sigma)i\pi/2}\; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \right.$$
$$\left. + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2}\; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2}\right\}. \qquad (32)$$

From (32), it follows the asymptotic form for

$$\Phi_1(Z) = \left\{\frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}\; e^{-(-1/4+i\sigma)i\pi/2}\; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \right.$$
$$\left. + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2}\; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2}\right\}. \qquad (33)$$

As it should be, we notice the sum of the two conjugate terms.

In a similar manner, we study at infinity the function $F(a-c+1, 2-c; Z)$:

$$F(a-c+1, 2-c, Z) = \left(\frac{\Gamma(2-c)}{\Gamma(1-a)}(-Z)^{-a+c-1} + ...\right) + \left(\frac{\Gamma(2-c)}{\Gamma(a-c+1)} e^Z Z^{a-1} + ...\right)\,.$$

Taking into account identities

$$(-Z)^{-a+c-1} = (-iZ_0)^{-3/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2}\right)^{-3/4+i\sigma} = e^{-(-3/4+i\sigma)i\pi/2}\, e^{(-3/4+i\sigma)\ln Z_0}\,,$$

$$Z^{a-1} = (iZ_0)^{-3/4-i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-3/4-i\sigma} = e^{+(-3/4-i\sigma)i\pi/2}\, e^{(-3/4-i\sigma)\ln Z_0}\,,$$

$$\frac{\Gamma(2-c)}{\Gamma(1-a)} = \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}, \qquad \frac{\Gamma(2-c)}{\Gamma(a-c+1)} = \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)},$$

we derive the asymptotic formula

$$F(a-c+1, 2-c, Z) = e^{iZ_0/2}\times\left\{\frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}\; e^{-(-3/4+i\sigma)i\pi/2}\; e^{(-3/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \right.$$
$$\left. + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2}\; e^{(-3/4-i\sigma)\ln Z_0} e^{+iZ_0/2}\right\}.$$

From this, for the function $\Phi_2(Z)$ we infer[12]

$$\Phi_2(Z) = \sqrt{Z} Z^{1/2} F(a-c+1, 2-c, Z) = e^{i\pi/4} \times$$

$$\times \left\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)} \; e^{-(-3/4+i\sigma)i\pi/2} \; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \right.$$
$$\left. + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2} \; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} . \qquad (34)$$

This results agrees with the previously obtained formula (31).

We can construct linearly independent solutions which do not behave at infinity as (quasi-)real superpositions of complex-valued functions. To this end, we should employ another pair of linearly independent solutions

$$Y_5(Z) = \Psi(a, c; Z), \qquad Y_7(Z) = e^Z \Psi(c-a, c; -Z) .$$

The two pairs $\{Y_5, Y_7\}$ and $\{Y_1, Y_2\}$ relate to each other by the Kummer formulas

$$Y_5 = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \, Y_1 + \frac{\Gamma(c-1)}{\Gamma(a)} \, Y_2 \, , \qquad Y_7 = \frac{\Gamma(1-c)}{\Gamma(1-a)} \, Y_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)} \, e^{i\pi c} \, Y_2 \, .$$

For large Z, ($| \, Z \, | \to \infty$), the following asymptotic formula is valid

$$Y_5 = \Psi(a, c; Z) = Z^{-a} = (iZ_0)^{-1/4+i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\sigma} ,$$
$$Y_7(Z) = e^Z \Psi(c-a, c; -Z) = e^Z (-iZ_0)^{a-c} = e^{iZ_0} (-iZ_0)^{-1/4-i\sigma} = e^{iZ_0} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\sigma} .$$

These formulas – after passing to the functions $\Phi(Z)$ – take the form

$$\Phi_5 = e^{-Z/2} Y_5 = e^{-iZ_0/2} \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\sigma} ,$$
$$\Phi_7 = e^{-Z/2} Y_7(Z) = e^{+iZ_0/2} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\sigma} .$$

We see that these functions are conjugate to each other; only these ones enter the superpositions (33) and (34).

[12] We recall that $\sqrt{Z} = e^{(1/2)(\ln Z_0 + i\pi/2)}$.

6. Spin 1 Particle with Vanishing Electric Ctge

Let us derive the corresponding result for the case of a neutral particle. Formally, this can be obtained ny means of the following limiting procedure

$$e \to 0 , \quad \frac{2E}{M}\lambda \to \infty , \quad \Gamma = \pm \frac{2eE}{M}\lambda \to \frac{2E}{M}\Lambda ,$$

where λ is a dimensionless parameter; the new Λ has the electric charge dimension. We consider below only the main relations:

$$\hat{a} = \tfrac{1}{\sqrt{2}} \left(+ \tfrac{d}{dx_3} + i\epsilon \right) , \quad \hat{b} = \tfrac{1}{\sqrt{2}} \left(- \tfrac{d}{dx_3} + i\epsilon \right) ,$$

$$(\hat{a}\hat{b} + \hat{b}\hat{a}) = -\tfrac{d^2}{dz^2} - \epsilon^2 , \quad (\hat{a}\hat{b} - \hat{b}\hat{a}) = 0 ;$$

$$f_{[12]} = \tfrac{1}{iM} (p_2 f_1 - p_1 f_2) , \qquad \left(\tfrac{d^2}{dz^2} + \epsilon^2 - p^2 - M^2 \right) (p_2 f_1 - p_1 f_2) = 0 ;$$

$$F = f_{[34]} , \qquad G = p_1 f_1 + p_2 f_2 ,$$

$$\Gamma^{-1} \left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2 \right] G = p^2 F ,$$

$$\Gamma^{-1} \left[(\hat{a}\hat{b} + \hat{b}\hat{a}) + p^2 + M^2 \right] F = -G + \Gamma F ;$$

$$\Phi_1 = G - \lambda_1 F , \quad \Phi_2 = G - \lambda_2 F ,$$

$$\lambda_1 = \tfrac{1}{2} (\Gamma + \sqrt{\Gamma^2 - 4p^2}) , \quad \lambda_2 = \tfrac{1}{2} (\Gamma - \sqrt{\Gamma^2 - 4p^2}) ;$$

$$\left(\tfrac{d^2}{dz^2} + \epsilon^2 - p^2 - M^2 + \Gamma \lambda_{1,2} \right) \Phi_{1,2}(z) = 0 .$$

Let us introduce the notation

$$\Delta = \epsilon^2 - p^2 - M^2 > 0 , \quad \Delta + \Gamma \lambda_{1,2} = p_z^2 ,$$

where

$$\Gamma \lambda_{1,2} = \frac{\Gamma}{2} (\Gamma \pm \sqrt{\Gamma^2 - 4p^2}) = \frac{i\Gamma_0}{2} \left(i\Gamma_0 \pm \sqrt{-\Gamma_0^2 - 4p^2} \right) = -\frac{\Gamma_0}{2} \left(\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2} \right) .$$

The solutions will have the form of plane waves $\Phi_{1,2}(z) = e^{\pm i p_3 z}$, only if

$$p_z^2 = \Delta - \frac{1}{2} \Gamma_0 \left(\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2} \right) > 0 .$$

Let us study this inequality. It is convenient to consider separately the following four subcases:
1. upper sign $(+)$, $\Gamma_0 > 0$;
2. lower sign $(-)$, $\Gamma_0 > 0$;
3. upper sign $(+)$ $\Gamma_0 < 0$;
4. lower sign $(-)$ $\Gamma_0 < 0$.

Consider variant 1:

$$\Gamma_0 > 0, \qquad 2\Delta > \Gamma_0 \left(\Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2} \right), \qquad 2\Delta - \Gamma_0^2 > \Gamma_0 \sqrt{\Gamma_0^2 + 4p^2} \; ;$$

here we must impose the obvious restriction

$$\Gamma_0^2 < 2\Delta \; ,$$

and we further derive

$$4\Delta^2 - 4\Delta\Gamma_0^2 + \Gamma_0^4 > \Gamma_0^2(\Gamma_0^2 + 4p^2) \quad \Longrightarrow \quad \Delta^2 - \Delta\Gamma_0^2 > \Gamma_0^2 p^2,$$

that is $\Gamma_0^2 < \frac{\Delta^2}{\Delta + p^2}$. We can readily check the inequality:

$$2\Delta > \frac{\Delta^2}{\Delta + p^2},$$

and thus conclude by the restriction

$$1. \qquad 0 < \Gamma_0 < \frac{\Delta}{\sqrt{\Delta + p^2}}, \qquad \Gamma_0 > 0 \; . \quad \Delta = \epsilon^2 - p^2 - M^2 > 0 \; .$$

Now consider variant 2:

$$2. \qquad \Gamma_0 > 0, \qquad 2\Delta > \Gamma_0 \left(\Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2} \right) \; ;$$

evidently, this relationship is always valid.

Then, variant 3:

$$3. \qquad \Gamma_0 < 0, \qquad 2\Delta > \Gamma_0 \left(\Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2} \right) \; ;$$

this relationship is always valid.

Finally, we address variant 4:

$$4. \qquad \Gamma_0 < 0, \qquad 2\Delta > \Gamma_0 \left(\Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2} \right) = (-\Gamma_0) \left((-\Gamma_0) + \sqrt{\Gamma_0^2 + 4p^2} \right) ,$$

where by using the results for the case 1, we obtain

$$\Gamma_0^2 < \frac{\Delta^2}{\Delta + p^2}, \qquad \Gamma_0 < 0 \; .$$

By summing, we conclude that the parameter Γ_0 must lay within the following bounds:

$$\Gamma_0 < + \frac{\Delta}{\sqrt{\Delta + p^2}} \; , \qquad \Delta = \epsilon^2 - p^2 - M^2 > 0 \; .$$

Conclusion

Within the matrix 10-dimensional Duffin-Kemmer-Petiau formalism applied to the Shamaly-Capri field, we study the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components Ψ_0, Ψ_+, Ψ_+. It is enough to solve the equation for the main component Φ_0, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein–Fock–Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

References

[1] E. S. Fradkin. To the theory of particles with higher spins. *JETP*. 1950. Vol. 20. P. 27–38.

[2] V. Ya. Feinberg. To interaction of the particlesof higher spins with electromagnetic fields. *Trudy FIAN. USSR*. 1955. Vol. 6. P. 269–332.

[3] M. Petras. A note to Bhabha's equation for a particle with maximum spin 3/2 M. Petras // *Czech. J. Phys.*-1955. Vol. 5. Vol. 3. P. 418–419.

[4] I. Ulehla. Anomalous equations for spin 1/2 particles. *JETP*. 1957. Vol. 33. P. 473–477.

[5] F. I. Fedorov, V. A. Pletyukhov. Wave equatioons withe repeated representations of the Lorentz group Intejer spin. Vesti of NASB. Ser. fiz.-mat. nauk. 1969. Vol. 6. P. 81–88; Wave equatioons withe repeated representations of the Lorentz group Half-intejer spin. Vesti of NASB. Ser. fiz.-mat. nauk. 1970. Vol. 3. P. 78–83; Wave equatioons withe repeated representations of the Lorentz group for spin 0 particle. Vesti of NASB. Ser. fiz.-mat. nauk. 1970. Vol 2. P. 79–85; Wave equatioons withe repeated representations of the Lorentz group for spin 1 particle. *Vesti of NASB. Ser. fiz.-mat. nauk.* 1970. Vol. 3. P. 84–92.

[6] A. Z. Capri. Nonuniqueness of the spin 1/2 equation. *Phys. Rev.* 1969. Vol. 178. Vol. 5. P. 181–1815; First order wave equations for half-odd-integral spin. *Phys. Rev.* 1969. Vol. 178. P. 2427–2433; Electromagnetic properties of a new spin-1/2 field. *Progr. Theor. Phys.* 1972. Vol. 48. P. 1364–1374.

[7] A. Shamaly, A. Z. Capri. First-order wave equations for integral spin. Nuovo Cimento. B. 1971. Vol. 2. P. 235–253; Unified theories for massive spin 1 fields. *Can. J. Phys.* 1973. Vol. 51. P. 1467–1470.

[8] M. A. K. Khalil. Properties of a 20-component spin 1/2 relativictic wave equation. Phys. Rev. D. 1977. Vol. 15. P. 1532–1539: Barnacle equivalence structure in relativistic wave equation. Progr. Theor. Phys. 1978. Vol. 60. P. 1559–1579; An equivalence of relativistic field equations. Nuovo Cimento. A. 1978. Vol. 45. P. 389–404; Reducible relativistic wave equations. *J. Phys. A. Math. and Gen.* 1979. Vol. 12. P. 649–663.

[9] A. A. Bogus, V. V. Kisel. Equations with epeated representations of the Lorent group and interaction of the Pauli type. *Izvestia Wuzov. Phizika*. 1984. Vol. 1. P. 23–27.

[10] A. A. Bogus, V. V. Kisel, L. G. Moroz, M. I. Levchuk. To desription pf scalar paticle with polarizability. Covariant methods in theoretical physics. *Elementary particles physics and relativity theory.* IP NAN BSSR. Minsk, 1981. P. 81–90.

[11] A. A. Bogus, V. V. Kisel, F. I. Fedorov. On interpretation of subsidiary components of wave funcrtions in electromagnetic interaction. *Doklady NAN BSSR*. 1984. Vol. 277. Vol. 2. P. 343–346.

[12] V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. *Petrash theory for particles with spin 1/2 in curved space-time.* Preprint 737. Institute of Physics of NAS of Belarus: Minsk, 2002.

[13] A. A. Bogush, V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. Petras theory for particles with spin 1/2 in curved space-time. *Vesti of NASB. Ser. fiz.-mat. nauk.* 2002. Vol. 1. P. 63–68.

[14] V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. Spin 1/2 particle with anomalous magnetic moment in a curved space-time, non-relativistic approximation. Pages 36-42 in: Proceedings of 11-th International School and Conference "Foundation and Advances in Nonlinear Science", Eds.: Kuvshinov V.I., Krylov G.G., Minsk, 2004.

[15] A. A. Bogush, V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. *Petras Theory of a Spin-1/2 Particle in Electromagnetic and Gravitational Fields*. 44 pages, 18 Apr 2006. – arXiv:hep-th/0604109.

[16] V. V. Kisel. *Relativistic wave equations with extended sets of represetations*. Kandid. dissert. Minsk, 1984. 11 p.

[17] V. V. Kisel, V. M. Red'kov. Shamaly-Capri equation and additional interaction of a vector particle with gravitational field. Covariant methods in theoretical physics. *Elementary particles physics and relativity theory*. Institute of Physics, NAS of Belarus. Minsk, 2001. P. 107–112.

[18] V. A. Pletyukhov, V. M. Red'kov, V. I. Strazhev. *Relativistic wave equations and intrinsic degrees of fredom*. Belarusian Science: Minsk, 2015. 328 p. (in Russian).

[19] V. M. Red'kov. Fields of particles in Riemannian space and the Lorentz group. Belarusian Science: Minsk, 2009. – 486 p. (in Russian).

[20] F. I. Fedorov. *The Lorentz group*. – Moscow, 1979. – 384 p.

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Chapter 3

TECHNIQUES OF PROJECTIVE OPERATORS USED TO CONSTRUCT SOLUTIONS FOR A SPIN 1 PARTICLE WITH ANOMALOUS MAGNETIC MOMENT IN THE EXTERNAL UNIFORM MAGNETIC FIELD

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Abstract

Within *the matrix 10-dimensional Duffin–Kemmer-Petiau formalism* applied to *the Shamaly–Capri field*, we study the behavior of a vector

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particle with anomalous magnetic moment in presence of an *external uniform magnetic field.*

The separation of variables in the wave equation is performed using projective operator techniques and the theory of DKP-algebras. The problem is reduced to a system of 2-nd order differential equations for three independent functions, which is solved in terms of confluent hypergeometric functions. Three series of energy levels are found, of which two substantially differ from those for spin 1 particles without anomalous magnetic moment. For assigning to them physical sense for all the values of the main quantum number $n = 0, 1, 2, ...$, one must impose special restrictions on a parameter related to the anomalous moment. Otherwise, only some part of the energy levels corresponds to bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in the occurrence of a space scaling of the arguments of the wave functions, compared to a particle without such a moment.

Keywords Duffin–Kemmer–Petiau algebra, projective operators, spin 1 particle, anomalous magnetic moment, magnetic field, exact solutions, bound states

1. Introduction

Commonly, we shall use only the simplest wave equations for fundamental particles of spin $0, 1/2, 1$. Meanwhile, it is known that other more complicated equations can be proposed for particles with such spins, which are based on the application of extended sets of Lorentz group representations (see [1]-[16]). Such generalized wave equations allow to describe more complicated objects, which have besides mass, spin, and electric charge, other electromagnetic characteristics, like polarizability or anomalous magnetic moment. These additional characteristics manifest themselves explicitly in presence of external electromagnetic fields.

In particular, within this approach Petras [3] proposed a 20-component theory for spin 1/2 particle, which-after excluding 16 subsidiary components - turns to be equivalent to the Dirac particle theory modified by presence of the Pauli interaction term. In other words, this theory describes a spin 1/2 particle with anomalous magnetic moment.

A similar equation was proposed by Shamaly–Capri [6, 7] for spin 1 particles (also see [16, 17]). In the following, we investigate and solve this wave equation in presence of the external uniform magnetic field. The generalized formulas for Landau energy levels are derived, and the corresponding wave functions are constructed. The new formulas for energies in presence of external magnetic field, in principle, allow to experimentally distinguish such a particle. The restriction to the case of neutral vector boson (the uncharged spin 1/2 particle with anomalous magnetic moment) is performed in Sec. 2 – Sec. 6.

In Section 7 we give some details of the general theory of the Shamaly–Capri particle; in particular, we describe some features of this theory extended to General Relativity.

2. The Separation of Variables

The wave equation for spin 1 particle with anomalous magnetic moment [6, 7] may be formulated in the form

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} P J_{[\mu\nu]} + M\right)\Psi = 0\,, \tag{1}$$

where the 10-dimensional wave function and the DKP-matrices are used:

$$\Psi = \left|\begin{array}{c}\Psi_\mu \\ \Psi_{[\mu\nu]}\end{array}\right|, \qquad J_{[\mu\nu]} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu\,,$$

where P stands for a projective operator separating from Ψ its vector component Ψ_μ; $D_\mu = \partial_\mu - ieA_\mu$; λ_3 denotes an arbitrary complex number (see notation in Sec. 7). In tensor form, (1) is[1]:

$$D_\mu\Psi_\nu - D_\nu\Psi_\mu + M\Psi_{[\mu\nu]} = 0\,,$$

$$D_\nu\Psi_{[\mu\nu]} \pm 2\tfrac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]}\Psi_\nu + M\Psi_\mu = 0\,.$$

By using DKP-matrices, we apply the method of generalized Kronecker's symbols [20] [2]:

$$\beta_\mu = e^{\nu,[\nu\mu]} + e^{[\nu\mu],\nu}, \quad P = e^{\nu,\nu}\,,$$

$$(e^{A,B})_{CD} = \delta_{AC}\delta_{BD}\,, \quad e^{A,B}e^{C,D}\delta_{BC}e^{A,D}\,,$$

$$\delta_{[\mu\nu],[\rho\sigma]} = \tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\,,$$

[1] In Minkowski space, the metric with imaginary unit is used, since $x_4 = ict$.

[2] The indexes $A(B, C, D, ...)$ take the values $1, 2, 3, 4$, [23], [31], [12], [14], [24], [34].

and the main relationships in the DKP algebra read:

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\rho + \delta_{\rho\nu} \beta_\mu \, , \quad [\beta_\lambda, J_{\rho\sigma}]_- = \delta_{\lambda\rho} \beta_\sigma - \delta_{\lambda\sigma} \beta_\rho \, ;$$

We use the following representation for DKP-matrices:

$$\beta_1 = \left| \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right| ,$$

$$\beta_2 = \left| \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right| ,$$

$$\beta_3 = \left| \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right| ,$$

$$\beta_4 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} .$$

A uniform magnetic field is specified by the relations

$$A_1 = -\frac{1}{2}Bx_2, \quad A_2 = \frac{1}{2}Bx_1, \quad A_3 = 0, \quad A_4 = 0\,, \quad \vec{B} = (0,0,B)\,,$$

$$F_{[\mu\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{[12} = -F_{[21]} = B\,, \quad F_{[13]} = 0, ...$$

The non-minimal interaction through the anomalous magnetic moment is given by the term

$$\pm\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} PJ_{[\mu\nu]} = \pm 2\frac{ie}{M}\lambda_3\lambda_3^* BPJ_{[12]}\,.$$

Correspondingly, the main equation (1) is written as

$$\Big[\beta_1(\partial_1 + \frac{ie}{2}Bx_2) + \beta_2(\partial_2 - \frac{ie}{2}Bx_1) + \beta_3\partial_3 + \beta_4\partial_4 \pm$$

$$\pm 2\frac{ie}{M}\lambda_3\lambda_3^* BPJ_{[12]} + M\Big]\,\Psi = 0\,. \qquad (2)$$

Let us introduce the matrix $Y = iJ_{[12]} = i(\beta_1\beta_2 - \beta_2\beta_1)$; it satisfies the minimal polynomial equation $Y(Y-1)(Y+1) = 0$, which permits to define tree projective operators:

$$P_0 + P_- + P_+ = I, \quad P_0 = 1 - Y^2, \quad P_+ = \frac{1}{2}Y(Y+1), \quad P_- = \frac{1}{2}Y(Y-1)$$

and resolve the wave function into three components:

$$\Psi_0 = P_0\Psi, \quad \Psi_+ = P_+\Psi, \quad \Psi_- = P_-\Psi\,, \quad \Psi = \Psi_- + \Psi_0 + \Psi_+\,.$$

By transforming (2) to cylindric coordinates

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad \tan\phi = \frac{x_2}{x_1},$$

$$\frac{\partial}{\partial x_1} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial x_2} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi};$$

we get

$$\Big[\beta_1(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} + iB_0 r\sin\phi) +$$
$$+\beta_2(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - iB_0 r\cos\phi) + (\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M)\Big]\Psi = 0, \quad (3)$$

where we use the following shortening notation:

$$\frac{eB}{2} = B_0, \quad \pm 4\frac{B_0}{M}\lambda_3\lambda_3^* = \Gamma. \quad (4)$$

We further act on (3) by P_0; by applying the identities

$$P_0\beta_3 = \beta_3 P_0, \quad P_0\beta_4 = \beta_4 P_0, \quad PY = YP,$$

$$P_0\beta_1 = \beta_1(1 - P_0) = \beta_1(P_+ + P_-), \quad P_0\beta_2 = \beta_2(1 - P_0) = \beta_2(P_+ + P_-),$$

we get

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\Psi_0 +$$

$$+\Big[\beta_1(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}) + \beta_2(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}) + iB_0 r\sin\phi\beta_1 - iB_0 r\cos\phi\beta_2\Big]\Psi_+$$

$$+\Big[\beta_1(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}) + \beta_2(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}) + iB_0 r\sin\phi\beta_1 - iB_0 r\cos\phi\beta_2\Big]\Psi_- = 0,$$

where we took into account the identities:

$$YP_0 \equiv 0 \quad \Longrightarrow \quad \Gamma Y\Psi_0 = \Gamma(YP_0)\Psi = 0.$$

By introducing the notation

$$\beta_+ = \frac{1}{\sqrt{2}}(\beta_1 + i\beta_2), \quad \beta_- = \frac{1}{\sqrt{2}}(\beta_1 - i\beta_2),$$

we can transform the previous equation to the form

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\,\Psi_0 +$$

$$+\frac{1}{\sqrt{2}}\left[e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r) + e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r)\right]\Psi_+ +$$

$$+\frac{1}{\sqrt{2}}\left[e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r) + e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r)\right]\Psi_- = 0\,.$$

By making use of the projective operators

$$P_\pm = \frac{1}{2}[\beta_1\beta_1 - 2\beta_1\beta_1\beta_2\beta_2 \pm i(\beta_1\beta_2 - \beta_2\beta_1)],$$

and the commutation relations for DKP-matrices, we prove the identities $\beta_- P_+ = \beta_+ P_- = 0$; so the above equation is written simpler

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\,\Psi_0 +$$

$$+\tfrac{1}{\sqrt{2}}e^{-i\phi}\beta_+(\tfrac{\partial}{\partial r} - \tfrac{i}{r}\tfrac{\partial}{\partial\phi} - B_0 r)\Psi_+ + \tfrac{1}{\sqrt{2}}e^{+i\phi}\beta_-(\tfrac{\partial}{\partial r} + \tfrac{i}{r}\tfrac{\partial}{\partial\phi} + B_0 r)\Psi_- = 0\,. \tag{5}$$

Now, we act on (3) by $1 - P_0 = P_+ + P_-$; this gives

$$(1-P_0)\beta_1\left(\cos\phi\tfrac{\partial}{\partial r} - \tfrac{\sin\phi}{r}\tfrac{\partial}{\partial\phi} + iB_0 r\sin\phi\right)\Psi +$$

$$+(1-P_0)\beta_2\left(\sin\phi\tfrac{\partial}{\partial r} + \tfrac{\cos\phi}{r}\tfrac{\partial}{\partial\phi} - iB_0 r\cos\phi\right)\Psi +$$

$$+\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)(\Psi_+ + \Psi_-) = 0\,.$$

By using the identities $(1-P_0)\beta_1 = \beta_1 P_0$ and $(1-P_0)\beta_2 = \beta_2 P_0$, we transform this equation to

$$\beta_1(\cos\phi\tfrac{\partial}{\partial r} - \tfrac{\sin\phi}{r}\tfrac{\partial}{\partial\phi})\Psi_0 + iB_0 r\sin\phi\beta_1\Psi_0 +$$

$$+\beta_2(\sin\phi\tfrac{\partial}{\partial r} + \tfrac{\cos\phi}{r}\tfrac{\partial}{\partial\phi})\Psi_0 - iB_0 r\cos\phi\beta_2\Psi_0 +$$

$$+\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)(\Psi_+ + \Psi_-) = 0\,,$$

from which follows that

$$\tfrac{1}{\sqrt{2}}\left[e^{-i\phi}\beta_+(\tfrac{\partial}{\partial r} - \tfrac{i}{r} - B_0 r) + e^{+i\phi}\beta_-(\tfrac{\partial}{\partial r} + \tfrac{i}{r} + B_0 r)\right]\Psi_0 +$$
$$+\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)(\Psi_+ + \Psi_-) = 0\,. \tag{6}$$

Let us act now on (6) by $\frac{1}{2}(1+Y)$. Because

$$\frac{1}{2}(1+Y)P_+ = P_+, \quad \frac{1}{2}(1+Y)P_- = 0, \quad Y\beta_- = \beta_- P_0, \qquad Y\beta_+ = -\beta_+ P_0,$$

the above equation simplifies to

$$(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M)\,\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r} + B_0 r)\Psi_0 = 0\,. \quad (7)$$

Similarly, by multiplying (6) by $\frac{1}{2}(1-Y)$ and taking into account the identities

$$\frac{1}{2}(1-Y)P_+ = 0, \quad \frac{1}{2}(1-Y)P_- = P_-, \quad Y\beta_- = \beta_- P_0, \qquad Y\beta_+ = -\beta_+ P_0,$$

we derive

$$(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M)\,\Psi_- + \frac{1}{\sqrt{2}}e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r} - B_0 r)\Psi_0 = 0\,. \quad (8)$$

Now, by considering the relations

$$YP_+ = \frac{1}{2}(Y^3 + Y^2) = \frac{1}{2}(1 + Y^2) = P_+, \quad YP_- = \frac{1}{2}(Y^3 - Y^2) = \frac{1}{2}(1 - Y^2) = -P_-,$$

we transform (5), (7) and (8) to the form

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\,\Psi_0 +$$

$$+\frac{1}{\sqrt{2}}e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r)\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r)\Psi_- = 0\,,$$

$$(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma P + M)\,\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r} + B_0 r)\Psi_0 = 0\,,$$

$$(\beta_3\partial_3 + \beta_4\partial_4 - \Gamma P + M)\,\Psi_- + \frac{1}{\sqrt{2}}e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r} - B_0 r)\Psi_0 = 0\,. \quad (9)$$

To separate the variables, we search for three components of the wave function in the form

$$\Psi_0 = e^{ip_4x_4}e^{ip_3x_3}e^{im\phi}f_0(r)\,, \quad \Psi_\pm = e^{ip_4x_4}e^{ip_3x_3}e^{i(m\pm1)\phi}f_+(r)\,.$$

The resulting from (9) radial equations are written in symbolic form as

$$(ip_3\beta_3 + ip_4\beta_4 + M)\,f_0 +$$

$$+\frac{1}{\sqrt{2}}\beta_{+}(\frac{d}{dr}+\frac{m+1}{r}-B_0 r)f_{+}+\frac{1}{\sqrt{2}}\beta_{-}(\frac{d}{dr}-\frac{m-1}{r}+B_0 r)f_{-}=0\,,$$

$$(ip_3\beta_3+ip_4\beta_4+\Gamma P+M)\,f_{+}+\frac{1}{\sqrt{2}}\beta_{-}(\frac{d}{dr}-\frac{m}{r}+B_0 r)f_0=0\,,$$
$$(ip_3\beta_3+ip_4\beta_4-\Gamma P+M)\,f_{-}+\frac{1}{\sqrt{2}}\beta_{+}(\frac{d}{dr}+\frac{m}{r}-B_0 r)f_0=0\,. \qquad (10)$$

3. The Radial System

By using the notations

$$\hat{a}_m=\frac{1}{\sqrt{2}}(+\frac{d}{dr}+\frac{m-B_0r^2}{r}),\quad \hat{b}_m=\frac{1}{\sqrt{2}}(-\frac{d}{dr}+\frac{m-B_0r^2}{r}),\quad ip_3\beta_3+ip_4\beta_4=i\hat{p}\,,$$

the equations (10) are written shorter

$$(i\hat{p}+M)\,f_0+\beta_{+}\hat{a}_{m+1}f_{+}-\beta_{-}\hat{b}_{m-1}f_{-}=0\,, \qquad (11)$$
$$(i\hat{p}+\Gamma P+M)\,f_{+}-\beta_{-}\hat{b}_m f_0=0\,, \qquad (12)$$
$$(i\hat{p}-\Gamma P+M)\,f_{-}+\beta_{+}\hat{a}_m f_0=0\,.$$

We further act on (11) by the operator

$$\frac{1}{M+\Gamma}(M+\Gamma\bar{P}),\quad \text{where}\quad \bar{P}=1-P\,.$$

This yields

$$\left[\frac{1}{M+\Gamma}(M+\Gamma\bar{P})i\hat{p}+\frac{1}{M+\Gamma}(M+\Gamma\bar{P})(M+\Gamma P)\right]f_{+}-$$
$$-\frac{1}{M+\Gamma}(M+\Gamma\bar{P})\beta_{-}\hat{b}_m f_0=0\,.$$

We note the relation

$$\frac{(M+\Gamma\bar{P})(M+\Gamma P)}{M+\Gamma}=\frac{M^2+M\Gamma P+M\Gamma\bar{P}+\Gamma^2\bar{P}P}{M+\Gamma}=\frac{M^2+M\Gamma}{M+\Gamma}=M\,,$$

which is valid due to the identities $P+\bar{P}=1,\ P\bar{P}=\bar{P}P=0$. We introduce the notations:

$$\frac{M+\Gamma\bar{P}}{M+\Gamma}\,i\hat{p}=A\,,\quad \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_{-}=\beta'_{-}\,.$$

Then the above equation transforms to

$$(A+M)f_{+} - \beta'_{-}\hat{b}_m f_0 = 0\,.$$

Similarly, we act on (12) by the operator

$$\frac{1}{M-\Gamma}(M-\Gamma\bar{P}), \quad \bar{P} = 1 - P\,,$$

which yields

$$\left[\frac{1}{M-\Gamma}(M-\Gamma\bar{P})i\hat{p} + \frac{1}{M-\Gamma}(M-\Gamma\bar{P})(M-\Gamma P)\right] f_{-} +$$

$$+\frac{1}{M-\Gamma}(M-\Gamma\bar{P})\beta_{+}\hat{a}_m f_0 = 0\,.$$

Considering the identities

$$\frac{(M-\Gamma\bar{P})(M-\Gamma P)}{M-\Gamma} = \frac{M^2 - M\Gamma P - M\Gamma\bar{P} + \Gamma^2\bar{P}P}{M-\Gamma} = \frac{M^2 - M\Gamma}{M-\Gamma} = M$$

and the notations

$$\frac{(M-\Gamma\bar{P})}{M-\Gamma}\, i\hat{p} = C\,, \quad \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_{+} = \beta'_{+}\,,$$

the above equation becomes

$$(C+M)f_{-} + \beta'_{+}\hat{a}_m f_0 = 0\,.$$

Thus, the radial system can be written as

$$(i\hat{p}+M)\, f_0 + \beta_{+}\hat{a}_{m+1} f_{+} - \beta_{-}\hat{b}_{m-1} f_{-} = 0\,,$$

$$(A+M)f_{+} - \beta'_{-}\hat{b}_m f_0 = 0\,,$$

$$(C+M)f_{-} + \beta'_{+}\hat{a}_m f_0 = 0\,.$$

To proceed with these equations, we introduce the matrices[3] with the properties

$$\overline{(i\hat{p}+M)}(i\hat{p}+M) = p^2 + M^2,$$
$$\overline{(A+M)}(A+M) = p^2 + M^2,$$
$$\overline{(C+M)}(C+M) = p^2 + M^2\,. \qquad (13)$$

[3] We take in the account that $p^2 = p_3^2 + p_4^2$

In fact these formulas determine the inverse matrices to within numerical factors $(p^2+M^2)^{-1}$. Then the system of radial equations can be rewritten alternatively

$$
\begin{gathered}
(i\hat{p}+M)(p^2+M^2)f_0+\beta_+\hat{a}_{m+1}(p^2+M^2)f_+-\beta_-\hat{b}_{m-1}(p^2+M^2)f_-=0\,,\\
(p^2+M^2)f_+-\overline{(A+M)}\beta'_-\hat{b}_m f_0=0\,,\\
(p^2+M^2)f_-+\overline{(C+M)}\beta'_+\hat{a}_m f_0=0\,.
\end{gathered}
\tag{14}
$$

The first equation in (14), with the help of the other two ones, transforms into an equation on the component $f_0(r)$:

$$
\begin{gathered}
(i\hat{p}+M)(p^2+M^2)f_0+\\
+\beta_+\hat{a}_{m+1}\overline{(A+M)}\beta'_-\hat{b}_m f_0+\beta_-\hat{b}_{m-1}\overline{(C+M)}\beta'_+\hat{a}_m f_0=0\,;
\end{gathered}
\tag{15}
$$

while the two remaining ones do not change

$$
\begin{gathered}
(p^2+M^2)f_+-\overline{(A+M)}\beta'_-\hat{b}_m f_0=0\,,\\
(p^2+M^2)f_-+\overline{(C+M)}\beta'_+\hat{a}_m f_0=0\,.
\end{gathered}
\tag{16}
$$

In fact, the equations (16) mean that it suffices to solve (15) with respect to f_0; the two other components f_+ and f_- can be calculated by means of equations (16).

To proceed further, we need to know the explicit form of the inverse operators (13). To solve this task, we first establish the minimal polynomials for the relevant matrices. The minimal polynomial for $(i\hat{p})$ is [3]

$$
i\hat{p}\,[(i\hat{p})^2+p^2]=0\,. \tag{17}
$$

We further consider the operator A^2:

$$
A^2=\frac{1}{(M+\Gamma)^2}(iM\hat{p}+i\Gamma\bar{P}\hat{p})(iM\hat{p}+i\Gamma\bar{P}\hat{p})=
$$

$$
=\frac{1}{(M+\Gamma)^2}[-M^2\hat{p}^{\,2}-M\Gamma\hat{p}\bar{P}\hat{p}-M\Gamma\bar{P}\hat{p}^2-\Gamma^2\bar{P}\hat{p}\bar{P}\hat{p}]\,.
$$

Due to the identities

$$
\beta_\mu=P\beta_\mu+\beta_\mu P=\bar{P}\beta_\mu+\beta_\mu\bar{P},\beta_\mu P=P\beta_\mu,\bar{P}\beta_\mu=\beta_\mu\bar{P},
$$

$$P\beta_\mu P = \bar{P}\beta_\mu \bar{P} = 0, \beta_\mu\beta_\nu P = P\beta_\mu\beta_\nu, \beta_\mu\beta_\nu\bar{P} = \bar{P}\beta_\mu\beta_\nu,$$
$$P + \bar{P} = 1, P\bar{P} = \bar{P}P = 0,$$

we find

$$A^2 = \frac{1}{(M+\Gamma)^2}(-M^2\hat{p}^2 - M\Gamma\hat{p}^2) = -\frac{M\hat{p}^2}{M+\Gamma}\,.$$

Thus, we get the minimal polynomial for A

$$A^3 = -\frac{M}{(M+\Gamma)^2}(M+\Gamma\bar{P})(i\hat{p})\hat{p}^2 = -\frac{Mp^2}{(M+\Gamma)}\frac{(M+\Gamma\bar{P})}{M+\Gamma}(i\hat{p}) = -\frac{Mp^2}{M+\Gamma}A\,.$$

Similarly, we find

$$C^3 = -\frac{M\hat{p}^2}{M-\Gamma}C\,.$$

Therefore, the needed inverse operators must be quadratic with respect to the relevant matrices. They are given by the formulas:

$$\overline{(M+i\hat{p})} = \tfrac{1}{M}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right],$$
$$\overline{(A+M)} = \tfrac{p^2+M^2}{M}\left[1 - \tfrac{M+\Gamma}{p^2+M^2+M\Gamma}A + \tfrac{M+\Gamma}{M(p^2+M^2+M\Gamma)}A^2\right],$$
$$\overline{(C+M)} = \tfrac{p^2+M^2}{M}\left[1 - \tfrac{M-\Gamma}{p^2+M^2-M\Gamma}C + \tfrac{M-\Gamma}{M(p^2+M^2-M\Gamma)}C^2\right].$$

We need an explicit form for the powers of $i\hat{p}$:

$$i\hat{p} = i\left|\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & -p_3 & 0 & p_4 & 0 & 0\\
0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_3\\
0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & p_4 & -p_3 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|,$$

$$(i\hat{p})^2 = -\begin{vmatrix} p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_4^2 & -p_3p_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_3p_4 & p_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3^2 & 0 & 0 & 0 & p_3p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_3^2 & 0 & -p_3p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p_3p_4 & 0 & p_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3p_4 & 0 & 0 & 0 & p_4^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 \end{vmatrix},$$

$$(i\hat{p})^2 + p^2 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3^2 & p_3p_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3p_4 & p_4^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_4^2 & 0 & 0 & 0 & -p_3p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_4^2 & 0 & p_3p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_3p_4 & 0 & p_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_3p_4 & 0 & 0 & 0 & p_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix};$$

we may prove the validness of the identity

$$i\hat{p}[(i\hat{p})^2 + p^2] =$$

$$i\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -p_3 & 0 & p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_3 \\ 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_4 & -p_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \times$$

$$\times \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3^2 & p_3 p_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 p_4 & p_4^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_4^2 & 0 & 0 & 0 & -p_3 p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_4^2 & 0 & p_3 p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_3 p_4 & 0 & p_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_3 p_4 & 0 & 0 & 0 & p_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| \equiv 0 \,.$$

Let us turn back to equation for f_0, rewritten in the form

$$(p^2 + M^2)^2 f_0 + \qquad + \overline{(M + i\hat{p})}\, \beta_+ \, \hat{a}_{m+1} \overline{(A + M)}\, \beta'_- \, \hat{b}_m f_0 +$$
$$+ \overline{(M + i\hat{p})}\, \beta_- \, \hat{b}_{m-1} \overline{(C + M)}\, \beta'_+ \, \hat{a}_m f_0 = 0 \,.$$

Taking into account the explicit form for inverse operators, we get

$$(p^2 + M^2) f_0 + \tfrac{1}{M^2} \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2) \right] \, \beta_+ \, \hat{a}_{m+1} \times$$
$$\times \left[1 - \tfrac{M+\Gamma}{p^2+M^2+M\Gamma} A + \tfrac{M+\Gamma}{M(p^2+M^2+M\Gamma)} A^2 \right] \, \beta'_- \, \hat{b}_m f_0 +$$
$$+ \tfrac{1}{M^2} \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2) \right] \, \beta_- \, \hat{b}_{m-1} \times$$
$$\times \left[1 - \tfrac{M-\Gamma}{p^2+M^2-M\Gamma} C + \tfrac{M-\Gamma}{M(p^2+M^2-M\Gamma)} C^2 \right] \, \beta'_+ \, \hat{a}_m f_0 = 0 \,.$$

Now, by considering the formulas

$$A = \tfrac{M+\Gamma\bar{P}}{M+\Gamma}\, i\hat{p} \,, \qquad A^2 = -\tfrac{M\hat{p}^2}{M+\Gamma} \,,$$
$$C = \tfrac{M-\Gamma\bar{P}}{M-\Gamma}\, i\hat{p} \,, \qquad C^2 = -\tfrac{M\hat{p}^2}{M-\Gamma} \,,$$
$$\beta'_- = \tfrac{M+\Gamma\bar{P}}{M+\Gamma} \beta_- \,, \quad \beta'_+ = \tfrac{M-\Gamma\bar{P}}{M-\Gamma} \beta_+ \,,$$

we transform the above equation into following one

$$(p^2 + M^2) f_0 + \tfrac{1}{M^2} \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2) \right] \, \beta_+ \times$$
$$\times \left[1 - \tfrac{M+\Gamma\bar{P}}{p^2+M^2+M\Gamma}\, i\hat{p} + \tfrac{(i\hat{p})^2}{p^2+M^2+M\Gamma} \right] \, \tfrac{M+\Gamma\bar{P}}{M+\Gamma} \beta_- \, \hat{a}_{m+1} \hat{b}_m f_0 +$$
$$+ \tfrac{1}{M^2} \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2 + M^2) \right] \beta_- \times$$
$$\times \left[1 - \tfrac{M-\Gamma\bar{P}}{p^2+M^2-M\Gamma}\, i\hat{p} + \tfrac{(i\hat{p})^2}{p^2+M^2-M\Gamma} \right] \, \tfrac{M-\Gamma\bar{P}}{M-\Gamma} \beta_+ \, \hat{b}_{m-1} \hat{a}_m f_0 = 0 \,.$$

After some manipulation with the use of identity $\hat{p}\beta_+\hat{p} = \hat{p}\beta_-\hat{p} = 0$, this equation can be presented differently

$$\{ (p^2+M^2) + \hat{a}_{m+1}\hat{b}_m \tfrac{1}{M^2(p^2+M^2+M\Gamma)}\tfrac{1}{M+\Gamma}\times$$
$$\times[\,(p^2+M^2+M\Gamma)(i\hat{p})^2\beta_+ - M(p^2+M^2+M\Gamma)i\hat{p}\beta_+ +$$
$$+(p^2+M^2)(p^2+M^2+M\Gamma)\beta_+ - (p^2+M^2)\beta_+ i\hat{p}(M+\Gamma P)+$$
$$+(p^2+M^2)\beta_+(i\hat{p})^2]\,(M+\Gamma\bar{P})\beta_- + \hat{b}_{m-1}\hat{a}_m\tfrac{1}{M^2(p^2+M^2-M\Gamma)}\tfrac{1}{M-\Gamma}\times$$
$$\times[\,(p^2+M^2-M\Gamma)(i\hat{p})^2\beta_- - M(p^2+M^2-M\Gamma)i\hat{p}\beta_- +$$
$$+(p^2+M^2)(p^2+M^2-M\Gamma)\beta_- - (p^2+M^2)\beta_- i\hat{p}(M-\Gamma P)+$$
$$+(p^2+M^2)\beta_-(i\hat{p})^2]\,(M-\Gamma\bar{P})\beta_+\,\}f_0 = 0.$$

Now we take into account the explicit form of f_0, $i\hat{p}$, and matrices $\beta_+, \beta_-, \bar{P}$. Then we obtain

$$(p^2+M^2)\left|\begin{array}{c} 0\\ 0\\ f_3\\ f_4\\ 0\\ 0\\ f_{12}\\ 0\\ 0\\ f_{34} \end{array}\right| + \hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\times$$

$$\left\{ (p^2+M^2)(p^2+M^2+M\Gamma)\left|\begin{array}{c} 0\\ 0\\ (M+\Gamma)f_3\\ (M+\Gamma)f_4\\ 0\\ 0\\ Mf_{12}\\ 0\\ 0\\ 0 \end{array}\right| - iM(M+\Gamma)(p^2+M^2+M\Gamma)\left|\begin{array}{c} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ (p_4f_3-p_3f_4) \end{array}\right| - \right.$$

$$-(M+\Gamma)(p^2+M^2+M\Gamma)\left|\begin{array}{c}0\\0\\p_4(p_4f_3-p_3f_4)\\-p_3(p_4f_3-p_3f_4)\\0\\0\\0\\0\\0\\0\end{array}\right| - (p^2+M^2)\left|\begin{array}{c}0\\0\\p_3(M+\Gamma)(p_3f_3+p_4f_4)\\p_4(M+\Gamma)(p_3f_3+p_4f_4)\\0\\0\\p^2Mf_{12}\\0\\0\\0\end{array}\right| +$$

$$+M(M+\Gamma)(p^2+M^2)\left|\begin{array}{c}0\\0\\p_3f_{12}\\p_4f_{12}\\0\\0\\-(p_3f_3+p_4f_4)\\0\\0\\0\end{array}\right|\Bigg\} + \hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\times$$

$$\Bigg\{(p^2+M^2)(p^2+M^2-M\Gamma)\left|\begin{array}{c}0\\0\\(M-\Gamma)f_3\\(M-\Gamma)f_4\\0\\0\\Mf_{12}\\0\\0\\0\end{array}\right| - iM(M-\Gamma)(p^2+M^2-M\Gamma)\left|\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\0\\(p_4f_3-p_3f_4)\end{array}\right| -$$

$$-(M-\Gamma)(p^2+M^2-M\Gamma)\left|\begin{array}{c}0\\0\\p_4(p_4f_3-p_3f_4)\\-p_3(p_4f_3-p_3f_4)\\0\\0\\0\\0\\0\\0\end{array}\right| - (p^2+M^2)\left|\begin{array}{c}0\\0\\p_3(M-\Gamma)(p_3f_3+p_4f_4)\\p_4(M-\Gamma)(p_3f_3+p_4f_4)\\0\\0\\p^2Mf_{12}\\0\\0\\0\end{array}\right| -$$

$$-M(M-\Gamma)(p^2+M^2)\left|\begin{array}{c}0\\0\\p_3f_{12}\\p_4f_{12}\\0\\0\\-(p_3f_3+p_4f_4)\\0\\0\\0\end{array}\right|\Bigg\} = 0\,.$$

From these relations we derive four equations:

$$(p^2+M^2)f_3+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\times$$

$$\{(p^2+M^2)(p^2+M^2+M\Gamma)(M+\Gamma)f_3-$$
$$-p_4(M+\Gamma)(p^2+M^2+M\Gamma)(p_4f_3-p_3f_4)-$$
$$-p_3(p^2+M^2)(M+\Gamma)(p_3f_3+p_4f_4)+p_3M(M+\Gamma)(p^2+M^2)f_{12}\}+$$
$$+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\times$$
$$\{(p^2+M^2)(p^2+M^2-M\Gamma)(M-\Gamma)f_3-$$
$$-p_4(M-\Gamma)(p^2+M^2-M\Gamma)(p_4f_3-p_3f_4)-$$
$$-p_3(p^2+M^2)(M-\Gamma)(p_3f_3+p_4f_4)-p_3M(M-\Gamma)(p^2+M^2)f_{12}\}=0\,, \tag{18}$$

$$(p^2+M^2)f_4+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\times$$
$$\{(p^2+M^2)(p^2+M^2+M\Gamma)(M+\Gamma)f_4+$$
$$+p_3(M+\Gamma)(p^2+M^2+M\Gamma)(p_4f_3-p_3f_4)-$$
$$-p_4(p^2+M^2)(M+\Gamma)(p_3f_3+p_4f_4)+p_4M(M+\Gamma)(p^2+M^2)f_{12}\}+$$
$$+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\times$$
$$\{(p^2+M^2)(p^2+M^2-M\Gamma)(M-\Gamma)f_4+$$
$$+p_3(M-\Gamma)(p^2+M^2-M\Gamma)(p_4f_3-p_3f_4)-$$
$$-p_4(p^2+M^2)(M-\Gamma)(p_3f_3+p_4f_4)-$$
$$-p_4M(M-\Gamma)(p^2+M^2)f_{12}\}=0\,, \tag{19}$$

$$(p^2+M^2)f_{12}+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\times$$

$$\{M(p^2+M^2)(p^2+M^2+M\Gamma)f_{12}-M(p^2+M^2)p^2f_{12}-$$
$$-M(M+\Gamma)(p^2+M^2)(p_3f_3+p_4f_4)\}+$$
$$+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\times$$
$$\{M(p^2+M^2)(p^2+M^2-M\Gamma)f_{12}-M(p^2+M^2)p^2f_{12}+$$
$$+M(M-\Gamma)(p^2+M^2)(p_4f_3+p_3f_4)\}=0\,, \tag{20}$$

$$(p^2+M^2)f_{34}+$$
$$+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\{-iM(p^2+M^2+M\Gamma)(M+\Gamma)(p_4f_3-p_3f_4)\}+$$
$$+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\{-iM(p^2+M^2-M\Gamma)(M-\Gamma)(p_4f_3-p_3f_4)\}=0\,,$$

$$(p^2+M^2)f_{34}-\frac{i}{M}(\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m)(p_4f_3-p_3f_4)=0\,. \tag{21}$$

The equations (18) and (19) may be simplified to

$$(p^2+M^2)f_3+\frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\{(p^2+M^2)f_3-p_4(p_4f_3-p_3f_4)-$$
$$-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_3(p_3f_3+p_4f_4)+\frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_3f_{12}\}+$$
$$+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\{(p^2+M^2)f_3-p_4(p_4f_3-p_3f_4)-$$
$$-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_3(p_3f_3+p_4f_4)-\frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_3f_{12}\}=0\,,$$

$$(p^2+M^2)f_4+\frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\{(p^2+M^2)f_4+p_3(p_4f_3-p_3f_4)-$$
$$-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_4(p_3f_3+p_4f_4)+\frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_4f_{12}\}+$$
$$+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\{(p^2+M^2)f_4+p_3(p_4f_3-p_3f_4)-$$

$$-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_4(p_3f_3+p_4f_4)-\frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_4f_{12}\}=0\,.$$

By multiplying the first equation by p_4, and the second one by $-p_3$, and then summing the two results, we find

$$(p^2+M^2)(p_4f_3-p_3f_4)+$$

$$+\frac{\hat{a}_{m+1}\hat{b}_m}{M^2}[(p^2+M^2)(p_4f_3-p_3f_4)-p^2(p_4f_3-p_3f_4)]+$$

$$+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}[(p^2+M^2)(p_4f_3-p_3f_4)-p^2(p_4f_3-p_3f_4)]=0$$

or

$$\left[\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m+p^2+M^2\right](p_4f_3-p_3f_4)=0\,.$$

By taking into consideration (21):

$$(\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m)(p_4f_3-p_3f_4)=-iM(p^2+M^2)f_{34}\,.$$

we obtain

$$f_{34}=-\frac{i}{M}\,(p_4f_3-p_3f_4)$$

We further consider (20), which simplifies to te form

$$(p^2+M^2)f_{12}+\hat{a}_{m+1}\hat{b}_m\frac{(p^2+M^2)}{M(p^2+M^2+M\Gamma)}\{Mf_{12}-(p_3f_3+p_4f_4)\}+$$

$$+\hat{b}_{m-1}\hat{a}_m\frac{(p^2+M^2)}{M(p^2+M^2-M\Gamma)}\{Mf_{12}+(p_3f_3+p_4f_4)\}=0\,,$$

or

$$f_{12}+\tfrac{\hat{a}_{m+1}\hat{b}_m}{M(p^2+M^2+M\Gamma)}\{Mf_{12}-(p_3f_3+p_4f_4)\}+$$

$$+\tfrac{\hat{b}_{m-1}\hat{a}_m}{M(p^2+M^2-M\Gamma)}\{Mf_{12}+(p_3f_3+p_4f_4)\}=0\,,$$

whence after elementary transformation we get

$$\left[(\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m)+\tfrac{(p^2+M^2)^2-M^2\Gamma^2}{p^2+M^2}+\tfrac{2M\Gamma B_0}{p^2+M^2}\right]f_{12}+$$

$$+\left[\tfrac{\Gamma}{p^2+M^2}(\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m)+\tfrac{2B_0}{M}\right](p_3f_3+p_4f_4)=0\,;$$

where the following identity is used:

$$\hat{a}_{m+1}\hat{b}_m - \hat{b}_{m-1}\hat{a}_m = -2B_0\,.$$

Now, we turn again to (18) and (19):

$$(p^2+M^2)f_3 + \frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\{(p^2+M^2)f_3 - p_4(p_4f_3 - p_3f_4) -$$

$$-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_3(p_3f_3+p_4f_4) + \frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_3f_{12}\} +$$

$$+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\{\,(p^2+M^2)f_3 - p_4(p_4f_3-p_3f_4) -$$

$$-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_3(p_3f_3+p_4f_4) - \frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_3f_{12}\} = 0\,,$$

$$(p^2+M^2)f_4 + \frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\{\,(p^2+M^2)f_4 + p_3(p_4f_3-p_3f_4) -$$

$$-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_4(p_3f_3+p_4f_4) + \frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_4f_{12}\} +$$

$$+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\{(p^2+M^2)f_4 + p_3(p_4f_3-p_3f_4) -$$

$$-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_4(p_3f_3+p_4f_4) - \frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_4f_{12}\} = 0\,,$$

By multiplying the first relation by p_3, and the second one by p_4, and summing the results, we find

$$(p_3f_3+p_4f_4) + \hat{a}_{m+1}\hat{b}_m\frac{1}{M(p^2+^2+M\Gamma)}[(M+\Gamma)(p_3f_3+p_4f_4) + p^2f_{12}] +$$

$$+\hat{b}_{m-1}\hat{a}_m\frac{1}{M(p^2+M^2-M\Gamma)}[(M-\Gamma)(p_3f_3+p_4f_4)] - p^2f_{12}] = 0\,.$$

Thus, we have found two equations for $(p_3f_3+p_4f_4)$ and f_{12}:

$$\left[(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{(p^2+M^2)^2 - M^2\Gamma^2}{p^2+M^2} + \frac{2M\Gamma B_0}{p^2+M^2}\right] f_{12} +$$
$$+\left[\frac{\Gamma}{p^2+M^2}(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{2B_0}{M}\right](p_3f_3+p_4f_4) = 0\,;$$

$$\{(p^2+M^2)^2 - M^2\Gamma^2\}(p_3f_3+p_4f_4) + (p^2+M^2-\Gamma^2)(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m)(p_3f_3+p_4f_4) -$$
$$-\frac{2B_0\Gamma p^2}{M}(p_3f_3+p_4f_4) - p^2\left[\Gamma(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{2B_0(p^2+M^2)}{M}\right] f_{12} = 0.$$

These equations may be reduced to such a form, that the 2-nd order operator $(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m)$ acts on a single function:

$$\left(\tfrac{2B_0}{M} - \Gamma\right)(p_3 f_3 + p_4 f_4) + \left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 - \Gamma^2 - \tfrac{2B_0\Gamma}{M}\right] f_{12} = 0\,,$$
$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right](p_3 f_3 + p_4 f_4) + \left(p^2\Gamma - \tfrac{2B_0 p^2}{M}\right) f_{12} = 0\,.$$

Thus, the final form of the equations for the four functions f_3, f_4, f_{12}, f_{34} has the following relatively simple structure:

$$f_{34} = -\frac{i}{M}(p_4 f_3 - p_3 f_4)\,, \qquad (22)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right](p_4 f_3 - p_3 f_4) = 0\,, \qquad (23)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right](p_3 f_3 + p_4 f_4) = -p^2(\Gamma - \frac{2B_0}{M})f_{12}\,, \qquad (24)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right] f_{12} = \\ = \Gamma\left(\Gamma - \tfrac{2B_0}{M}\right) f_{12} + \left(\Gamma - \tfrac{2B_0}{M}\right)(p_3 f_3 + p_4 f_4)\,. \qquad (25)$$

The analysis of (22) and (23) can be now clearly done. The second subsystem (24)–(25) is solved through diagonalizing the mixing matrix.

To this aim, let us introduce the new functions

$$\Phi_1 = (p_3 f_3 + p_4 f_4) + \lambda_1 f_{12}\,, \quad \Phi_2 = (p_3 f_3 + p_4 f_4) + \lambda_2 f_{12}\,, \qquad (26)$$

where λ_1, λ_2 stand for the roots of the equation $\lambda^2 - \lambda\Gamma + p^2 = 0$:

$$\lambda_1 = \frac{1}{2}\left(\Gamma + \sqrt{\Gamma^2 - 4p^2}\right), \quad \lambda_2 = \frac{1}{2}\left(\Gamma - \sqrt{\Gamma^2 - 4p^2}\right).$$

So we get two separate equations:

$$\left(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 + \lambda'_{1,2}\right)\Phi_{1,2} = 0\,,$$

where $\lambda'_1 = (\frac{2B_0}{M} - \Gamma)\lambda_1$, and $\lambda'_2 = (\frac{2B_0}{M} - \Gamma)\lambda_2$.

4. The Energy Spectra

We note that, explicitly, the radial equations read

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \epsilon^2 - M^2 - p_3^2 - \lambda'_{1,2} - \frac{(m - B_0 r^2)^2}{r^2} \right) \Phi_{1,2} = 0\,.$$

In the variable $x = |B_0| r^2$, the equation for Φ_1 takes the form

$$\left[4|B_0| \left(x\frac{d^2}{dx^2} + \frac{d}{dx} \right) - \frac{|B_0|(m - xB_0/|B_0|)^2}{x} + \epsilon^2 - M^2 - p_3^2 - \lambda'_1 \right] \Phi_1 = 0\,.$$

First, let be $B_0 = -|B_0|$; then we have

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{(m+x)^2}{4x} + \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|} \right] \Phi_1 = 0\,.$$

With the substitution $\Phi_1 = x^A e^{-Cx} \bar{\Phi}_1,\ A = |m|/2,\ c = \frac{1}{2}$, we get

$$\left[x\frac{d^2}{dx^2} + (|m| + 1 - x)\frac{d}{dx} - \left(\frac{|m| + m + 1}{2} - \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|} \right) \right] \bar{\Phi}_1 = 0\,.$$

This is a confluent hypergeometric equation; to get polynomial solutions we must impose the restriction

$$\frac{|m| + m + 1}{2} - \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|} = -n\,;$$

whence it follows that

$$\epsilon^2 - M^2 - p_3^2 - \lambda'_1 = 2|B_0|(m + |m| + 1 + 2n)\,.$$

Hence, the energy spectra are

$$\Phi_{1,}, \qquad \epsilon_1^2 - M^2 - p_3^2 = 2|B_0|(m + |m| + 1 + 2n) + \lambda'_{1,2}\,.$$

By using the simplifying notations

$$2|B_0|(m + |m| + 1 + 2n) = N\,, \quad -p^2 = \epsilon^2 - p_3^2 = E > 0\,, \quad \frac{2B_0}{M} - \Gamma = x\,,$$

$$\lambda'_1 = \frac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E}), \quad \lambda'_2 = \frac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E}).$$

the formulas for energy levels read

$$\Phi_1, \quad B_0 = -|B_0|, \quad E - M^2 = N + \tfrac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E})\,,$$
$$\Phi_2, \quad B_0 = -|B_0|, \quad E - M^2 = N + \tfrac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E})\,.$$

We solve these equations for E:

$$2E - 2M^2 - 2N - x\Gamma = \pm x\sqrt{\Gamma^2 + 4E} \quad \Longrightarrow$$
$$z \equiv 2N + 2M^2 + x\Gamma, \quad E^2 - E(z + x^2) + \frac{z^2 - x^2\Gamma^2}{4} = 0\,;$$

and the roots are

$$E_1 = \frac{z + x^2}{2} + \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)},$$
$$E_2 = \frac{z + x^2}{2} - \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}. \tag{27}$$

To have both E_1 and E_2 real-valued and positive (such that these refer to physical energy levels), we require

$$z^2 - x^2\Gamma^2 > 0, \quad z + x^2 > 0, \quad (z + x^2)^2 - (z^2 - x^2\Gamma^2) > 0\,.$$

We consider the first inequality

$$z^2 - x^2\Gamma^2 = (z - x\Gamma)(z + x\Gamma) > 0 \quad \Longrightarrow (2N + 2M^2)(2N + 2M^2 + 2x\Gamma) > 0\,;$$

this holds true if we impose the following restriction[4]:

$$x\Gamma > 0 \quad \Longleftrightarrow \quad \left(\frac{2|B_0|}{M} + \Gamma\right)\Gamma < 0 \quad \Longleftrightarrow \quad -\frac{2|B_0|}{M} < \Gamma < 0\,. \tag{28}$$

The second inequality

$$z + x^2 = (2N + 2M^2) + x\Gamma + x^2 > 0;$$

is valid due to (28). The third inequality $2zx^2 + x^4 + x^2\Gamma^2 > 0$ is valid due to

$$z = 2N + 2M^2 + x\Gamma,\ x\Gamma > 0.$$

[4] We remind that $B_0 = -|B_0| < 0$.

Thus, we get one simple restriction on the parameter Γ:

$$B_0 = -|B_0|, \qquad -\frac{2|B_0|}{M} < \Gamma < 0\,, \tag{29}$$

which ensures that both spectra are physical (real and positive) for all the values of quantum numbers. In the case under consideration, $B_0 = -|B_0| < 0$, from (4) it follows

$$\Gamma = \pm 4 \frac{-|B_0|}{M} \lambda_3 \lambda_3^*\,;$$

therefore we have the only case when the upper sign is related to $\Gamma < 0$.

Similar results can be obtained for the case of the opposed orientation of the magnetic field, $B_0 = +|B_0|$:

$$\Phi_{1,2}, \qquad \epsilon_1^2 - M^2 - p_3^2 = 2|B_0|(-m + |m| + 1 + 2n) + \lambda'_{1,2}\,.$$

With the similar notation

$$2|B_0|(-m+|m|+1+2n) = N\,, \quad -p^2 = \epsilon^2 - p_3^2 = E > 0\,, \quad \frac{2B_0}{M} - \Gamma = x\,,$$

$$\lambda'_1 = \frac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E}), \quad \lambda'_2 = \frac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E}),$$

we formally derive the same formulas for energies:

$$E_1 = \frac{z + x^2}{2} + \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}\,,$$

$$E_2 = \frac{z + x^2}{2} - \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}.$$

In order to have energy values positive and real-valued, we must impose the following restrictions

$$z + x^2 > 0, \quad z^2 - x^2\Gamma^2 > 0, \quad (z + x^2)^2 - (z^2 - x^2\Gamma^2) > 0\,.$$

From the inequality

$$z^2 - x^2\Gamma^2 = (z - x\Gamma)(z + x\Gamma) > 0 \quad \Longrightarrow \quad (2N + 2M^2)(2N + 2M^2 + 2x\Gamma) > 0$$

we get the main restriction[5]:

$$x\Gamma > 0 \quad \Longleftrightarrow \quad (-\frac{2|B_0|}{M} + \Gamma)\Gamma > 0 \quad \Longleftrightarrow \quad \Gamma < 0\,.$$

[5] We remind that $B_0 = +|B_0| < 0$.

We note that the possibility of positive values $\Gamma > 0$, $\Gamma > 2|B_0|/M$ is ignored, because in this case the admissible region for Γ does not contain the close to zero values. The two remaining inequalities are valid as well:

$$z + x^2 = (2N + 2M^2) + x\Gamma + x^2 > 0;$$

$$2zx^2 + x^4 + x^2\Gamma^2 > 0 \qquad\qquad (z = 2N + 2M^2 + x\Gamma,\ x\Gamma > 0)\,.$$

5. Conclusion to Sections 2–4

Let us summarize the main results of the Sections 2–4. Three series of the energy levels are found; two of them substantially differ from those for spin 1 particles without anomalous magnetic moment.

The formula (27) and its restriction (29) provide us with two series for energy levels[6] in both cases $B_0 = -|B_0|$, and $B_0 = +|B_0|$:

$$E_1 = \tfrac{z+x^2}{2} + \tfrac{1}{2}\sqrt{(z+x^2)^2 - (z^2 - x^2\Gamma^2)},$$
$$E_2 = \tfrac{z+x^2}{2} - \tfrac{1}{2}\sqrt{(z+x^2)^2 - (z^2 - x^2\Gamma^2)}.$$

To assign to the energies E_1 and E_2 a physical sense for all the values of the main quantum number $n = 0, 1, 2, ...$, one must impose special restrictions – which are explicitly formulated – on the values of the anomalous magnetic moment. Without these restrictions, only some part of the energy levels correspond to bound states.

The third series of the energy levels (see (23)) has the form:

$$B_0 = -|B_0|: \quad E_3 = \epsilon^2 - M^2 - p_3^2 = 2|B_0|(\ m + |m| + 1 + 2n)\,,$$
$$B_0 = +|B_0|: \quad E_3 = \epsilon^2 - M^2 - p_3^2 = 2|B_0|(-m + |m| + 1 + 2n)\,;$$

in these states the anomalous magnetic moment does not manifest itself at all.

[6]We remember the formal change $m \Longrightarrow -m$, when inverting the orientation of the magnetic field

6. Neutral Spin 1 Particles with Anomalous Magnetic Moment

The case of a neutral vector boson exhibits a particular interest; now the radial system for f_3, f_4, f_{12}, f_{34} becomes simpler:

$$f_{34} = -\frac{i}{M}\,(p_4 f_3 - p_3 f_4)\,, \qquad (30)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right](p_4 f_3 - p_3 f_4) = 0\,, \qquad (31)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right](p_3 f_3 + p_4 f_4) = -p^2\Gamma f_{12}\,, \qquad (32)$$

$$\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right] f_{12} = \Gamma^2 f_{12} + \Gamma(p_3 f_3 + p_4 f_4)\,. \qquad (33)$$

Solving (30) and (31) is a trivial task. The system (31)–(33) can be solved trough the diagonalization of the mixing matrix. Let us introduce the notation

$$\Delta = \frac{1}{\Gamma}\left[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2\right], \ (p_3 f_3 + p_4 f_4) = \Phi_1, \quad f_{12} = \Phi_2\,;$$

then (32)–(33) reads in matrix form as follows

$$\Delta \left|\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}\right| = \left|\begin{array}{cc} 0 & -p^2 \\ 1 & \Gamma \end{array}\right| \left|\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}\right| \quad \Longrightarrow \quad \Delta\, S \left|\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}\right| = S \left|\begin{array}{cc} 0 & -p^2 \\ 1 & \Gamma \end{array}\right| S^{-1} S \left|\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}\right|$$

Requiring

$$S \left|\begin{array}{cc} 0 & -p^2 \\ 1 & \Gamma \end{array}\right| S^{-1} = \left|\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right|, \qquad S = \left|\begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array}\right|;$$

we derive

$$\left|\begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array}\right| \left|\begin{array}{cc} 0 & -p^2 \\ 1 & \Gamma \end{array}\right| = \left|\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right| \left|\begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array}\right|,$$

which is equivalent to two sub-systems:

$$\left|\begin{array}{cc} -\lambda_1 & 1 \\ -p^2 & (\Gamma - \lambda_1) \end{array}\right| \left|\begin{array}{c} s_{11} \\ s_{12} \end{array}\right| = 0, \qquad \left|\begin{array}{cc} -\lambda_2 & 1 \\ -p^2 & (\Gamma - \lambda_2) \end{array}\right| \left|\begin{array}{c} s_{21} \\ s_{22} \end{array}\right| = 0$$

We use solutions of the form

$$\lambda_1 = \tfrac{1}{2}(\Gamma + \sqrt{\Gamma^2 - 4p^2}), s_{11} = 1, \quad s_{12} = \lambda_1\,;$$

$$\lambda_2 = \tfrac{1}{2}(\Gamma - \sqrt{\Gamma^2 - 4p^2})\,,\ s_{21} = 1,\ s_{22} = \lambda_2\,.$$

Thus, for the functions Φ_1 and Φ_2, we get the *separated* equations

$$\left(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 - \Gamma\lambda_{1,2}\right)\Phi_{1,2} = 0 \ .$$

In explicit form, these read

$$\epsilon^2 - M^2 - p_3^2 + \Gamma\lambda_{1,2} \equiv D \ , \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + D - \frac{m^2}{r^2}\right)\Phi_{1,2} = 0 \ .$$

Let us search for solutions of the form $\Phi = r^A e^{Br} f(r)$; for $f(r)$, we derive

$$\frac{d^2 f}{dr^2} + \left(\frac{2A+1}{r} + 2B\right)\frac{df}{dr} + \left(\frac{A^2 - m^2}{r^2} + \frac{2AB + B}{r} + B^2 + D\right) f = 0 \ .$$

By imposing the following restrictions on A, B:

$$A^2 - m^2 \equiv 0 \quad \Longrightarrow \quad A = \pm \mid m \mid ; \quad B^2 = -D \quad \Longrightarrow \quad B = \pm i\sqrt{D} \ ,$$

the above equation simplifies to

$$r\frac{d^2 f}{dr^2} + (2A + 1 + 2Br)\frac{df}{dr} + (2AB + B) f = 0.$$

If we take the positive case $A = + \mid m \mid$, then the solutions are vanishing near the point $r = 0$. Moreover, from physical considerations, we must require the parameter D be positive, in order to agree with the correspondence principle:

$$\Gamma = 0 \quad \Longrightarrow \quad D \to D_0 = \epsilon^2 - M^2 - p_3^2 > 0 \ .$$

Without loss of generality, assume that $B = +i\sqrt{D}$. In the new variable, the above equation will read as a confluent hypergonetric equation

$$2Br = -x, \quad x\frac{d^2 f}{dx^2} + (2A + 1 - x)\frac{df}{dx} - \left(A + \frac{1}{2}\right) f = 0$$

that is

$$F'' + (c - x)F' - aF = 0, \quad a = A + 1/2, \quad c = 2A + 1 = 2a,$$

where

$$x = -2Br = -2i\sqrt{M^2 - p_3^2 + \Gamma\lambda_{1,2}} \, .$$

Thus, for a neutral particle, no bound states exist, and the qualitative manifestation of the anomalous magnetic moment is mainly revealed by appearing of space scaling of the arguments of the wave functions, in comparison with the case of particles without the magnetic moment. Formally, we have two sorts of states depending on the sign of Γ:

$$\lambda_{1,2}, \quad x = -2Br = -2i\sqrt{M^2 - p_3^2 + \Gamma\frac{1}{2}(\Gamma \pm \sqrt{\Gamma^2 - 4p^2})}\,,$$

There exists a third type of states in which the parameter Γ does not manifest itself in any way (see (31)):

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \epsilon^2 - M^2 - p_3^2 - \frac{m^2}{r^2}\right)(p_4 f_3 - p_3 f_4) = 0\ ;$$

for these states, the solutions depend on the ordinary (non-modified) argument x:

$$x = -2Br = -2i\sqrt{M^2 - p_3^2}\ .$$

7. Shamaly–Capri Theory and General Relativity

First, let us show that in Minkowski space, the Shamaly–Capri 20-component model for the spin 1 particle in absence of external electromagnetic field is reduced to ordinary DKP 10-component theory. We start with a free particle wave equation

$$(\,i\,\Gamma^a \partial_a \,-\, m)\,\Psi(x) = 0\,, \qquad (34)$$

where the 20-component wave function includes the tensors Φ, Φ_a, $\Phi_{[ab]}$, $\Phi_{(ab)}$, and transforms by the following representation of the Lorentz group $SO(3,1)$

$$T = (0,0) \oplus (1/2,1/2) \oplus (0,1) \oplus (1,0) \oplus (1,1)$$

and Γ^a are 20×20-matrices

$$\Gamma^a = -i\,(\,\lambda_1\,e^{4,a} \,-\, \lambda_1^*\,e^{a,4} \,+$$
$$+\lambda_2\,g_{kn}\,e^{n,[ka]} \,-\, \lambda_2^*\,g_{kn}\,e^{[ka],n} \,-\, \lambda_3\,g_{kn}\,e^{n,(ka)} \,-\, \lambda_3^*\,g_{kn}\,e^{(ka),n}\,; \qquad (35)$$

($*$) stands for complex conjugation, $(g_{ab}) = \mathrm{diag}(+1, -1, -1, -1)$. In (35) numerical parameters λ_i are arbitrary obeying to the following set of restrictions (see [18]):

$$\lambda_1 \lambda_1^* - \frac{3}{2} \lambda_3 \lambda_3^* = 0 \, , \; \lambda_2 \lambda_2^* - \lambda_3 \lambda_3^* = 1 \, . \tag{36}$$

We determine the explicit form of the matrices Γ^a by using basic elements of the relevant matrix algebra $e^{A,B}$:

$$(e^{\;A,B})_C^{\;D} = \delta_C^A \, g^{B,D}, \; e^{A,B} e^{C,D} = g^{B,C} \, e^{A,D} \, ,$$

$$A, \, B, ... = 0, \, a, \, [ab], \, (ab) \, ,$$

where δ_A^B is the generalized Kronecker symbol. The symbols with upper indexes $g^{A,B}$ are derived from δ_B^A with the help of the Minkowski metric tensor. We use the following Kronecker symbols:

$$\delta_{[cd]}^{\;[ab]} = \delta_c^a \, \delta_d^b \; - \; \delta_d^a \, \delta_c^b \, , \; g^{[ab],[cd]} = g^{ac} \, g^{bd} \; - \; g^{ad} \, g^{bc} \, ;$$

$$\delta_{(cd)}^{\;(ab)} = \delta_c^a \, \delta_d^b + \delta_d^a \, \delta_c^b - \frac{1}{2} \, g^{ab} \, g_{cd} \, , \quad g^{(ab),(cd)} = g^{ac} \, g^{bd} + g^{ad} \, g^{bc} - \frac{1}{2} g^{ab} \, g^{cd} \, ;$$

and also the generators J^{ab} for the Lorentz group representation

$$J^{ab} = (e^{a,b} - e^{b,a}) + g_{kn}(e^{[ak],[bn]} - e^{[bk],[an]}) + g_{kn}(e^{(ak),(bn)} - e^{(bk),(an)}) \, .$$

Let us transform now (34) to its tensor form with respect to Φ, Φ_a, $\Phi_{[ab]}$, $\Phi_{(ab)}$:

$$\begin{gathered} \lambda_1 \, \partial^a \Phi_a = m \, \Phi \, , \\ - \lambda_1^* \, \partial_b \Phi \; + \; \lambda_2 \, \partial^a \Phi_{[ba]} \; - \; \lambda_3 \, \partial^a \Phi_{(ba)} = m \, \Phi_b \, , \\ \lambda_2^* \, (\, \partial_b \Phi_a \; - \; \partial_a \Phi_b \,) = m \, \Phi_{[ba]} \, , \\ - \lambda_3^* \, (\, \partial_b \Phi_a \; + \; \partial_a \Phi_b \; - \; \frac{1}{2} \, g_{ab} \, \partial^c \Phi_c \,) = m \, \Phi_{(ab)} \, . \end{gathered} \tag{37}$$

From the first and fourth equations in (37), by considering the relations (36), we obtain

$$- \lambda_1^* \, \partial_b \Phi \; - \; \lambda_3 \, \partial^a \Phi_{(ba)} = \frac{1}{m} \lambda_3 \lambda_3^* \, \partial^a (\partial_a \Phi_b - \partial_b \Phi_a)$$

or

$$- \lambda_1^* \, \partial_b \Phi \; - \; \lambda_3 \, \partial^a \Phi_{(ba)} = \frac{\lambda_3 \lambda_3^*}{\lambda_2^*} \, \partial^a \Phi_{[ab]} \, .$$

Then (see (37) and (36)) we get

$$\frac{1}{\lambda_2^*}\,\partial^a \Phi_{[ba]} = m\,\Phi_b\,.$$

Defining now

$$\Psi_a = \lambda_2^* \Phi_a\,, \quad \Psi_{[ab]} = \Phi_{[ab]}\,, \tag{38}$$

we obtain the *ordinary Proca tensor equations*

$$\partial^b \Psi_{[ab]} = m\Psi_a\,, \quad \partial_a \Psi_b - \partial_b \Psi_a = m\Psi_{[ab]}\,. \tag{39}$$

The last equation can be represented in *DKP matrix form*

$$(\,i\,\beta^a \partial_a - m)\Psi = 0\,, \quad \Psi = \left| \begin{array}{c} \Psi_a \\ \Psi_{[ab]} \end{array} \right|\,,\ \beta^a = -i g_{bc}(e^{c,[ba]} - e^{[ba],c})\,. \tag{40}$$

So, the equations (34) and (40) are equivalent from physical standpoint, because their solutions must be unambiguously mutually related.

The generalization of (34) to the case of arbitrary curved space-time with the metric $g_{\alpha\beta}(x)$ and any relevant $e^{\mu}_{(a)}(x)$, may be performed in accordance with *the tetrad method of Tetrode–Weyl–Fock–Ivanenko* [19]. Such an equation has the form

$$[\,i\Gamma^{\mu}(\partial_\mu + B_\mu)\ -\ m\,]\,\Psi = 0\,, \tag{41}$$

or

$$(\,i\Gamma^a \partial_{(a)}\ +\ \frac{i}{2}\Gamma^a J^{cd}\gamma_{cda}\ -\ m\,)\,\Psi = 0\,. \tag{42}$$

We use the notation

$$\Gamma^\mu = \Gamma^a e^{\mu}_{(a)}\,,\ B_\mu = \frac{1}{2} J^{ab} e^{\nu}_{(a)} \nabla_\mu e_{(b)\nu}\,,$$

$$\partial_{(a)} = e^{\mu}_{(a)}\partial_\mu\,,\ \gamma_{abc} = -(\nabla_\beta e_{(a)\alpha}) e^{\alpha}_{(b)} e^{\beta}_{(c)}\,;$$

here ∇_μ represents the covariant derivative, while γ_{abc} stand for the Ricci rotation coefficients.

Let us show that any two equations of the type (41)

$$[\,i\Gamma^{\mu}(\partial_\mu + B_\mu) - m\,]\,\Psi = 0\,,\ [\,i\Gamma'^{\mu}(\partial_\mu + B'_\mu) - m\,]\,\Psi' = 0\,, \tag{43}$$

referring to the respective tetrads related by a local Lorentz transformation $e'^{\mu}_{(a)}(x) = L_a^{\ b}(x) e^{\mu}_{(b)}(x)$, are mutually translated to each other by means of local transformation of the form $\Psi'(x) = S(x)\Psi(x)$:

$$\left| \begin{array}{l} \Phi' \\ \Phi'_a \\ \Phi'_{[ab]} \\ \Phi'_{(ab)} \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & L_a^{\ c} & 0 & 0 \\ 0 & 0 & L_a^{\ c} L_b^{\ d} & 0 \\ 0 & 0 & 0 & L_a^{\ c} L_b^{\ d} \end{array} \right| \left| \begin{array}{l} \Phi \\ \Phi_c \\ \Phi_{[cd]} \\ \Phi_{(cd)} \end{array} \right| , \tag{44}$$

if and only if the following two relations are valid

$$S \Gamma^{\mu} S^{-1} = \Gamma'^{\mu} \,, \; S B_{\mu} S^{-1} + S \partial_{\mu} S^{-1} = B'_{\mu} \,. \tag{45}$$

The first one can be rewritten as

$$S \Gamma^{a} e_{(a)}^{\ \mu} S^{-1} = \Gamma^{b} e'^{\ \mu}_{(b)} \implies S \Gamma^{a} S^{-1} = \Gamma^{b} L_b^{\ a},$$

which is a known relation, which ensures the Lorentz invariance of the wave equation (34) in Special Relativity. To prove (45), in its form

$$S B_{\alpha} S^{-1} = \frac{1}{2} S J^{ab} S^{-1} e_{(a)}^{\ \beta} \left(\nabla_{\alpha} e_{(b)\beta} \right)$$

we express the tetrad $e_{(a)\alpha}$ in terms of primed tetrad. This leads to

$$S B_{\alpha} S^{-1} = \frac{1}{2} (S J^{ab} S^{-1}) \, (L^{-1})_a^{\ k} \, [\, (\partial_{\alpha} (L^{-1})_b^{\ l} g_{kl} + (L^{-1})_b^{\ l} e'^{\beta}_{(k)} \nabla_{\alpha} e'_{(l)\beta} \,] \, .$$

By using the explicit form of S (see (44)) and of J^{ab}, we prove

$$S J^{ab} S^{-1} = J^{mn} L_m^{\ a} L_n^{\ b}.$$

Then we obtain

$$S B_{\alpha} S^{-1} = \frac{1}{2} J^{mn} L_n^{\ b} \partial_{\alpha} (L_{mb}) + B'_{\alpha}.$$

We can infer the conclusion provided that

$$S B_{\mu} S^{-1} = \frac{1}{2} J^{mn} L_n^{\ b} \partial_{\mu} (L_{mb}) \, .$$

But the last relationship can by proved by using the known pseudo-orthogonality condition for Lorentz transformations and the explicit form of S and J^{ab}. Thus, (43) are mutually related to each other by a local transformation of the type (44).

The general covariant matrix wave equation (42) may be translated to the tetrad tensor form

$$\begin{aligned} \lambda_1 \, (\partial^{(a)} \, + \, \gamma^{ba}_{\ \ b}) \, \Phi_a \, = m \, \Phi \, , \\ -\lambda_1^* \, \partial_{(r)} \Phi \, + \, \lambda_2 \, (\, \partial^{(a)} \Phi_{[ra]} \, + \, \gamma_r^{\ bc} \Phi_{[bc]} \, + \, \gamma^{bc}_{\ \ c} \Phi_{[br]} \,) \, - \\ -\lambda_3 \, (\, \partial^{(a)} \Phi_{(ra)} \, + \, \gamma_r^{\ dc} \Phi_{(dc)} \, + \, \gamma^{dc}_{\ \ c} \Phi_{(dr)} \,) \, = m \, \Phi_r \, , \\ \lambda_2^* \, (\, \partial_{(r)} \Phi_s \, - \, \partial_{(s)} \Phi_r \, + \, \gamma^{d}_{\ rs} \Phi_d \, - \, \gamma^{d}_{\ sr} \Phi_d \,) \, = m \, \Phi_{[rs]} \, , \\ -\lambda_3^* \, [\, (\, \partial_{(r)} \Phi_s \, + \, \partial_{(s)} \Phi_r \, + \, \gamma_r^{\ d}{}_s \Phi_d \, + \, \gamma_s^{\ d}{}_r \Phi_d) \, - \\ - \, \frac{1}{2} g_{rs} \, (\, \partial^{(a)} \Phi_a \, + \, \gamma^{ad}_{\ \ a} \Phi_d \,) \,] \, = m \, \Phi_{(rs)} \, . \end{aligned} \tag{46}$$

Let us eliminate the components, and obtain the equation for the main components Φ_a and $\Phi_{[cd]}$. To this end, from the first and the fourth equation in (46) we express Φ and $\Phi_{(ad)}$ and substitute the results into the second one. Due to the conditions (36) and the third equation in (46), we get

$$\begin{aligned} - \, \lambda_1^* \, \partial_{(r)} \Phi \, - \, \lambda_3 \, [\, \partial^{(a)} \Phi_{(ra)} \, + \, \gamma_r^{\ da} \Phi_{(da)} \, + \, \gamma^{ad}_{\ \ a} \Phi_{(dr)}] = \\ = - \frac{\lambda_3 \lambda_3^*}{\lambda_2^*} \, [\, \partial^{(a)} \Phi_{[ra]} \, + \, \gamma_r^{\ bc} \Phi_{[bc]} \, + \, \gamma^{bc}_{\ \ c} \Phi_{[br]} \,] \, - \\ - \, \frac{2 \lambda_3 \lambda_3^*}{m} \, [\, \gamma^{b}_{\ ar}{}^{,(a)} \, \Phi_b \, + \, \gamma^{ba}_{\ \ b,(r)} \, \Phi_a \, - \\ - \, \gamma^{ab}_{\ \ r} \, \partial_{(a)} \Phi_b \, - \, \gamma_r^{\ ab} \, \partial_{(a)} \Phi_b \, + \, \gamma_r^{\ ab} \gamma^{d}_{\ ba} \Phi_d \, - \, \gamma^{ab}_{\ \ b} \gamma^{c}_{\ ar} \Phi_c \,] \, . \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \frac{1}{\lambda_2^*} \, (\, \partial^{(a)} \Phi_{[ra]} \, + \, \gamma_r^{\ bc} \, \Phi_{[bc]} \, + \gamma^{bc}_{\ \ c} \, \Phi_{[br]} \,) \, - \\ - \, \frac{2 \lambda_3 \lambda_3^*}{m} \, [\, \gamma^{b}_{\ ar}{}^{,(a)} \, \Phi_b \, + \, \gamma^{ba}_{\ \ b,(r)} \, \Phi_a \, - \, \gamma^{ab}_{\ \ r} \, \partial_{(a)} \Phi_b \, - \, \gamma_r^{\ ab} \, \partial_{(a)} \Phi_b \, + \\ + \, \gamma_r^{\ ab} \, \gamma^{d}_{\ ba} \, \Phi_d \, - \, \gamma^{ab}_{\ \ b} \, \gamma^{c}_{\ ar} \, \Phi_c \,] = m \Phi_r \, . \end{aligned} \tag{47}$$

It remains to translate the equations (46) and (47) to the variables Ψ_a and Ψ_{ab} according to (38). In the end, we derive *the tetrad generalized Proca system*:

$$\partial^{(a)}\Psi_{[ra]} + \gamma_r{}^{bc}\,\Psi_{[bc]} + \gamma^{bc}_{\;\;c}\,\Psi_{[br]} - \\ - 2\lambda_3\lambda_3^* m^{-1}\,[\,\gamma^{b}_{\;ar}{}^{,(a)}\,\Psi_b + \gamma^{ba}_{\;\;b,(r)}\,\Psi_a - \gamma^{ab}_{\;\;r}\,\partial_{(a)}\Psi_b - \gamma_r{}^{ab}\,\partial_{(a)}\Psi_b + \\ + \gamma_r{}^{ab}\gamma^{d}_{\;ba}\,\Psi_d - \gamma^{ab}_{\;\;b}\gamma^{c}_{\;ar}\,\Psi_c\,] = m\,\Psi_r\,, \quad (48)$$

$$\partial_{(a)}\Psi_b - \partial_{(b)}\Psi_a + \gamma^{d}_{\;ab}\,\Psi_d - \gamma^{d}_{\;ba}\,\Psi_d = m\,\Psi_{[ab]}\,. \quad (49)$$

In (48), the term proportional to $\frac{2\lambda_3\lambda_3^*}{m}$ determines an additional interaction term for a a generalized vector particle with the gravitational field.

If we take into account the tetrad form of the Riemann and Ricci tensors through the Ricci rotation coefficients (48) can be written as:

$$R_{abcd} = -\gamma_{abc,(d)} + \gamma_{abd,(c)} + \gamma_{akc}\gamma^{k}_{\;bd} + \gamma_{abn}\gamma^{n}_{\;cd} - \gamma_{akd}\gamma^{k}_{\;bc} - \gamma_{abn}\gamma^{n}_{\;dc}\,,$$

$$R^{b}_{\;r} = R^{ab}_{\;\;ra} = -\,\gamma^{ab}_{\;\;r,(a)} + \gamma^{ab}_{\;\;a,(r)} + \gamma_{rna}\gamma^{ban} - \gamma^{a}_{\;ka}\gamma^{kb}_{\;\;r}\,.$$

We finally get

$$\partial^a\Psi_{[ra]} + \gamma_r{}^{bc}\Psi_{[bc]} + \gamma^{bc}_{\;\;c}\Psi_{[br]} - - \frac{2\lambda_3\lambda_3^*}{m}\,[\,R^{b}_{\;r}\Psi_b - \gamma^{ab}_{\;\;r}\,\partial_{(a)}\Psi_b - \gamma_r{}^{ab}\,\partial_{(a)}\Psi_b\,] - m\Psi_r = 0\,.$$

Like in (39), the system (48)-(49) can be represented in *matrix DKP form*:

$$\{\,i\beta^a\partial_{(a)} + \frac{i}{2}\beta^a J^{cd}_{(0)}\gamma_{cda} - \frac{\lambda_3\lambda_3^*}{m}\,[\,(\gamma^{a}_{\;bk} - \gamma^{a}_{\;kb})\,(e^{b,k} - e^{k,b})\partial_{(a)} + R_{bk}\,(e^{b,k} + e^{k,b})\,] - m\,\}\,\Psi = 0\,.$$

It can be readily proved that the tetrad system (46) can be translated to the generally covariant tensor form[7]:

$$\lambda_1\,D_\alpha\Phi^\alpha = m\,\Phi\,, \\ -\lambda_1^*\,D_\beta\Phi + \lambda_2\,D^\alpha\Phi_{[\beta\alpha]} - \lambda_3\,D^\alpha\Phi_{(\alpha\beta)} = m\,\Phi_\beta\,, \\ \lambda_2^*\,(D_\alpha\Phi_\beta - D_\beta\Phi_\alpha) = m\,\Phi_{[\alpha\beta]}\,, \\ -\,\lambda_3^*\,[\,D_\alpha\Phi_\beta + D_\beta\Phi_\alpha - \frac{1}{2}g_{\alpha\beta}\nabla^\rho\Phi_\rho\,] = m\,\Phi_{(\alpha\beta)}\,. \quad (50)$$

The relations between the tetrad and the tensor components are:

$$\Phi_\alpha = e^{(a)}_\alpha\Phi_a\,, \qquad \Phi_{[\alpha\beta]} = e^{(a)}_\alpha e^{(b)}_\beta\Phi_{[ab]}\,, \qquad \Phi_{(\alpha\beta)} = e^{(a)}_\alpha e^{(b)}_\beta\Phi_{(ab)}\,.$$

[7] We consider the notation ∇_α described by the equality $D_\alpha = \nabla_\alpha - ieA_a(x)$.

As for (46), the system (50) can be reduced to the minimal form

$$\frac{1}{\lambda_2^*} D^\alpha \, \Phi_{[\beta\alpha]} \, + \, \frac{2\lambda_3\lambda_3^*}{m} \, [D_\alpha, D_\beta]_- \, \Phi^\alpha \, = m \, \Phi_\beta \, ,$$
$$\lambda_2^* \, (D_\alpha \Phi_\beta \, - \, D_\beta \Phi_\alpha) \, = m \, \Phi_{[\alpha\beta]} \, .$$

or, alternatively, to

$$D^\alpha \Psi_{[\beta\alpha]} \, + \, \frac{2\lambda_3\lambda_3^*}{m} \, [D_\alpha, D_\beta]_- \, \Psi^\alpha = m \, \Psi_\beta \, ,$$
$$D_\alpha \Psi_\beta - D_\beta \Psi_\alpha = m \, \Psi_{[\alpha\beta]} \, . \qquad (51)$$

Taking into account that

$$[D_\alpha, D_\beta]_- \, \Psi^\alpha = (-ieF_{\alpha\beta} + R_{\alpha\beta}) \Psi^\alpha \, ,$$

we conclude that the parameter $\frac{\lambda_3\lambda_3^*}{m}$ in (51) determines both the *anomalous magnetic moment* of the spin 1 particle and *the additional interaction term with non-Euclidean space-time background* through the Ricci tensor $R_{\alpha\beta}$.

Conclusion

By applying the matrix 10-dimensional Duffin–Kemmer-Petiau formalism to the Shamaly–Capri field, the behavior of a vector particle with anomalous magnetic moment is studied in the presence of external uniform magnetic field. The separation of variables in the wave equation is performed by using projective operator techniques and the DKP-algebra theory. The problem is reduced to a system of 2-nd order differential equations for three independent functions, which are solved in terms of confluent hypergeometric functions. Three series of the energy levels are found; two of them substantially differ from those for spin 1 particle without anomalous magnetic moment. To assign to them physical sense for all the values of the main quantum number $n = 0, 1, 2, ...$ one must impose special restrictions on a parameter related to the anomalous moment. Otherwise, the energy levels corresponds only partially of to bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in the occurrence of space scaling of the arguments of the wave functions, in comparison with a particle which has no such moment. Some features of theory of the Shamaly–Capri particle within General Relativity are given.

References

[1] E. S. Fradkin. To the theory of particles with higher spins. *JETP*. 1950. Vol. 20. P. 27–38.

[2] V. Ya. Feinberg. To interaction of the particlesof higher spins with electromagnetic fields. *Trudy FIAN. USSR*. 1955. Vol. 6. P. 269–332.

[3] M. Petras. A note to Bhabha's equation for a particle with maximum spin 3/2 M. Petras // *Czech. J. Phys.*-1955. Vol. 5. Vol. 3. P. 418–419.

[4] I. Ulehla. Anomalous equations for spin 1/2 particles. *JETP*. 1957. Vol. 33. P. 473–477.

[5] F. I. Fedorov, V. A. Pletyukhov. Wave equatioons withe repeated representations of the Lorentz group Intejer spin. *Vesti of NASB. Ser. fiz.-mat. nauk.* 1969. Vol. 6. P. 81–88; Wave equatioons withe repeated representations of the Lorentz group Half-intejer spin. *Vesti of NASB. Ser. fiz.-mat. nauk.* 1970. Vol. 3. P. 78–83; Wave equatioons withe repeated representations of the Lorentz group for spin 0 particle. *Vesti of NASB. Ser. fiz.-mat. nauk.* 1970. Vol 2. P. 79–85; Wave equatioons withe repeated representations of the Lorentz group for spin 1 particle. *Vesti of NASB. Ser. fiz.-mat. nauk.* 1970. Vol. 3. P. 84–92.

[6] A. Z. Capri. Nonuniqueness of the spin 1/2 equation. *Phys. Rev.* 1969. Vol. 178. Vol. 5. P. 181–1815; First order wave equations for half-odd-integral spin. *Phys. Rev.* 1969. Vol. 178. P. 2427–2433; Electromagnetic properties of a new spin-1/2 field. *Progr. Theor. Phys.* 1972. Vol. 48. P. 1364–1374.

[7] A. Shamaly, A. Z. Capri. First-order wave equations for integral spin. *Nuovo Cimento. B.* 1971. Vol. 2. P. 235–253; Unified theories for massive spin 1 fields. *Can. J. Phys.* 1973. Vol. 51. P. 1467–1470.

[8] M. A. K. Khalil. Properties of a 20-component spin 1/2 relativictic wave equation. Phys. Rev. D. 1977. Vol. 15. P. 1532–1539: Barnacle equivalence structure in relativistic wave equation. Progr. Theor. Phys. 1978. Vol. 60. P. 1559–1579; An equivalence of relativistic field equations. Nuovo Cimento. A. 1978. Vol. 45. P. 389–404; Reducible relativistic wave equations. *J. Phys. A. Math. and Gen.* 1979. Vol. 12. P. 649–663.

[9] A. A. Bogus, V. V. Kisel. Equations with epeated representations of the Lorent group and interaction of the Pauli type. *Izvestia Wuzov. Phizika*. 1984. Vol. 1. P. 23–27.

[10] A. A. Bogus, V. V. Kisel, L. G. Moroz, M. I. Levchuk. To desription pf scalar paticle with polarizability. Covariant methods in theoretical physics. Elementary particles physics and relativity theory. *IP NAN BSSR*. Minsk, 1981. P. 81–90.

[11] A. A. Bogus, V. V. Kisel, F. I. Fedorov. On interpretation of subsidiary components of wave funcrtions in electromagnetic interaction. *Doklady NAN BSSR*. 1984. Vol. 277. Vol. 2. P. 343–346.

[12] V. V. Kisel, N. G. Tokarevskaya, V. M Red'kov. *Petrash theory for particles with spin 1/2 in curved space-time.* Preprint 737. Institute of Physics of NAS of Belarus: Minsk, 2002.

[13] A. A. Bogush, V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. Petras theory for particles with spin 1/2 in curved space-time. *Vesti of NASB. Ser. fiz.-mat. nauk.* 2002. Vol. 1. P. 63–68.

[14] V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. Spin 1/2 particle with anomalous magnetic moment in a curved space-time, non-relativistic approximation. Pages 36-42 in: Proceedings of 11-th International School and Conference "Foundation and Advances in Nonlinear Science", Eds.: Kuvshinov V.I., Krylov G.G., Minsk, 2004.

[15] A. A. Bogush, V. V. Kisel, N. G. Tokarevskaya, V. M. Red'kov. *Petras Theory of a Spin-1/2 Particle in Electromagnetic and Gravitational Fields.* 44 pages, 18 Apr 2006. – arXiv:hep-th/0604109.

[16] V. V. Kisel. *Relativistic wave equations with extended sets of represetations.* Kandid. dissert. Minsk, 1984. 11 p.

[17] V. V. Kisel, V. M. Red'kov. Shamaly-Capri equation and additional interaction of a vector particle with gravitational field. Covariant methods in theoretical physics. *Elementary particles physics and relativity theory.* Institute of Physics, NAS of Belarus. Minsk, 2001. P. 107–112.

[18] V. A. Pletyukhov, V. M. Red'kov, V. I. Strazhev. *Relativistic wave equations and intrinsic degrees of fredom.* Belarusian Science: Minsk, 2015. 328 p. (in Russian).

[19] V. M. Red'kov. *Fields of particles in Riemannian space and the Lorentz group.* Belarusian Science: Minsk, 2009. – 486 p. (in Russian).

[20] F. I. Fedorov. *The Lorentz group*. – Moscow, 1979. – 384 p.

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Chapter 4

A QUATERNION APPROACH IN THE ESTIMATION OF THE ATTITUDE OF AN INDUSTRIAL ROBOT

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Abstract

Traditionally, the automotive industry has been the largest employer of robots, but their control is inline and programmed to follow planning trajectories. In this case, in the department motor's test of Volkswagen Mexico a semi-autonomous robot is developed. To date, some critical technical problems must be solved in a number of areas, including in dynamics control. Generally, the attitude estimation and the measurement of the angular velocity are a requirements for the attitude control. As a result, the computational cost and the complexity of the control loop is relatively hight. This chapter deals with the implementation of a cheap Micro AHRS (Attitude and Heading Reference System) using low-cost inertial sensors. In the present chapter, the technique proposed is designed with attitude estimation and the prediction movement via the kinematic of a 4GDL robot. With this approach, only the measurements of at least two non-collinear directional sensors are needed. Since the control laws are highly simple and a model-based observer for angular velocity reconstruction is not needed, the proposed new strategy is very suitable for embedded implementations. The global convergence of the estimation and prediction techniques is proved. Simulation with some robustness tests is performed.

Keywords: estimation, quaternion, prediction of the movement, robot, attitude, AHRS

1. Introduction

Robots have considerable potential for application in Volkswagen plants. Looking at the four major sectors of a vehicle assembly operation, as follow:

1. PRESS. As a VW has installed high-speed pressed with integral part handling.
2. BODY. They are seeking for robots that provide speed, accuracy, more payload capacity and are being easy to integrate.
3. PAINTING. In this area they want to find robots that have the abilities to do such things as see and felt.

In the motor test area, the assemblies of all of the instrumentation and wiring systems, and the test per se, autonomous robotic and telerobotic systems have been suggested. Industrial Robot has been considered for the different test.

To date, some critical technical problem must be solved in a number of areas, including in dynamics and control. A prerequisite is state estimation where the states typically are position, velocity and orientation. State estimation is especially important where attitude control is needed. With attitude we refer to the robot's orientation relative to the gravity vector, usually described by pitch and roll movements. Attitude estimation is usually performed by combining measurements from three kinds of sensors: rate gyros, inclinometers and accelerometers.

Figure 1. Robot and telerobotic systems.

The attitude (orientation) of a rigid body can be parameterized by several methods: for instance, Euler's angles, Cardan angles and unit quaternion. The unit quaternion is a four parameter representation with one constraint. Therefore, it yields the lowest dimensionality possible for a globally non-singular representation of the attitude.

Several approaches have been applied to the attitude estimation problem. These estimators fall into three main families. The first one deals with a constraint least-square minimization problem proposed firstly by Wahba [1, 2], for finding the rotation matrix.

The second approach is within the framework of the EKF (Extended Kalman filter) [3]. Its major feature concerns the ability to fuse signals acquired from different sensor types. An excellent survey of these methods is given in Ref. [4].

The third approach issues from nonlinear theory, and non linear observers are applied to the attitude determination problem [5, 6, 7, 8, 9]. In this approach, the convergence of the error to zero is proved in a Lyapunov sense.

In this chapter, an attitude estimator using quaternion representation is studied. Two approaches are jointly used, namely a constraint least-square minimization technique and a prediction technique. Thus, no assumptions of the weakness (or not) of the accelerations are done. Therefore, the main advantage of the approach presented in this chapter compared to others approaches, is that the estimated attitude remains valid even in the presence of high accelerations over long time periods.

The chapter is organized as follows. First the algebra in a quaternion-based formulation of the orientation of rigid body is given. After, the problem statement and the kinematic model is formulated. Then attitude's estimation and prediction via quaternion is presented. Finally, some simulation results are given.

2. Mathematical Background

As mentioned in the introduction, the attitude of a rigid body can be represented by a unit quaternion, consisting of a unit vector $\vec{e}$ known as the Euler axis, and a rotation angle β about this axis. The quaternion $q$is then defined as follows:

$$q = \begin{pmatrix} cos\frac{\beta}{2} \\ \vec{e}sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix} \in \mathbb{H} \tag{1}$$

where

$$\mathbb{H} = \begin{matrix} \{q \mid {q_0}^2 + \vec{q}^T\vec{q} = 1 \\ q = [q_0\vec{q}^T]^T, q_0 \in \mathbb{R}, \vec{q} \in \mathbb{R}^3\} \end{matrix} \tag{2}$$

$\vec{q} = [q_1 q_2\, q_3]^T$ and q_0 are known as the vector and scalar partis of the quaternion recpectively. In attitude control aplications, the unit quaternion represents the rotation from an inertial coordinate system $N(x_n, y_n, z_n)$ located at some point in the space (for instance, the earth NED frame), to the body coordinate system $B(x_b, y_b, z_b)$ located on the center of mass of a rigid body.

If $\vec{r}$ is a vector expressed in N, then its coordinates in $\boldsymbol{B}$ are expressed by:

$$b = \bar{q} \otimes r \otimes q \tag{3}$$

where $b = \left[0\, \vec{b}^T\right]^T$ and $r = [0\, \vec{r}^T]^T$ are the quaternions associated to vectors $\vec{b}$ and $\vec{r}$ respectively. $\otimes$ denotes the quaternion multiplication and $\bar{q}$ is the conjugate quaternion multiplication of q, defined as:

$$\bar{q} = [q_0 - \vec{q}^T]^T \tag{4}$$

The rotation matrix C(q) corresponding to the attitude quaternion q, is computed as:

$$C(q) = ({q_0}^2 - \vec{q}^T\vec{q})I_3 + 2(\vec{q}\vec{q}^T - q_0[\vec{q}^x]) \tag{5}$$

where I_3 is the identity matrix and $[\xi^x]$is a skew symmetric tensor associated with the axis vector ξ:

$$[\xi^x] = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}^x = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \quad (6)$$

Thus, the coordinate of vector $\vec{r}$ expressed in the B frame is given by:

$$\vec{b} = C(q)\vec{r} \quad (7)$$

The quaternion attitude error used to quantify the mismatch between two attitudes q_1 and q_2 is computed by:

$$q_e = q_1 \otimes \bar{q}_2 \quad (8)$$

The reason that quaternions found applications in computer graphics, computer vision, robotics, navigation, molecular dynamics, flight dynamics, and orbital mechanics, is their two distinct geometric interpretations.

A. They can represent rotations in 3 and 4 space (and are called Rotation quaternions in this case).

B. They can represent an orientation (rotation relative to a reference position) in 3D space, they are called orientation quaternions or attitude quaternions. Algorithmically they require a less number of operations to implement rotations (when compared with the multiplying a vector with a rotation matrix that involves Euler angles). Any rotation of a 4D vector can be represented by the product of two Quaternions with that vector. Two excellent references describing these details from an application perspective are:

Note

Each of the algebras of complex numbers, Quaternions, Octonions, Sedenions, etc. (or 2n-ons) are generalizations algebras of the previous in the list and they contain all the previous ones as sub-algebras, while all are special cases of Cayley-Dixon Algerbas or Hyperlinear Algebras as some are calling them.

Although Quaternions are attributed to Hamilton (1843) it was C.F. Gauss who first discovered them earlier (1819) but published his work in 1900. Also, in 1864 J.C. Maxwell in his first paper (before the book) on Electromagnetism (James Clerk Maxwell, “A Dynamical Theory of the Electromagnetic Field” (Royal Society Transactions, Vol. CLV, 1865, p 459) formulated

Electromagnetism in terms of 20 quaternion equations that also appeared in the 1873 edition of “A Treatise on Electricity and Magnetism” aka The BOOK.

3. Problem Statement

Good models of industrial robots are necessary in a variety of applications, such as mechanical design, performance simulation, control, diagnosis, supervision and offline programming. This motivates the need for good modelling tools. In the first part of this book the foreword kinematic modelling of serial industrial robots is studied, the focus is on modelling the foreword kinematics. The main interest is the principal structure, and issues regarding efficiently implementation have not been considered. The work is based on homogeneous transformations using the Denavit-Hartenberg (D-H) representation, which gives coordinate frames adapted to the robot structure and a quaternion approach in the attitude estimation modelling is presented.

Attitude is a term used to describe the orientation of a vehicle in three-dimensional space. The attitude of the articulated arm can be described using several different parameters (e.g., quaternions, Euler angles, direction cosine matrices, etc.) that have been described in some detail in earlier chapters of this book. Robotic attitude determination systems provide a means for measuring or estimating these parameters that describe the end effector and the robot’s orientation. The outputs of these systems are used for vehicle guidance, navigation, and in this case, control and attitude determination. The focus of this chapter is attitude determination systems for small robots, which, for the purposes here, are defined to be robot with a total mass between 150 kg and 300 kg and fit in a volume envelope roughly 50 cm × 50 cm × 60 cm in dimension. Designing attitude determination systems for these small articulated robots represents a challenging and specialized task; sensors and their processing must also be made to fit within the limited size, weight, and mass specifications while still performing to high accuracies for many applications. This chapter highlights these challenges (particularly for a robot like showed in Figure 3) and discusses current and future developments aimed at addressing them.

In the case of the attitude estimation, one seeks to estimate the attitude and accelerations of a rigid body. From now on, it is assumed that the system (AHRS) is equipped with a triaxis accelerometer, three magnetometery and three rate gyros mounted orthogonally.

In this chapter, we describe the body’s kinematic of the model. In order to estimate the arm’s robot position with respect to an inertial frame, a module containing three rate gyros, three accelerometer and three magnetometery assembled in tri-axis, are positioned in the extreme of the arm's robot. Thus, the attitude for the articulation is estimated. The combination of this information jointly to a knowledge a priori of the robot makes possible to obtain information on the end effector respect to the base.

3.1. Inertial Sensors

Inertial sensors describe a pair of measurement devices used to determine a subset of the kinematic state of the body to which they are attached. The sensors are accelerometers, magmetometers and gyrometers. Accelerometers measure specific force—the algebraic sum of linear acceleration and gravitational acceleration normalized by mass. The name "accelerometer" is somewhat of a misnomer, as the sensor actually measures force rather than acceleration. However, by proper data scaling, a measurement of acceleration can be produced. A triad of accelerometers arranged orthogonally will measure the specific force vector of a vehicle. Gyrometers (or gyros for short) measures angular rate or integrated rate. Integrated rate is sometimes called "incremental angle" or simply "delta theta" $\Delta\theta$. Lower cost (and quality) gyros tend to be rate gyros measuring rate, while higher end sensors are rate-integrating sensors whose output is $\Delta\theta$ Similarly, a triad of gyros will provide a measurement of the angular velocity vector. Normally, a triad of accelerometers and gyros are packaged together to form what is called an inertial measurement unit (IMU/AHRS).

3.2. IMU/AHRS Kinematic Equations

As opposed to the limited acceleration data from accelerometers, magnetometers and gyros can provide continuous useful information for attitude determination. The output of the triad of rate gyros is a measurement of the angular velocity $\omega_{b/i}^{b}$. If rate-integrating gyros are used, then the output will be $\Delta\theta_{b/i}^{b}$. The subscript b/i indicates that these sensors measure the angular rate of the body frame relative to the inertial frame. The superscript indicates that this measurement is expressed in $\mathcal{F}^{b}$ Without a loss of generality, the discussion below will focus on the use of rate gyros and, thus, $w = \omega_{b/i}^{b}$ as the basic measurement processed. This is motivated, in part, by the fact that rate gyros are more typical in robotics applications.

The equation describing the relation between the quaternion and the body's kinematic is given in introducing the angular variation $\vec{w} = \left[w_x w_y w_z\right]^T$ from this, it follows.

$$\dot{q} = \frac{1}{2}\Omega(\vec{w})q(t) = \frac{1}{2}\Xi(q)\vec{w}(t) \quad (9)$$

where $\Omega(\vec{w})$ y $\Xi(q)$ are defined as:

$$\Omega(\vec{w}) = \begin{bmatrix} -[\vec{w}x] & \dots & \vec{w} \\ \dots & \dots & \dots \\ -\vec{w}^T & \dots & 0 \end{bmatrix} \quad (10)$$

$$\Xi(q) = \begin{bmatrix} q_0 I_3 x_3 + [\vec{q}x] \\ \dots . \\ -\vec{q}^T \end{bmatrix} \quad (11)$$

The matrix $[\vec{w}x]$ and $[\vec{q}x]$are obtained by the cross product issue of $\vec{d} \times \vec{b} = \left[\vec{d} \times\right]b$ with $\left[\vec{d} \times\right] \in \mathbb{R}^{3x3}$:

$$\left[\vec{d} \ \times\right] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (12)$$

The quaternion must be:

$$q^T q = \vec{q}^T \vec{q} + q_0{}^2 = 1 \quad (13)$$

In the other hand, the matrix $\Xi(q)$ has the relation:

$$\Xi^T(q)\Xi(q) = q^T q I_{3x3} \quad (14)$$

$$\Xi(q)\Xi^T(q) = q^T q I_{4x4} - q^T q$$

$$\Xi^T(q)(q) = 0_{3x1}$$

Generally $\Xi^T(q)\lambda = -\Xi^T(\lambda)q$, for any $\lambda \in \mathbb{H}$.

$$A(q) = (q_0{}^2 - \vec{q}^T\vec{q})I_{3x3} + 2\vec{q}\vec{q}^T - 2q_0[\vec{q} \times] \quad (15)$$

That is denoted like the orientation matrix 3-D of dimension 3x3.

4. Robot Configuration

The robot links form a kinematic chain. When the kinematic chain is open, every link is connected to every other link by one and only one chain. If, on the other hand, a sequence of the links forms one or more loops, the robot contains closed kinematic chains. In Figure 2 example of configurations of robots is showed.

IRB2400 has closed kinematic chains

IRB340 FlexPicker has closed kinematic chains

IRB6600 has an open kinematic structure

Figure 2. Robot configurations found among the ABB robots.

The robot has a closed kinematic chain due to the so-called parallelogram-linkage structure, represented by a mechanical coupling between motor, placed on the foot of the robot, and the actual link 3. It can also be seen from the figure (from the ABB internet web site) that has an open kinematic-chain structure. Robots having an open kinematic chain can be divided into the following types, based on geometry. The Cartesian robot has three prismatic joints and the links are mutually orthogonal, which gives that each degree of mobility corresponds to a degree of freedom in the Cartesian space. Changing the first prismatic joint to a revolute joint gives a cylindrical geometry, where each degree of mobility corresponds to a degree of freedom in cylindrical coordinates. Replacing the first two prismatic joints by revolute joints gives a spherical robot, where the degrees of freedom are in spherical coordinates, similar to the cases above. SCARA stands for selective compliance assembly robot arm, and a SCARA robot has a mechanical structure with high stiffness to vertical load and compliance to horizontal load. The anthropomorphic robot has three revolute joints, and has similarities to a human arm. The workspace is approximately a portion of a sphere and the robot structure can be seen in many industrial applications. The types are explained in more detail in Sciavicco and Siciliano (2000) and Spong et al. (2006), among others. The work in this thesis is limited to serial robots, that is, robots with an open kinematic structure, and the class of parallel robots that can be rewritten to this structure using a bilinear transformation. Especially serial robots having only revolute joints, so-called anthropomorphic robots, are studied.

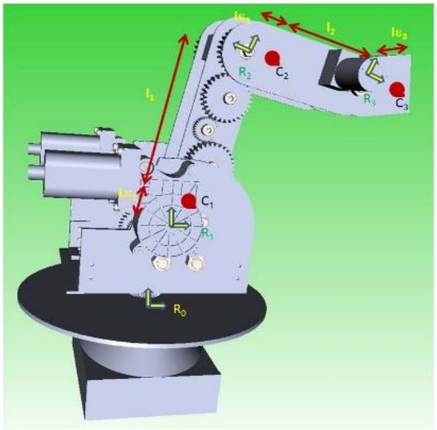

Figure 3. Virtual robot.

In this chapter we obtain the attitude from the robot show in Figure 3 with an open kinematic structure.

Using coordinate frames attached to each joint, shown in Figure 3, the position and orientation of the robot tool can be defined in the Cartesian coordinates Ci, with respect to the base frame R_0 of the robot by successive coordinate transformations. This results in the relation.

$I_{\varepsilon n}, n = 1{:}3$, Distance between the center of rotation and the center of attitude sensor module.

$I_n, n = 1{:}3$, Distance between the position of the attitude sensor module and the end of the segment in consideration.

$C_n, n = 1{:}3$, is the attitude sensor module.

The point 0 is supposed to be fixe.

Table 1. Coordinate systems

link	θ_i	d_i	a_i	α_i
1	θ_1	0	a_1	$\frac{-\pi}{2}$
2	θ_2	l_1	0	0
3	θ_3	l_2	0	$\frac{\pi}{2}$

Table 2. Quaternion-based model

Quaternion	$vector$
$Q = \left[\cos\frac{\theta_1}{2} \quad 0 \quad 0 \quad \sin\frac{\theta_1}{2}\right]$	V_1
$R = \left[\cos\frac{\theta_2}{2} \quad \sin\frac{\theta_2}{2} \quad 0 \quad 0\right]$	V_2
$R = \left[\cos\frac{\theta_3}{2} \quad \sin\frac{\theta_3}{2} \quad 0 \quad 0\right]$	V_3

The angular velocity $\vec{\omega}_{CI}$ is obtained by finite differences from equation (9) at the instants k and k-1 (k estimation instant).

$$\vec{\omega}_{CI} = 2\Xi^T(q_{CI})q_{CI} \tag{16}$$

According to the robot, the model for foreword kinematics based on the convention Denavit-Hartenberg is defined for each one of the coordinate systems in Table 1.

This model can be compared with the quaternion-based model, which is Simplified in Table 2, so it is possible to observe that the quaternion-based model is significantly more compact, which reduces the number of operations significantly.

4.1. Modeling Sensors

1) Rate Gyros: The angular velocity $\vec{\omega} = [\omega_1\omega_2\omega_3]^T$ is measured by the rate gyros, which are supposed to be orthogonally mounted. The output signal of a rate gyro is influenced by various factors, such as bias drift and noise. In the absence of rotation, the output signal can be modeled as the sum of a white Gaussian noise and of a slowly varying function. Since an integration step is required in order to obtain the current attitude quaternion (9), even the smallest variation of the rate gyro measurement will produce a wrong estimation of the attitude. The bias is denoted by $\vec{v}$, belonging to space R^3. The rate gyro measurements are modeled by (Brown y Hwang, 1997):

$$\vec{\omega}_G = \vec{\omega} + \vec{\upsilon} + \vec{\eta}_G \tag{17}$$

$$\dot{\vec{\upsilon}} = -T^{-1}\vec{\upsilon} + \vec{\eta}_\upsilon \tag{18}$$

where $\vec{\eta}_G$ and are $\vec{\eta}_\upsilon$ supposed by Gaussian white noises and $T = \tau I_3$ is a diagonal matrix of time constants. In this case, the constant τ which has been set to 100 s. The bias vector $\vec{\upsilon}$ will be estimated online, using the observer presented in the following section.

2) Accelerometers: Since the 3-axis accelerometer is fixed to the body, the measurements are expressed in the body frame B. Thus, the accelerometer output can be written as:

$$\vec{b}_A = C(q)\left(\vec{a} - \vec{g}\right) + \vec{\eta}_A \tag{19}$$

where $\vec{g} = \begin{bmatrix} 0 & 0 & g \end{bmatrix}^T$ and $\vec{a} \in R^3$ are the gravity vector and the inertial accelerations of the body respectively. Both are expressed in frame N. g = 9:81 m/sec2 denotes the gravitational constant and $\vec{\eta}_A \in R^3$ is the vector of noises that are supposed to be white Gaussian.

3) Magnetometers: The magnetic field vector $\vec{h}_M$ is expressed in the N frame it is supposed to be $\vec{h}_M = \begin{bmatrix} h_{Mx} & 0 & h_{Mz} \end{bmatrix}^T$. Since the measurements take place in the body frame B, they are given by:

$$\vec{b}_M = C(q)\vec{h}_M + \vec{\eta}_M \tag{20}$$

where $\vec{\eta}_M \in R^3$ denotes the perturbing magnetic field. This perturbation vector is supposed to be modeled by Gaussian white noises.

5. Non Linear Attitude Observer

The attitude nonlinear observer that includes the bias and the error update is given by:

$$\dot{\hat{q}} = \frac{1}{2}\Xi(\hat{q})\left[\vec{\omega}_G - \hat{\vec{\upsilon}} + K_1\vec{\varepsilon}\right] \tag{21}$$

$$\dot{\hat{\vec{\upsilon}}} = -T^{-1}\hat{\vec{\upsilon}} - K_2\vec{\varepsilon}\,. \tag{22}$$

where T has been defined in (18) and Ki; i = 1; 2 are positive constant parameters $\hat{q}$ is the prediction of the attitude at time t. It this obtained via the integration of the kinematics equation (14) using the measured angular velocity $\vec{\omega}_G$, the bias estimate $\hat{\vec{\upsilon}}$ and $\vec{\varepsilon} = \vec{q}_e$ which is the vector part of the quaternion error qe. Remember that qe measures the discrepancy between $\hat{q}$ and the pseudo measured attitude q_{ps} (17). In this chapter, qps is obtained thanks to an appropriate treatment of the accelerometer and magnetometer measurements and it will be explained in the next section.

Combining (17), (18), (21) and (22) the error model is expressed as:

$$\dot{q}_e = \frac{1}{2}\begin{pmatrix} 0 & \vec{\gamma}^T \\ -\vec{\gamma} & \left[2\vec{\omega}^\times\right] + \left[\vec{\gamma}^\times\right] \end{pmatrix}\begin{pmatrix} q_{e0} \\ \vec{q}_e \end{pmatrix} \tag{23}$$

$$\dot{\tilde{\upsilon}} = -T^{-1}\tilde{\upsilon} + K_2\vec{\varepsilon} \tag{24}$$

where $\vec{\gamma} = \tilde{\upsilon} + K_1\vec{\varepsilon}$ and $\tilde{\upsilon} = \vec{\upsilon} - \hat{\vec{\upsilon}}$. The system (22)-(23) admits two equilibrium points ($q_{e0} = 1,\ \vec{q}_e = 0, \tilde{\upsilon} = 0$) and ($q_{e0} = -1,\ \vec{q}_e = 0, \tilde{\upsilon} = 0$). This is due to fact that quaternions q and -q represent the same attitude. From (1), one obtains:

$$q_{e0} = 1 \Rightarrow \beta = 0$$

$$q_{e0} = -1 \Rightarrow \beta = 2\pi\left(generally \qquad 2n\pi\right)$$

that is, there is only one equilibrium point in the physical 3D space.

We can observe that the global asymptotically convergence of the error to zero ($q_{e0}=1$, $\vec{q}_e=0$, $\tilde{\upsilon}=0$) and consequently the convergence of $\hat{q}$ to the real q is given by:

$$\vec{\eta}_G=\vec{\eta}_\upsilon=0 \qquad \hat{q}_{ps}\approx q \tag{25}$$

where q is the "true" attitude quaternion of the rigid body. Thus, the convergence is guaranteed if and only if:

$$|q_{e0}|\to 1, \quad \vec{q}_e\to 0, \quad \tilde{\upsilon}=\vec{\upsilon}-\hat{\vec{\upsilon}}\to 0 \tag{26}$$

Theorem 1. Consider the equilibrium states of the system (21)-(22) and let $\vec{\omega}_G$ be the measured angular velocity. Then, the equilibrium point ($q_{e0}=1, \vec{q}_e=0, \tilde{\upsilon}=0$) is globally asymptotically stable.

Proof. Consider the candidate Lyapunov function V which is positive definite, radially unbounded and which belongs to the class C^2:

$$V=K_2\left(\left(1-q_{e0}\right)^2+\vec{q}_e^T\vec{q}_e\right)+\frac{1}{2}\tilde{\upsilon}^T\tilde{\upsilon} \tag{27}$$

The derivative of (27), together with (23) and (24), is given by (28):

$$\begin{aligned}\dot{V} &= -2K_2\dot{q}_{e0}+\tilde{\upsilon}^T\dot{\tilde{\upsilon}} \\ &= -K_2\vec{\gamma}^T\vec{q}_e+\tilde{\upsilon}^T\left(-T^{-1}\tilde{\upsilon}+K_2\vec{\varepsilon}\right) \\ &= -K_2\left(\tilde{\upsilon}^T+K_1\vec{\varepsilon}^T\right)\vec{q}_e-\tilde{\upsilon}^T T^{-1}\tilde{\upsilon}+K_2\tilde{\upsilon}^T\vec{\varepsilon}\end{aligned} \tag{28}$$

Since $\vec{\varepsilon}=\vec{q}_e$ and $\tilde{\upsilon}^T\vec{q}_e=\vec{q}_e^T\tilde{\upsilon}$, it comes that:

$$\dot{V}=-K_2K_1\vec{q}_e^T\vec{q}_e-\tilde{\upsilon}^T T^{-1}\tilde{\upsilon}\le 0 \tag{29}$$

6. Computation of the Attitude's Estimation and Prediction

The attitude estimator uses quaternion representation. Two approaches are jointly used, namely an estimation with a constraint least-square minimization technique and a prediction of the estate at the instant *k*. The prediction is performed in order to produce an estimate of the accelerations and the attitude quaternion.

Actually, this latter problem is divided in three steps. First, the body accelerations are estimated from the previously computed quaternion. Then, the influence of the body accelerations is predicted from the accelerometer measurements together with the magnetometer measurements, a measure estate is estimated via an optimization technique. In this way, the quaternion that is obtained by the estimation with a constraint least-square is insensitive to the body accelerations. Thus, no assumptions of the weakness (or not) of the accelerations are done, and no switching procedure from one model to another one is necessary. Therefore, the main advantage of the approach presented in this chapter compared to others approaches, is that the estimated attitude remains valid even in the presence of high accelerations over long time periods.

In this chapter, a critter that takes in account the evolution of the attitude state via determination of $x = [q_o, q_1, q_2, q_3, a_x, a_y]^T$ in the function $f(x)$ is proposed. The mínimum error is chosen, but it takes in account the prediction of the state $\hat{x}$ and the coefficients of weight for the state μ and the measures estimated (MesEstimated = MS) at the instant *k*.

$$f(x) = \frac{1}{2}\left[\mu\left(\sum_{j=1}^{n}\left(MesEstimated - v_{mes}(j)\right)^2\right) + \|\hat{x} - x\|_2^2\right] \qquad (30)$$

with $q^T q - 1 = 0$.

The process of Estimation and Prediction needs the determination of his gradient; this one is obtained by equation (24)

$$H_q = \left[\frac{\partial}{\partial q}\left(\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\right]$$
$$H_q = \left[\frac{\partial^2 f}{\partial q^2}\frac{\partial q}{\partial x}\right]^T \qquad (31)$$

Similarly, the gradient of the state for the case of acceleration is obtained.

Finally, the total Gradient is obtained by the fusion between the calculation show for the quaternion case and the gradient omitted for the acceleration case.

$$F(x) = \begin{bmatrix} H_q \frac{\partial_{g_q}}{\partial a} \\ \frac{\partial_{g_a}}{\partial x} H_a \end{bmatrix} \tag{32}$$

For the prediction's process of $\hat{x}$, several tehcnique have been validated, for purpose of simplicity, the prediction via spline is chosen. Cubic spline is a spline constructed of piecewise third-order polynomials which pass through a set of n control points.

Suppose we are n+1 data points $(\hat{x}_k, MS_k)$ such that.

$a = x_0 <..< x_n$, Then the coefficients of the vector μ exists cubic polynomials with coefficients $\mu_{i,j} 0 \leq i \leq 3$ such that the following hold.

$$\mu(\hat{x}) = \mu_j(\hat{x}) = \sum_{j=0}^{3} \left[\hat{x} - x_j\right]^i \ \forall \hat{x} \in \left[\hat{x} - x_{j+1}\right] 0 \leq k \leq n-1$$

$$\mu(x_j) = y_k \ 0 \leq k \leq n-1$$

$$\mu_j(x_{j+1}) = \mu_{j+1}(\mu_{j+1}) \ 0 \leq k \leq n-2$$

$$\mu_j'(x_{j+1}) = \mu_{j+1}'(x_{j+1}) \ 0 \leq k \leq n-2$$

$$\mu_j''(x_{j+1}) = \mu_{j+1}''(x_{j+1}) \ 0 \leq k \leq n-2$$

So we see that the cubic spline not only interpolates the data $(\hat{x}_k, MS_k)$ but matches the first and second derivatives at the knots. Notice, from the above definition, one is free to specify constrains on the endpoints. The end point constrain $\mu''(a) = 0 \ \mu''(b) = 0$ is chosen.

7. Results

In this section, some simulation and results of the articulated arm (Figure 4) are represented in order to show the performance of the proposed control laws via quaternion. A rigid body with low momentum of inertia is taken as the experimental system. In fact, the low momentum of inertia makes the system vulnerable to high angular accelerations which prove the importance to apply the control.

Figure 4. Robot for the experiment.

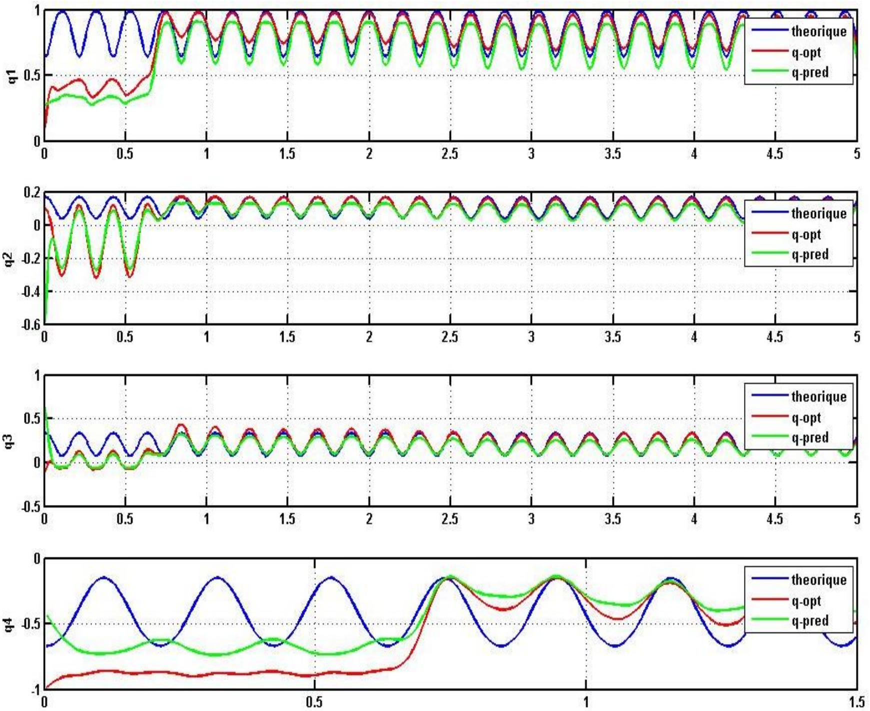

Figure 5. Estimation and prediction of the quaternion for the end effector.

The proposed technique is compared to the existing methods (namely, the MEKF (multiplicative extended Kalman filter) [10], (Control Force) [7] and the AEKF (additive Kalman filter) [11].

Initial conditions are set to extreme error values in order to assess the effectiveness of attitude estimation. These results are depicted in Figures 5 and 6.

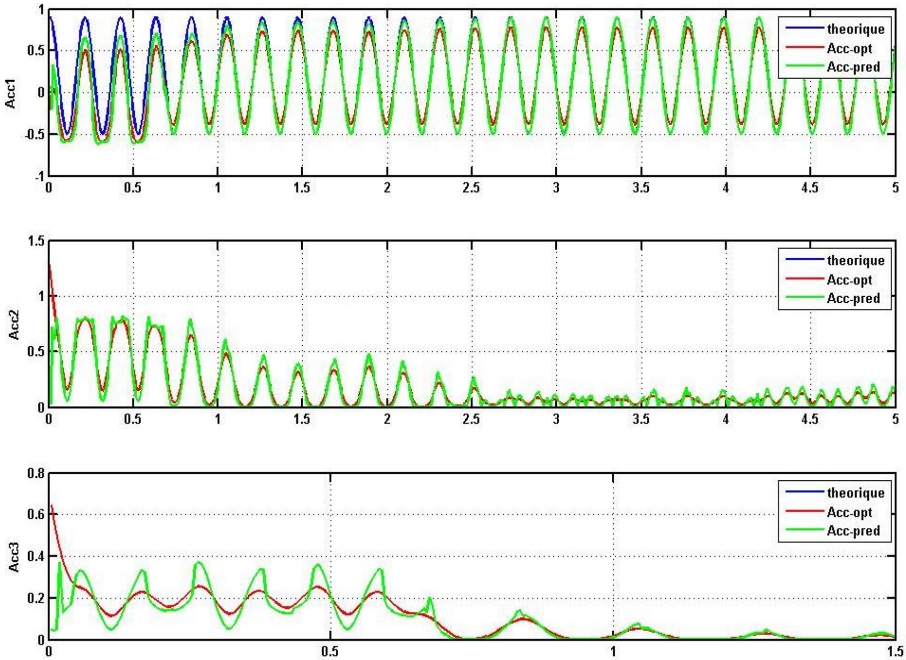

Figure 6. Estimation and prediction of the acceleration of the end effector.

The proposed method performances are similar to those of the Extended Kalman Filter (Multiplicative and Additive). However, for example errors the convergence rate for our estimation-prediction is higher (As can be appreciated in Figure 7, the estimation has been made in a PC with 2GB in RAM, Intel®, Core™ Duo CPU T6400 @2.00Ghz 2.00GhZ).

7.1. Experimental Results

A Commercial Micro AHRS (Attitude and Heading Reference System) [12,13] is used to acquire the data instead of the MEMS sensor presented in section III ((Robot showed in virtual reality Figure 3) and prototype in Figure 3. This AHRS also provides the Euler angles. The methodology of estimation and prediction are implemented in real-time using the LabView environment.

Profile Summary

Generated 28-Feb-2014 12:19:25 using cpu time.

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
Prueba_tiempo	1	0.835 s	0.475 s	
QRQ	1001	0.294 s	0.182 s	
Qpro	5005	0.058 s	0.058 s	
Qrot	6006	0.054 s	0.054 s	
axis	1	0.028 s	0.025 s	
xlabel	2	0.011 s	0.008 s	
view	1	0.008 s	0.002 s	
zlabel	2	0.008 s	0.008 s	
grid	1	0.005 s	0.005 s	
view>ViewCore	1	0.005 s	0.005 s	
ylabel	2	0.005 s	0.005 s	
axescheck	6	0.003 s	0.003 s	
axis>LocSetLimits	1	0.002 s	0.002 s	
axis>allAxes	1	0.001 s	0.001 s	
view>isAxesHandle	1	0.001 s	0.001 s	
deg2rad	2	0.001 s	0.001 s	

Self time is the time spent in a function excluding the time spent in its child functions. Self time also includes overhead resulting from the process of profiling.

Figure 7. Estimation time Profile Summary.

Remember that the attitude estimate is computed using a unit quaternion formulation. For comparison purpose, the estimate quaternion is converted into the rotation matrix. As can be shown, after large angular velocity change over a long period, the AHRS has a low convergence rate (approx. 1 min, Figure 8) compared to the one archived with our proposed methodology, onother advantage is that the problem of the “gimbal lock” is avoided (Figure 9 and Figure 10).

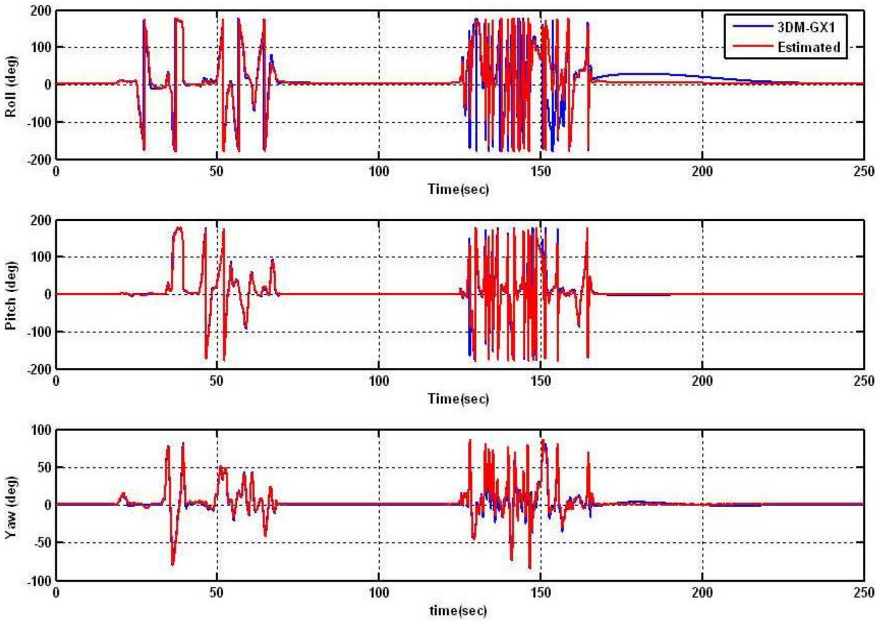

Figure 8. Experimental data.

On the other hand, this system doesn't provide the acceleration of the body so for validation we have done slowly movement and abrupt movement to appreciate the effect of the acceleration in our method.

Conclusion

The first key to keep in mind from this chapter is that hybrid methodogyes of attitude estimation via queternion, constitutes a viable alternative for improving the overall performance and robustness of embedded attitude estimation systems dealing with faulty sensor measurements.

By modeling the sensor fusion problem via queternion within the hybrid systems framework, we are able to exploit the redundancy of information emerging from the different sensors in order to perform real-time diagnosis of their modes of operation, therefore allowing the attitude estimation system to compensate for both methodogies and unmodeled faulty behavior.

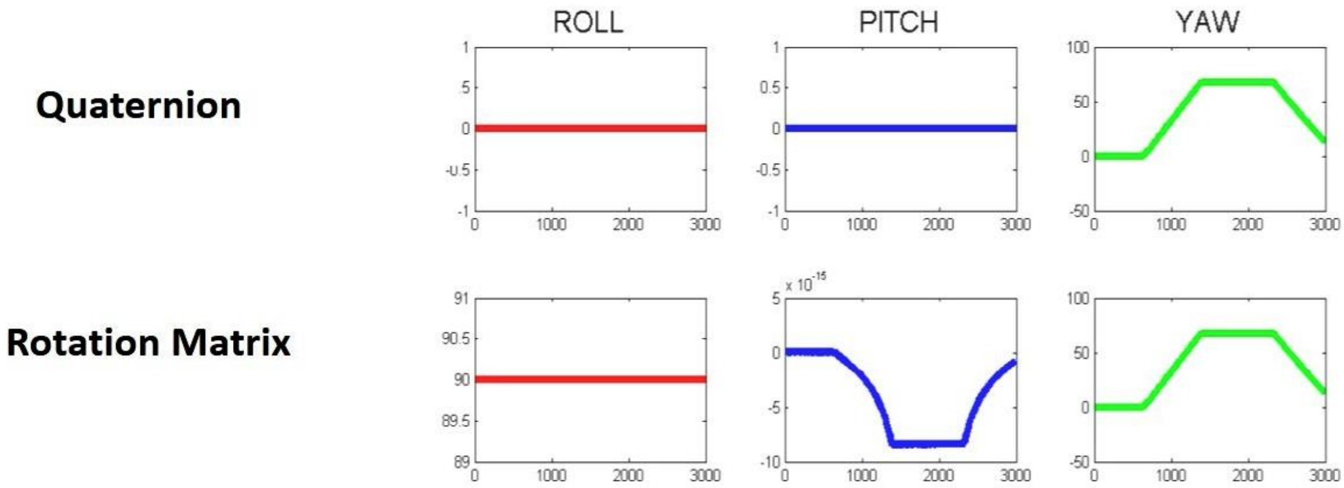

Figure 9. Attitude measured from the first articulation.

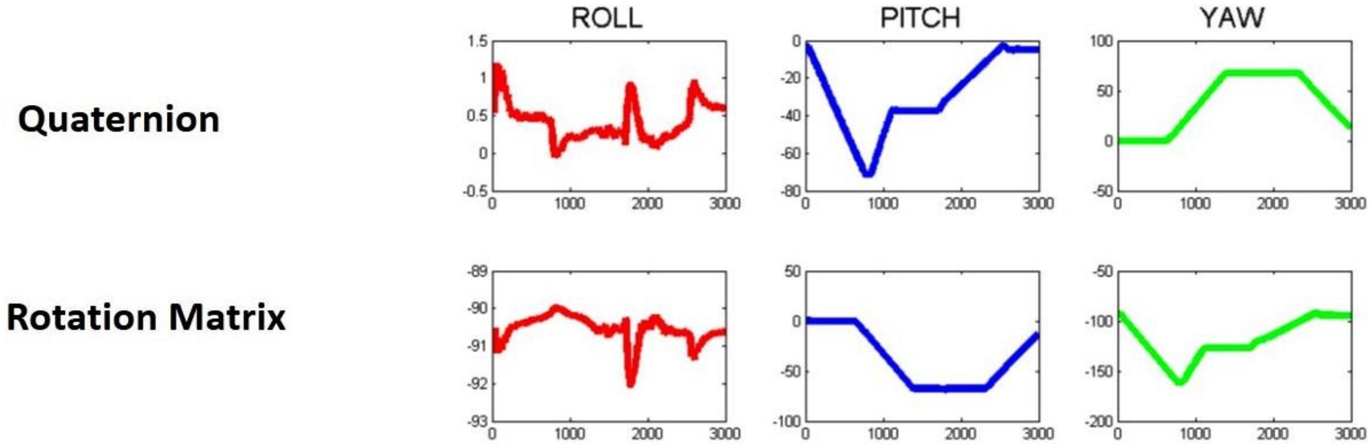

Figure 10. Attitude measured from the end effect.

This chapter presents a new strategy for attitude estimation of possibly non-symmetric rigid bodies. Two globally methods of calculation of the body's attitude are proposed, namely one methodology fussing data information with a three-axis accelerometer, three magnetometer and three rate gyros mounted orthogonally jointly, with prediction of the movement via cubic splines are studied and simulated. Furthermore, the attitude estimation is independent of the body's inertia. The numerical simulations have shown the effectiveness of the proposed methodologies and their robustness with respect to sensors noise and far initial points. Moreover, their simplicity makes them suitable for embedded implementation. This control estimation is tested in real application, consisting in a set of ABB 6 Degrees of Freedom robot mounted in the laboratory's Motor. This later is located in the laboratory: "Departamento de Pruebas de Motores" Volkswagen México.

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References

[1] G. Whaba, A least squares estimate of spacecraft attitude, *SIAM Review*, Vol. 7, No. 3, July 1965, p. 409.

[2] John L. Crassidis and F. Landis Markley. Minimum Model Error Approach for Attitude Estimation, *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 6 (1997), pp. 1241-1247.

[3] Lefferts, E. J., F. L. Markley, and M. D. Shuster, Kalman Filtering for Spacecraft Attitude Estimation, *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 5, Sept.-Oct. 1982, pp. 417-429.

[4] M. D. Shuster, A survey of attitude representations, *Journal of the Astronautically Sciences* (1993) 439-5177.

[5] Markley, F. L., New Quaternion Attitude Estimation Method, *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 2, March-April 1994, pp. 407-409.

[6] Pires, JN, and Sá da Costa JMG, A Real Time System for Position/Force Control of Mechanical Manipulators, *Proceedings of the 7th International Machine Design Conference,* Ankara, Turkey, 1996.

[7] Siciliano B., Villani L., Robot Force Control, Kluwer Academic Publishers International Series in Engineering and Computer Science, Boston, MA, 1999.

[8] Thienel, J.; Sanner, R.M.; “A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise,” *Automatic Control, IEEE Transactions on*, vol.48, no. 11, pp. 2011-2015, Nov. 2003.

[9] R. L. Bellaire, E. W. Kamen and S. M. Zabin “A new nonlinear iterated filter with applications to target tracking,” Proc. AeroSense: 8th Int. Symp. Aerospace/Defense Sensing, Simulation and Controls, vol. 2561, pp. 240 1995.

[10] H. Rehbinder, X.Hu, Drif-free attitude estimation for accelerated rigid bodies, *Automatica* (2004) 653-659.

[11] B.B Salmerón-Quiroz, C.F. Mendez-Barrios, J.F. Guerrero-Castellanos, G. Villegas-Medina, S.A. Rodríguez-Paredes. “Towards Human Capture Movement: Estimation of Anatomical Movements of the Shoulder,” *Discrete Dynamics in Nature and Society*, Vol. 2013, Article ID 586480, 13 pages, 2013. doi:10.1155/2013/586480., ISSN: 1026-0226.

[12] S. Lesecq, A. *Barraud, Capture de mouvement: Report*, October 2004-July 2005, GIPSA-lab - CEA-Leti, 2005, Grenoble, France.

[13] K. Xianwen and C. M. Gosselin, “Type synthesis of 3T1R 4-DOF parallel manipulators based on screw theory,” *IEEE Trans. Robot. Autom*, vol. 20, no. 2, pp. 181-190, Apr. 2004.

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Chapter 5

2D Hermite-Gaussian and Gauss-Laguerre Circular Harmonic Eigenfunction of the Quaternion and Reduced Biquaternion Fourier Transform

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Abstract

The quaternions, reduced biquaternions (RBs) and their respective Fourier transformations, i.e., discrete quaternion Fourier transform (DQFT) and discrete reduced biquaternion Fourier transform (DRBQFT), are very useful for multi-dimensional signal processing and analysis. In this paper, the basic concepts of quaternion and RB algebra are reviewed, and we introduce the two dimensional Hermite-Gaussian functions (2D-HGF) as the eigenfunction of DQFT/DRBQFT, and the eigenvalues of 2D-HGF for three types of DQFT and two types of DRBQFT. After that, the relation between 2D-HGF and Gauss-Laguerre circular harmonic function (GLCHF) is given. From the aforementioned relation and some derivations, the GLCHF can be proved as the eigenfunction of

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DQFT/DRBQFT and its eigenvalues are summarized. These GLCHFs can be used as the basis to perform color image expansion. The expansion coefficients can be used to reconstruct the original color image and as a rotation invariant feature. The GLCHFs can also be applied to color matching applications.

1. Introduction

Properties of eigenvalues and eigenfunctions for Fourier transform and its variants are widely surveyed in the literatures. In 1982, Dickinson and Steiglitz [1] proposed a matrix which commutes with discrete Fourier transform (DFT) matrix and used this commuting matrix to compute orthonormal eigenfunctions for DFT. In the same year as [1], Grunbaum [2] discussed the Hermite function as eigenfunction of DFT. Later in 2006, Pei et al. [3] proposed a nearly tridiagonal commuting matrix to obtain orthonormal eigenfunctions for DFT with smaller error than [1]. In 2008, Santhanam et al. [4] inspired by the ideas from quantum mechanics in finite dimensions and presented an approach which computes commuting matrix whose eigenvalue spectrum is closely approximated to that of the Hermite-Gaussian differential operator. Recently in 2010, Pei et al. [5] derived the eigenvalues of discrete 2D-HGF for two-side DQFT [6] and two-side DRBQFT [7].

The quaternions have been applied to many research fields such as computer science, mathematics, signal processing and image processing, since Hamilton introduced the concept in 1843 [8]. The fundamental theorems are well developed, and mathematical operations like Fourier transform, wavelet transform, convolution of this four-dimensional, non-commutative algebra have been constructed maturely [9-16]. The usefulness and effectiveness of quaternions in dealing with multi-dimensional computations are demonstrated, especially those involving operations of computer graphics and image processing, like three-dimensional rotations and many other geometrical transformations. However, there are many other interesting variants of quaternions like the biquaternions [17], the reduced biquaternions (RBs) [18], and the quad-quaternions [19], which are very useful and possess special properties and abilities that quaternions don’t have.

These variants have also found several applications. Among these variants, in particular, we concentrate on discussing the quaternion, RBs and derive the eigenvalues and eigenfunctions for their DQFT/DRBFT. The multiplication rule of quaternions is non-commutative, on the other hand, the multiplication rule of RBs is commutative. The commutative property is one of the advantages of RBs over quaternions. Due to the commutativity of multiplications, many operations, such as Fourier transform, correlation, convolution, singular value decomposition (SVD), of

RB algebra are much simpler and more convenient to the users than the ones of quaternion algebra. In [7], the efficient implementation of DRBQFT, convolution, correlation, phase-only correlation and RB linear-time-invariant and symmetric multichannel complex system are developed and they are much simpler than the existing implementation of quaternions. The commutative property is important and useful because the commutative DRBQFT is much simpler than non-commutative DQFT. In [20], the SVD of RB matrices are introduced. Compared with the quaternion matrix SVD, the complexity of the RB matrix SVD is reduced to a small factor of one-fourth. From the above discoveries, we can tackle with many arithmetic problems more efficiently in signal and image processing by using RBs. The RBs also have their limitations. The algebra of RBs is not a division algebra, and their geometric meaning is unfamiliar to most engineers. However, these problems have almost no influence on signal and image processing applications. We will briefly summarize the comparison of quaternions and RBs in the following section. In this work, we derive the eigenvalues of 2D-HGF for three types of DQFT and two types of DRBQFT, i.e., right-side, left-side and two-side for DQFT and type1, type2 for DRBQFT. By applying the relation between 2D-HGF and GLCHF mentioned in [21], we extended the conventional GLCHF in quaternion and RB spaces. We found that the GLCHF is the eigenfunction of left-side and right-side DQFT (type 1 and type2 DRBQFT), and the modified GLCHF is eigenfunction of two-side DQFT (type 1 and type2 DRBQFT). Over all, the major contribution of this paper are:

1) Three types of DQFT (right-side, left-side, and two-side) and two types of DRBQFT (type1 and type2) are introduced.
2) 2D Hermite-Gaussian and Gauss-Laguerre circular harmonic eigenfunctions and eigenvalues are derived for the above quaternion and reduced biquaternion Fourier transforms.
3) Discrete 2D Gauss-Laguerre circular harmonic eigenfunctions can be efficiently computed using the linear combination coefficients and 2D discrete Hermite-Gaussian eigenfunctions. Both 2D eigenfunctions forms a complete and orthonormal basis in the 2D plane.
4) DQFT and DRBQFT can be efficiently implemented using the conventional 2D FFT.
5) Discrete 2D Gaussian-Laguerre circular harmonic functions are suitable for circular pattern analysis and expansions of the color images.
6) Color image expansion, reconstruction, rotation invariant features, and color shape matching are proposed and demonstrated using GLCHFs for color image processing applications.

This paper is organized as follows. In section 2, the background knowledge and the fundamentals of the quaternions, RBs, 2D-HGF and GLCHF are briefly reviewed. Three types of DQFT and two types of DRBQFT are introduced and the eigenvalue derivation of 2D-HGF, GLCHF, and modified GLCHF for these transformations are discussed in section 3.In section 4, we demonstrate the spatial and spectral domain results of GLCHF and modified GLCHF to justify our proposition of eigenvalues and derivations in section 2. The GLCHFs are also used to perform color image expansion and color matching. Two reconstruction methods are proposed for different purpose and apply these methods to reconstruct color image. We also found that the expansion coefficients can also be used as a rotation invariant feature. Section 5 concludes this work.

2. Preliminaries

2.1. Quaternions

The quaternions can be viewed as a four-dimensional vector space defined over real numbers. The quaternions are also generalizations of complex numbers. A quaternion consists of four components, i.e., one real part and three imaginary parts. A quaternion is often represented as the following form:

$$q = q_r + q_i i + q_j j + q_k k \tag{1}$$

where q_r, q_i, q_j, q_k are all real numbers, and the elements {1, i, j, k} form the basis of the quaternion vector space. The {1, i, j, k} obeys following multiplication rules:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, \; jk = -kj = i, ki = -ik = j \tag{2}$$

The conjugate of a quaternion is defined as:

$$q^* = q_r - q_i i - q_j j - q_k k \tag{3}$$

The norm of a quaternion can be written as:

$$|q| = \sqrt{qq^*} = (q_r^2 + q_i^2 + q_j^2 + q_k^2)^{1/2} \quad (4)$$

Table 1. Complexity of quaternions, RBs, RBs e_1-e_2 form in multiplication

	Real multiplications	Real additions
Qs direct calculation	16	12
RBs direct calculation	16	12
RBs e_1-e_2 form	8	16

2.2. Reduced Biquaternions

The idea of reduced biquaternions (RBs) was suggested by Schtte and Wenzel [18] in 1990. Other similar ideas can be found in the articles about bicomplex algebra [22-24]. They [18] proposed to apply the RBs to the implementation of digital filter, and demonstrated that fourth order real filter can be realized by means of a first order RB filter. The RBs are another types of four-dimensional hypercomplex numbers, and are also represented as the form of (1), but {1, i, j, k} obeys different multiplication rules from those of (2). The rules are given by:

$$i^2 = k^2 = -1, j^2 = 1, ij = ji = k, jk = kj = i, ki = ik = -j \quad (5)$$

As (5) shows, the major difference between RBs and quaternions is the multiplication rules. If we define the norm of the RBs as $|q| = (q_r^2 + q_i^2 + q_j^2 + q_k^2)^{1/2}$, then $|q_1 q_2| \neq |q_1||q_2|$, where q_1 and q_2 are two arbitrary RB, and if we define the conjugate of the RBs as $q^* = q_r - q_i i - q_j j - q_k k$, then qq^* is still a RB. There are three different definitions of conjugate for the RBs in [2, 3], but the product of a RB and one of these three conjugates are still not real, therefore, we define the modulus and conjugate of RB as follows:

$$|q| = \sqrt[4]{\delta} = ((q_r^2 + q_i^2 + q_j^2 + q_k^2)^2 - 4(q_r q_j + q_i q_k)^2)^{1/4} \geq 0 \tag{6}$$

where δ is the determinant of the matrix representation of the RBs (M_q).

M_q can be written as $\begin{bmatrix} q_r & -q_i & q_j & -q_k \\ q_i & q_r & q_k & q_j \\ q_j & -q_k & q_r & -q_i \\ q_k & q_j & q_i & q_r \end{bmatrix}$, and $\delta = |M_q| = \begin{vmatrix} q_r & -q_i & q_j & -q_k \\ q_i & q_r & q_k & q_j \\ q_j & -q_k & q_r & -q_i \\ q_k & q_j & q_i & q_r \end{vmatrix} =$ $= (q_r^2 + q_i^2 + q_j^2 + q_k^2)^2 - 4(q_r q_j + q_i q_k)^2 \geq 0$. If $\delta = 0$, then the inverse of M_q and q (RB) do not exist. The matrix representation is useful to analyze many concepts of RBs like its inverse, addition, multiplication and norm, etc. We define $|q| = \sqrt[4]{\delta}$ because $|q_1 q_2| = |q_1||q_2|$ is satisfied and $|q| = (q_r^2 + q_i^2)^{1/2}$ if $q_j = q_k = 0$ (compatible to complex numbers). The only property different from that of complex numbers is the Schwartz triangle inequality, i.e., $|q_1 + q_2| > |q_1| + |q_2|$ is not satisfied for some special cases. For example, two RB numbers $(1+j)/2$ and $(1-j)/2$ have zero norm, but the sum of them are equal to one. The conjugate of the RBs can be represented as:

$$q^* = \sqrt{\delta} \cdot q^{-1} = |q|^2 / q \tag{7}$$

The matrix representation of q^* is $\sqrt{\delta} \cdot (M_q)^{-1}$, therefore, we can also write down q^* as:

$$q^* = \frac{1}{\sqrt{\delta}} \{ \begin{vmatrix} q_r & q_k & q_j \\ -q_k & q_r & -q_i \\ q_j & q_i & q_r \end{vmatrix} - i \begin{vmatrix} q_i & q_k & q_j \\ q_j & q_r & -q_i \\ q_k & q_i & q_r \end{vmatrix} + j \begin{vmatrix} q_i & q_r & q_j \\ q_j & -q_k & -q_i \\ q_k & q_j & q_r \end{vmatrix} - k \begin{vmatrix} q_i & q_r & q_k \\ q_j & -q_k & q_r \\ q_k & q_j & q_i \end{vmatrix} \} \tag{8}$$

The reason for choosing $q^* = \sqrt{\delta} \cdot q^{-1}$ is that qq^* is a real number, $qq^* = |q|^2$ and $(q_1 q_2)^* = q_1^* q_2^*$.If $\delta = 0$, then the inverse of q (RB) and q do not exist.

Table 2. Comparison of quaternions and RBs

Property	Quaternions	Reduced Biquaternions
Commutativity	Non-commutative multiplication rules. (Therefore, the arithmetic operations are more complex than RBs.)	Commutative multiplication rules. (Therefore, the arithmetic operations are simpler than quaternions.)
Complexity	The FT, SVD [20]-[21] are more complicated than RBs.	The FT, SVD [20]-[21], are simpler than quaternions.
Applicability	Popular in signal and image processing areas, especially in the aerospace, computer vision, and multi-dimensional signal processing field.	Less known but RBs can perform almost same tasks as quaternions do in signal and image processing areas because of their similarity.
Algebraic property and geometric meaning	Division algebra and familiar geometric meaning.	Non-division algebra and unfamiliar geometric meaning. but these have no influence on signal and image processing applications. [20].

2.3. e_1 - e_2 Form of RBs, Complexity Analysis and Comparison of Quaternion and RB

In [25], Davenport had found that there exists two special nonzero numbers e_1 and e_2 in HCA_4 such that $e_1 e_2 = 0$, $e_1^n = e_1^{n-1} = \ldots = e_1$, and $e_2^n = e_2^{n-1} = \ldots = e_2$. Therefore, e_1 and e_2 are both idempotent elements and divisors of zero. For complex numbers and quaternions, the idempotent elements are only 0 and 1, and the divisor of zero is only the number 0. For RBs, $e_1 = (1+j)/2$, $e_2 = (1-j)/2$. Any RB can be represented by the linear combination of e_1 and e_2:

$$q = (q_r + q_i i) + (q_j + q_k i)j = q_a + q_b j \equiv q_{a+b} e_1 + q_{a-b} e_2 \tag{9}$$

where $q_{a+b} = q_a + q_b, q_{a-b} = q_a - q_b$. We name (9) the e_1 - e_2 form of RBs. This form is the irreducible representation for RBs. The complexity of many operations about RBs, such as multiplication and Fourier transform, can be reduced by the use of e_1 and e_2, and the analysis about RBs become easier. For example, the multiplication of two RBs q_1 and q_2 can be computed by following equation:

$$q_1 q_2 = q_2 q_1 = (q_{1a} + q_{1b})(q_{2a} + q_{2b})e_1 + (q_{1a} - q_{1b})(q_{2a} - q_{2b})e_2 \quad (10)$$

We only need two instead of four complex multiplications to calculate the multiplication of two RBs.

However, the real addition operations are increased from 12 to 16. The complexity of multiplication for quaternions, RBs, and e_1-e_2 form of RBs is summarized in Table 1. The comparison of quaternions and RBs is shown in Table 2 for readers to understand their main differences.

2.4. 2D Hermite-Gaussian Function and Gauss-Laguerre Circular Harmonic Function

The 2D-HGF $H_{ab}(m,n)$ form the complete orthonormal set in L_2 space. We can define it as the following 1D separable form:

$$H_{ab}(m,n) = H_a(m)H_b(n)$$
$$H_a(m) = \frac{1}{\sqrt{2^a a!\sqrt{\pi}}} e^{-\frac{m^2}{2}} h_a(m) \quad (11)$$

where a=0,1,2,… and $h_a(m)$ are a-th order Hermite polynomials [26]. (m,n) is spatial location in Cartesian coordinate. The GLCHFs can be linearly combinated by using 2D-HGFs, that is:

$$L_{s-t,t}(m,n) = \sum_{g=0}^{s} \mathrm{T}_{t,g}^{(s)} H_{s-g,g}(m,n) \quad (12)$$

where $s \geq 0$. For $t \geq s - t$, and:

$$\mathrm{T}_{t,g}^{(s)} = j^g \sqrt{\frac{(s-g)!g!}{2^s (s-t)!t!}} \left(\sum_{k=\max(0,g-t)}^{\min(g,m-t)} (-1)^k \binom{s-t}{s-t-k} \binom{t}{t-g+k} \right) \quad (13)$$

(13) are the coefficients of linear combination which can be written in a matrix form (see the complex numbers in Figure 1) and the readers may refer to [21] for detailed discussion about 2D-HGF and GLCHF. The relation between 2D-HGFs and GLCHFs is illustrated in Figure 1. For example, GLCHF L1 (i.e.,L_{50}) can be written as a linear combination form using fifth order 2D-HGFs as follows:

$$\begin{aligned} &\mathrm{L1}=\mathrm{L}_{50}=\mathrm{Re}\{\mathrm{L}_{50}\}+\mathrm{i}\cdot\mathrm{Im}\{\mathrm{L}_{50}\}=0.1768\cdot\mathrm{H}_{50}-0.3953\mathrm{i}\cdot\mathrm{H}_{41}-0.5590\cdot\mathrm{H}_{32} \\ &+0.5590\mathrm{i}\cdot\mathrm{H}_{23}+0.3953\cdot\mathrm{H}_{14}-0.1768\mathrm{i}\cdot\mathrm{H}_{05} \end{aligned} \tag{14}$$

3. Eigenfunctions and Eigenvalues of DQFT and DRBQFT

3.1. The Definitions of DQFT and the Derivation of Their Eigenvalues

We define three types of DQFT used to derive their eigenvalues as follows (only foreword transforms are given):

Left-side DQFT (denote as L-DQFT):

$$F_{QL}(u,v)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_a(\frac{um}{M}+\frac{vn}{N})}f(m,n) \tag{15}$$

Right-side DQFT (denote as R-DQFT):

$$F_{QR}(u,v)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)e^{-2\pi\mu_a(\frac{um}{M}+\frac{vn}{N})} \tag{16}$$

Two-side DQFT (denote as T-DQFT):

$$F_{QT}(u,v)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_a\frac{um}{M}}f(m,n)e^{-2\pi\mu_b\frac{vn}{N}} \tag{17}$$

where $\{\mu_a,\mu_b\}$ are unit pure quaternions (real=0 and norm=1).

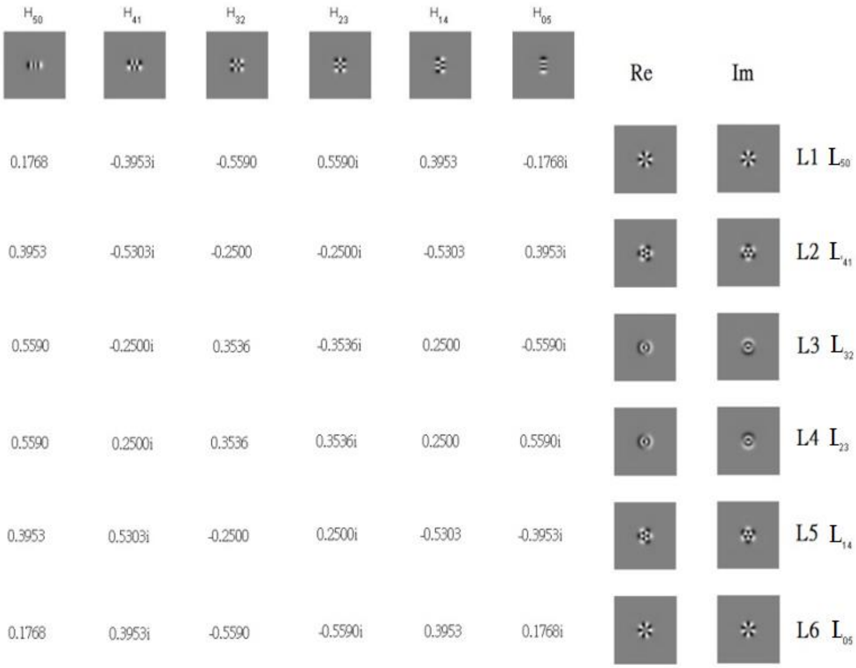


Figure 1. Fifth order 2D-HGFs in top row can be transformed to the corresponding GLCHFs in rightmost columns (L1 to L6, as shown in real and imaginary parts) by using linear combination with the coefficients in matrix form.

3.1.1. Eigenvalue Derivation of 2D-HGF $H_{ab}(m,n)$ for Left-Side QFT

For 2D-HGF $H_{ab}(m,n)$ and the transformation axis μ is unit pure quaternion:

$$\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu\left(\frac{um}{M}+\frac{vn}{N}\right)}H_{ab}(m,n)=\frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}e^{-2\pi\mu\frac{um}{M}}H_a(m)\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}e^{-2\pi\mu\frac{vn}{N}}H_b(n) \quad (18)$$
$$=(-\mu)^a H_a(m)(-\mu)^b H_b(n)=(-\mu)^{a+b}H_{ab}(m,n)=H_{ab}(m,n)(-\mu)^{a+b}$$

The eigenvalue of 1D-HGF for DFT is well-known and its $(-\mu)^a$, where a is the order of HGF and μ is transform axis of DFT. Therefore, from the derivation of (18), we know that the eigenvalue of 2D-HGF $H_{ab}(m,n)$ for left-side QFT is $(-\mu)^{a+b}$.

3.1.2. Eigenvalue Derivation of GLCHF$L_{ab}(m,n)$ for Left-Side QFT

Take GLCHF L_{30} for example:

$$L_{30}(m,n) = A * H_{30}(m,n) + B * H_{21}(m,n) + C * H_{12}(m,n) + \\ + D * H_{03}(m,n) = R + \mu * I,$$

where the transformation axis μ are unit pure quaternion, (A,B,C,D) are the coefficients of linear combination obtained from (13), the original imaginary axis i in (13) is now extended to unit pure quaternion μ, R is real part, I is imaginary part (R and I are also linear combination of 2D-HGFs). Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} L_{ab}(m,n) = \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}(R+\mu I) \\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} R + \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} \mu I \\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} R + (\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} I)\mu \\
&= (-\mu)^{a+b} R + (-\mu)^{a+b} I\mu \\
&= (-\mu)^{a+b}(R+\mu I) = (-\mu)^{a+b} L_{ab}(m,n) = L_{ab}(m,n)(-\mu)^{a+b}
\end{aligned} \tag{19}$$

From (19), we know that the eigenvalue of GLCHF$L_{ab}(m,n)$ for left-side QFTis $(-\mu)^{a+b}$.

3.1.3. Eigenvalue Derivation of 2D-HGF $H_{ab}(m,n)$ for Right-Side QFT

For 2D-HGF $H_{ab}(m,n)$ and the transformation axis μ is unit pure quaternion:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_{ab}(m,n) e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})} = \frac{1}{\sqrt{M}}\sum_{m=0}^{M-1} H_a(m) e^{-2\pi\mu\frac{um}{M}} \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} H_b(n) e^{-2\pi\mu\frac{vn}{N}} \\
&= H_a(m)(-\mu)^a H_b(n)(-\mu)^b = H_{ab}(m,n)(-\mu)^{a+b} = (-\mu)^{a+b} H_{ab}(m,n)
\end{aligned} \tag{20}$$

From (20), we know that the eigenvalue of 2D-HGF $H_{ab}(m,n)$ for right-side QFT is $(-\mu)^{a+b}$.

3.1.4. Eigenvalue Derivation of GLCHF$L_{ab}(m,n)$ for Right-Side QFT

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}L_{ab}(m,n)e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(R+\mu I)e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}\\
&=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}R\cdot e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}+\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\mu I\cdot e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}\\
&=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}R\cdot e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})}+\mu(\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}I\cdot e^{-2\pi\mu(\frac{um}{M}+\frac{vn}{N})})\qquad(21)\\
&=R(-\mu)^{a+b}+\mu I(-\mu)^{a+b}\\
&=(R+\mu I)(-\mu)^{a+b}=L_{ab}(m,n)(-\mu)^{a+b}=(-\mu)^{a+b}L_{ab}(m,n)
\end{aligned}$$

From (21), we can see that the eigenvalue of GLCHF$L_{ab}(m,n)$ for right-side QFTis $(-\mu)^{a+b}$.

3.1.5. Eigenvalue Derivation of 2D-HGF $H_{ab}(m,n)$ for Two-Side QFT

For 2D-HGF $H_{ab}(m,n)$ and two transformation axes μ_1 and μ_2 are unit pure quaternions:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}H_{ab}(m,n)e^{-2\pi\mu_2\frac{vn}{N}}=\frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}e^{-2\pi\mu_1\frac{um}{M}}H_a(m)\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}H_b(n)e^{-2\pi\mu_2\frac{vn}{N}}\qquad(22)\\
&=(-\mu_1)^aH_a(m)H_b(n)(-\mu_2)^b=(-\mu_1)^a(-\mu_2)^bH_{ab}(m,n)=H_{ab}(m,n)(-\mu_1)^a(-\mu_2)^b
\end{aligned}$$

From (22), we realize that the eigenvalue of 2D-HGF $H_{ab}(m,n)$ for two-side QFT is $(-\mu_1)^a(-\mu_2)^b$.

3.1.6. Eigenvalue Derivation of Modified GLCHFs $L_{ab}(m,n)\cdot(\mu_1+\mu_2)$ and $(\mu_1+\mu_2)\cdot L_{ab}(m,n)$ for Two-Side QFT

For two-side QFT we have to change the form of original GLCHF $L_{ab}(m,n)$ in order to obtain eigenvalues. The modified GLCHFs are defined as $L_{ab}(m,n)\cdot(\mu_1+\mu_2)$ and $(\mu_1+\mu_2)\cdot L_{ab}(m,n)$. $L_{ab}(m,n)$ can be written as $R+\mu_1 I$ or $R+\mu_2 I$, where $\mu_1$and μ_2 are unit pure quaternions and they are also transformation axes, R and I are linear combinations of 2D-HGFs. We denote modified GLCHF of first kind as MGLCHF I, and GLCHF of second kind as MGLCHF II. Before we proceed to find the eigenvalues of modified GLCHFs for two-side QFT, a simple theorem is introduced first and it may be helpful for us to derive the eigenvalues.

Theorem 1. For any unit pure quaternions uand v, we have:

$$(-\mathrm{u})^n(\mathrm{u}+\mathrm{v}) = (\mathrm{u}+\mathrm{v})(-\mathrm{v})^n \tag{23}$$

Proof. When $\mathrm{n} = 0$, (23) is obviously true. Assume $\mathrm{n} = \mathrm{k}$, we have:

$$(-\mathrm{u})^k(\mathrm{u}+\mathrm{v}) = (\mathrm{u}+\mathrm{v})(-\mathrm{v})^k \tag{24}$$

Then, $(-\mathrm{u})^{k+1}(\mathrm{u}+\mathrm{v}) = (-\mathrm{u})^k(-\mathrm{u})(\mathrm{u}+\mathrm{v}) = (-\mathrm{u})^k(1-\mathrm{uv}) = (-\mathrm{u})^k \cdot (-\mathrm{uv}+1) = (-\mathrm{u})^k(\mathrm{u}+\mathrm{v})(-\mathrm{v}) = (\mathrm{u}+\mathrm{v})(-\mathrm{v})^k(-\mathrm{v}) = (\mathrm{u}+\mathrm{v})(-\mathrm{v})^{k+1}$(by (24)), where $\mathrm{u}^2 = \mathrm{v}^2 = -1$. Therefore, according to the mathematical induction, the proof is completed. In what follows, we derive the eigenvalues of MGLCHFs for two-side QFT:

For MGLCHF of first kind, $L_{ab}(m,n)\cdot(\mu_1+\mu_2)$:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}L_{ab}(m,n)(\mu_1+\mu_2)e^{-2\pi\mu_2\frac{vn}{N}} = \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}(R+\mu_1 I)(\mu_1+\mu_2)e^{-2\pi\mu_2\frac{vn}{N}}\\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot R\mu_1\cdot e^{-2\pi\mu_2\frac{vn}{N}} + \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot R\mu_2\cdot e^{-2\pi\mu_2\frac{vn}{N}}\\
&+ \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot \mu_1\mu_1 I\cdot e^{-2\pi\mu_2\frac{vn}{N}} + \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot \mu_1\mu_2 I\cdot e^{-2\pi\mu_2\frac{vn}{N}}\\
&= \mu_1(-\mu_1)^a(-\mu_2)^b R + (-\mu_1)^a(-\mu_2)^b\mu_2 R + \mu_1(-\mu_1)^a(-\mu_2^*)^b\mu_1 I + \mu_1(-\mu_1)^a(-\mu_2)^b\mu_2 I\\
&= R\mu_1(-\mu_1)^a(-\mu_2)^b + \mu_1 I\mu_1(-\mu_1)^a(-\mu_2)^b + R(-\mu_1)^a(-\mu_2)^b\mu_2 + \mu_1 I(-\mu_1)^a(-\mu_2)^b\mu_2\\
&= (R+\mu_1 I)(\mu_1(-\mu_1)^a(-\mu_2)^b + (-\mu_1)^a(-\mu_2)^b\mu_2)\\
&= (R+\mu_1 I)(-\mu_1)^a(\mu_1+\mu_2)(-\mu_2)^b = (R+\mu_1 I)(\mu_1+\mu_2)(-\mu_2)^a(-\mu_2)^b\\
&(\because (-\mu_1)^a(\mu_1+\mu_2) = (\mu_1+\mu_2)(-\mu_2)^a (Thm\quad 1))\\
&= (R+\mu_1 I)(\mu_1+\mu_2)(-\mu_2)^{a+b}\\
&= L_{ab}(m,n)(\mu_1+\mu_2)(-\mu_2)^{a+b}
\end{aligned} \tag{25}$$

For MGLCHF of second kind, $(\mu_1+\mu_2)\cdot L_{ab}(m,n)$:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}(\mu_1+\mu_2)L_{ab}(m,n)e^{-2\pi\mu_2\frac{vn}{N}}=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}(\mu_1+\mu_2)(R+\mu_2 I)e^{-2\pi\mu_2\frac{vn}{N}}\\
&=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot\mu_1 R\cdot e^{-2\pi\mu_2\frac{vn}{N}}+\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot\mu_2 R\cdot e^{-2\pi\mu_2\frac{vn}{N}}\\
&+\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot\mu_1\mu_2 I\cdot e^{-2\pi\mu_2\frac{vn}{N}}+\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}e^{-2\pi\mu_1\frac{um}{M}}\cdot\mu_2\mu_2 I\cdot e^{-2\pi\mu_2\frac{vn}{N}}\\
&=\mu_1(-\mu_1)^a(-\mu_2)^b R+(-\mu_1)^a(-\mu_2)^b\mu_2 R+\mu_1(-\mu_1)^a(-\mu_2)^b\mu_2 I+\mu_2(-\mu_1^*)^a(-\mu_2)^b\mu_2 I\\
&\quad=\mu_1(-\mu_1)^a(-\mu_2)^b R+\mu_1(-\mu_1)^a(-\mu_2)^b\mu_2 I+(-\mu_1)^a(-\mu_2)^b\mu_2 R+(-\mu_1)^a(-\mu_2)^b\mu_2\mu_2 I\\
&\quad=(\mu_1(-\mu_1)^a(-\mu_2)^b+(-\mu_1)^a(-\mu_2)^b\mu_2)(R+\mu_2 I)\\
&\quad=(-\mu_1)^a(\mu_1+\mu_2)(-\mu_2)^b(R+\mu_2 I)=(-\mu_1)^a(-\mu_1)^b(\mu_1+\mu_2)(R+\mu_2 I)\\
&\quad(\because(-\mu_1)^n(\mu_1+\mu_2)=(\mu_1+\mu_2)(-\mu_2)^n(Thm\quad 1))\\
&\quad=(-\mu_1)^{a+b}(\mu_1+\mu_2)(R+\mu_2 I)\\
&\quad=(-\mu_1)^{a+b}(\mu_1+\mu_2)L_{ab}(m,n)
\end{aligned}\tag{26}$$

From (25), we know that the eigenvalue of MGLCHF of first kind, $L_{ab}(m,n)\cdot(\mu_1+\mu_2)$ for two-side QFT is $(-\mu_2)^{a+b}$. On the other hand, from (26), we can see that the eigenvalue of MGLCHF of second kind, $(\mu_1+\mu_2)\cdot L_{ab}(m,n)$ for two-side QFT is $(-\mu_1)^{a+b}$.

3.2. The Definitions of DRBQFT and the Derivation of Their Eigenvalues

We define two types of DRBQFT used to derive their eigenvalues as follows (only foreword transforms are given):

DRBFT of type 1 (denote as DRBFT I, two imaginary unit axis,$u_1^2=u_2^2=-1$*)*:

$$F_{RB1}(p,s)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})}\tag{27}$$

DRBFT of type 2 (denote as DRBFT II, one imaginary unit axis,$u_1^2=-1$*)*:

$$F_{RB2}(p,s)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})}\tag{28}$$

3.2.1. Eigenvalue Derivation of 2D-HGF $H_{ab}(m,n)$for DRBQFT of Type 1

For 2D-HGF $H_{ab}(m,n)$ and two imaginary unit axes, $\mathrm{u}_1^2 = \mathrm{u}_2^2 = -1$:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_{ab}(m,n)e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})} \\
&= \frac{1}{\sqrt{M}}\sum_{m=0}^{M-1} H_a(m)e^{-2\pi u_1\frac{pm}{M}} \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} H_b(n)e^{-2\pi u_2\frac{sn}{N}} \\
&= H_a(m)(-u_1)^a H_b(n)(-u_2)^b = H_{ab}(m,n)(-u_1)^a(-u_2)^b = (-u_1)^a(-u_2)^b H_{ab}(m,n)
\end{aligned} \tag{29}$$

From (29), we see that the eigenvalue of 2D-HGF $H_{ab}(m,n)$ for DRBQFT of type 1 is $(-u_1)^a(-u_2)^b$.

3.2.2. Eigenvalue Derivation of 2D-HGF $H_{ab}(m,n)$for DRBQFT of Type 2

For 2D-HGF $H_{ab}(m,n)$ andone imaginary unit axis, $\mathrm{u}_1^2 = -1$:

$$\begin{aligned}
&\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_{ab}(m,n)e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})} \\
&= \frac{1}{\sqrt{M}}\sum_{m=0}^{M-1} H_a(m)e^{-2\pi u_1\frac{pm}{M}} \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} H_b(n)e^{-2\pi u_1\frac{sn}{N}} \\
&= H_a(m)(-u_1)^a H_b(n)(-u_1)^b = H_{ab}(m,n)(-u_1)^{a+b} = (-u_1)^{a+b} H_{ab}(m,n)
\end{aligned} \tag{30}$$

From (30), we see that the eigenvalue of 2D-HGF $H_{ab}(m,n)$ for DRBQFT of type 1 is $(-u_1)^{a+b}$.

3.2.3. Eigenvalue Derivation of GLCHF$L_{ab}(m,n)$ for DRBQFT of Type 1

$$L_{ab}(m,n) = H_1(m,n) + u\cdot H_2(m,n) = H_{ab}^R(m,n) + u\cdot H_{ab}^I(m,n)$$

where $u \in \{i,j,k\}$.

$$\begin{aligned}
L_{RB1}(p,s) &= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} L_{ab}(m,n) e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})} \\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_1(m,n) e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})} + u\cdot\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_2(m,n) e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})} \\
&= (-u_1)^a(-u_2)^b H_1(m,n) + u\cdot(-u_1)^a(-u_2)^b H_2(m,n) \\
&= (-u_1)^a(-u_2)^b (H_1(m,n) + u\cdot H_2(m,n)) \\
&= (-u_1)^a(-u_2)^b (H_{ab}^R(m,n) + u\cdot H_{ab}^I(m,n)) \\
&= (-u_1)^a(-u_2)^b L_{ab}(m,n)
\end{aligned} \tag{31}$$

Thus, eigenvalue of GLCHF$L_{ab}(m,n)$ for RBQFT of type 1 is ($u_1^2 = u_2^2 = -1$):

$$(-u_1)^a(-u_2)^b.$$

3.2.4. Eigenvalue Derivation of GLCHF$L_{ab}(m,n)$ for DRBQFT of Type 2

$$\begin{aligned}
L_{RB2}(p,s) &= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} L_{ab}(m,n) e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})} \\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_1(m,n) e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})} + u\cdot\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} H_2(m,n) e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})} \\
&= DFT_{u_1}(H_1(m,n)) + u\cdot DFT_{u_1}(H_2(m,n)) \\
&= (-u_1)^{a+b} H_1(m,n) + u\cdot(-u_1)^{a+b} H_2(m,n) \\
&= (-u_1)^{a+b}(H_1(m,n) + u\cdot H_2(m,n)) \\
&= (-u_1)^{a+b}(H_{ab}^R(m,n) + u\cdot H_{ab}^J(m,n)) \\
&= (-u_1)^{a+b} L_{ab}(m,n)
\end{aligned} \tag{32}$$

Thus, eigenvalue of GLCHF$L_{ab}(m,n)$ for DRBQFT of type 2 is ($u_1^2 = -1$):

$$(-u_1)^{a+b}.$$

As mentioned in sec. 2.3, the complexity of DRBQFT implementation can be reduced by using e_1 - e_2 form of RB. Take DRBQFT of type 1 for example, we demonstrate the procedure as follows:

Assume:

$$u_1 = i, u_2 = k, u = j, e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}$$

$$\begin{aligned} e^{-2\pi(u_1 \frac{pm}{M} + u_2 \frac{sn}{N})} &= \cos(2\pi \frac{sn}{N}) e^{-2\pi u_1 \frac{pm}{M}} + u \cdot (-u_1) \cdot \sin(2\pi \frac{sn}{N}) e^{-2\pi u_1 \frac{pm}{M}} \\ &= e^{-2\pi u_1 (\frac{pm}{M} + \frac{sn}{N})} \cdot e_1 + e^{-2\pi u_1 (\frac{pm}{M} - \frac{sn}{N})} \cdot e_2 \end{aligned} \quad (33)$$

Table 3. Summarization of eigenvalues of 2D-HGF, GLCHF, and MGLCHF I&II for three types of DQFT and two types of DRBQFT. ($\{\mu, \mu_1, \mu_2\} \in$ unit purequaternions. $\{u, u_1, u_2, u_3\} \in$ RBs and $u^2 = u_1^2 = u_2^2 = u_3^2 = -1$, $\{a,b\}$ are orders of functions)

	L-DQFT	R-DQFT	T-DQFT	DRBQFT I	DRBQFT II
2D-HGF	$(-\mu)^{a+b}$	$(-\mu)^{a+b}$	$(-\mu_1)^a(-\mu_2)^b$	$(-u_1)^a(-u_2)^b$	$(-u_1)^{a+b}$
GLCHF	$(-\mu)^{a+b}$	$(-\mu)^{a+b}$	NA	$(-u_1)^a(-u_2)^b$	$(-u_1)^{a+b}$
MGLCHF I	NA	NA	$(-\mu_2)^{a+b}$	$(-u_1)^a(-u_3)^b$	$(-u)^{a+b}$
MGLCHF II	NA	NA	$(-\mu_1)^{a+b}$	$(-u_1)^a(-u_3)^b$	$(-u)^{a+b}$

$$\begin{aligned} L_{ab}(m,n) &= H_1(m,n) + j \cdot H_2(m,n) \\ &= H_{1+2}(m,n) \cdot e_1 + H_{1-2}(m,n) \cdot e_2 \end{aligned} \quad (34)$$

where

$$H_{1+2}(m,n) = H_1(m,n) + H_2(m,n)$$

$$H_{1-2}(m,n) = H_1(m,n) - H_2(m,n) \quad (35)$$

$$
\begin{aligned}
&L_{RB1}(p,s)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}L_{ab}(m,n)e^{-2\pi(u_1\frac{pm}{M}+u_2\frac{sn}{N})}\\
&=(\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_{1+2}(m,n)e^{-2\pi u_1(\frac{pm}{M}+\frac{sn}{N})})\cdot e_1+(\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_{1-2}(m,n)e^{-2\pi u_1(\frac{pm}{M}-\frac{sn}{N})})\cdot e_2\\
&=H_{1+2}(p,s)\cdot e_1+\cdot H_{1-2}(p,-s)\cdot e_2\\
&H_{1+2}(p,s)=DFT_{u_1}(H_{1+2}(m,n)),H_{1-2}(p,s)=DFT_{u_1}(H_{1-2}(m,n))
\end{aligned}
\tag{36}
$$

Therefore, we can use conventional 2D DFT to implement complex DRBQFT.

3.2.5. Eigenvalue Derivation of MGLCHF$\boldsymbol{L_{ab}(m,n)\cdot(u_1+u_3)}$ for Two-Side DRBQFT

Because the commutative multiplication rule in RB algebra, $L_{ab}(m,n)\cdot(u_1+u_3)$=$(u_1+u_3)\cdot L_{ab}(m,n)$. Therefore, the two MGLCHFs are equivalent.

Eigenvalue derivation of MGLCHF$L_{ab}(m,n)\cdot(u_1+u_3)$ for DRBQFT of type 1:

First, we can obtain the following result:

$$
\begin{aligned}
&L_{ab}(m,n)\cdot(u_1+u_3)=(H_1(m,n)+u_1\cdot H_2(m,n))(u_1+u_3)\\
&=-H_2(m,n)+H_1(m,n)\cdot u_1+H_2(m,n)\cdot u_2+H_1(m,n)\cdot u_3\\
&=(-H_2(m,n)+H_1(m,n)\cdot u_1)+(H_2(m,n)+H_1(m,n)\cdot u_1)\cdot u_2\\
&=H_f(m,n)+H_g(m,n)\cdot u_2
\end{aligned}
\tag{37}
$$

Then,

$$
\begin{aligned}
&L_{RB1}(p,s)=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}L_{ab}(m,n)\cdot(u_1+u_3)e^{-2\pi(u_1\frac{pm}{M}+u_3\frac{sn}{N})}\\
&=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_f(m,n)e^{-2\pi(u_1\frac{pm}{M}+u_3\frac{sn}{N})}+u_2\cdot\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_g(m,n)e^{-2\pi(u_1\frac{pm}{M}+u_3\frac{sn}{N})}\\
&=(-u_1)^a(-u_3)^bH_f(m,n)+u_2\cdot(-u_1)^a(-u_3)^bH_g(m,n)\\
&=(-u_1)^a(-u_3)^b(H_f(m,n)+u_2\cdot H_g(m,n))\\
&=(-u_1)^a(-u_3)^b(H_1(m,n)+u_1\cdot H_2(m,n))(u_1+u_3)\\
&=(-u_1)^a(-u_3)^bL_{ab}(m,n)\cdot(u_1+u_3)
\end{aligned}
\tag{38}
$$

Table 4. Summarization of parameters and methods for generating results in Figures 4-13

Case	Function	Order	Axes	Size	Eigenvalue	Method
1	H_{30}	(a,b)=(3,0)	$\mu = i$	121x121	i	L-DQFT
2	H_{21}	(a,b)=(2,1)	$\mu = \frac{i+j}{\sqrt{2}}$	121x121	$\frac{i+j}{\sqrt{2}}$	R-DQFT
3	H_{12}	(a,b)=(1,2)	$\mu_1 = \frac{i+j+k}{\sqrt{3}}$ $\mu_2 = \frac{i+j}{\sqrt{2}}$	121x121	$\frac{i+j+k}{\sqrt{3}}$	T-DQFT
4	L_{30}	(a,b)=(3,0)	$\mu = k$	121x121	k	L-DQFT
5	L_{30}	(a,b)=(3,0)	$\mu = \frac{i+j}{\sqrt{2}}$	121x121	$\frac{i+j}{\sqrt{2}}$	R-DQFT
6	$L_{30}(\mu_1 + \mu_2)$	(a,b)=(3,0)	$\mu_1 = \frac{i+j+k}{\sqrt{3}}$ $\mu_2 = \frac{i+j}{\sqrt{2}}$	121x121	$\frac{i+j}{\sqrt{2}}$	T-DQFT
7	$(\mu_1 + \mu_2)L_{30}$	(a,b)=(3,0)	$\mu_1 = \frac{i+j+k}{\sqrt{3}}$ $\mu_2 = \frac{i+j}{\sqrt{2}}$	121x121	$\frac{i+j+k}{\sqrt{3}}$	T-DQFT
8	H_{03}	(a,b)=(0,3)	$u_1 = k$	121x121	k	DRBQFT I
9	$L_{30}(u_1 + u_3)$	(a,b)=(3,0)	$u_1 = i$ $u_3 = k$	121x121	i	DRBQFT II

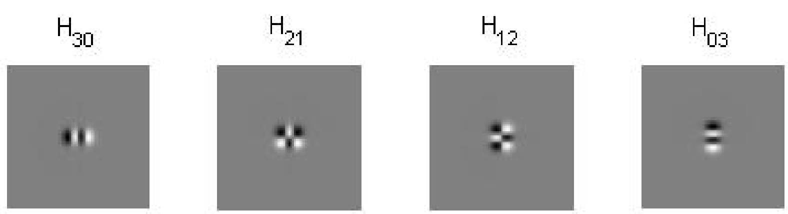


Figure 2. Input 2D-HGFs $\{H_{30}, H_{21}, H_{12}, H_{03}\}$, size:121x121 pixels.

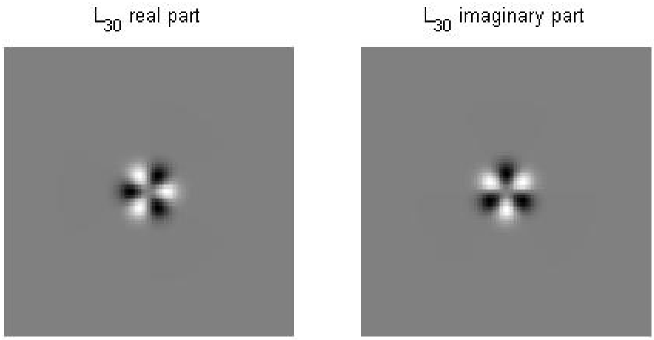


Figure 3. Input GLCHF L_{30}, size:121x121 pixels.

Thus, eigenvalue of modified GLCHF $L_{ab}(m,n)\cdot(u_1+u_3)$ for DRBQFT of type 1 is

$$(-u_1)^a(-u_3)^b.$$

Eigenvalue derivation of MGLCHF$L_{ab}(m,n)\cdot(u_1+u_3)$ for DRBQFT of type 2:

$$\begin{aligned}
L_{RB2}(p,s) &= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}L_{ab}(m,n)\cdot(u_1+u_3)e^{-2\pi u(\frac{pm}{M}+\frac{sn}{N})} \\
&= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_f(m,n)e^{-2\pi u(\frac{pm}{M}+\frac{sn}{N})} + u_2\cdot\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}H_g(m,n)e^{-2\pi u(\frac{pm}{M}+\frac{sn}{N})} \\
&= (-u)^{a+b}H_f(m,n)+u_2\cdot(-\mu)^{a+b}H_g(m,n) \\
&= (-u)^{a+b}(H_f(m,n)+u_2\cdot H_g(m,n)) \\
&= (-u)^{a+b}(H_1(m,n)+u_1\cdot H_2(m,n))(u_1+u_3) \\
&= (-u)^{a+b}L_{ab}(m,n)\cdot(u_1+u_3)
\end{aligned} \tag{39}$$

where

$$(u_2=u_1\cdot u_3, u_1^2=u_3^2=-1, u^2=-1)$$

Thus, eigenvalue of modified GLCHF $L_{ab}(m,n)\cdot(u_1+u_3)$ for DRBQFT of type 2 is $(-u)^{a+b}$. Before we leave this section, the above results are briefly summarized in Table 3.

4. Experimental Results

4.1. Verification of Derived Eigenvalue

First, we use Matlab, QTFM toolbox [27] developed by Sangwine et al. and our reduced biquaternion toolbox to perform DQFT and DRBQFT for 2D-HGF, GLCHF, MGLCHF I and MGLCHF II. The transformed spectrums of functions mentioned above and the ratio of spectrum to original function multiplied by derived eigenvalue, i.e., 1, will be demonstrated to verify the results summarized in Table 3. For QFT, we verify all cases in Table 3. Because of the page limit, we only verify 2D-HGF for DRBQFT II, and MGLCHF I&II for DRBQFT I. The parameters and methods used in these experiments are summarized in Table 4 and the experimental results are shown in Figure 4~13.

Case 1 to 3. 2D-HGF/L-DQFT, 2D-HGF/R-DQFT, 2D-HGF/T-DQFT

We can see from Figure 4 to 6 that the transformed spectrum L-DQFT(H_{30}) is equivalent to original function multiplied by derived eigenvalue $(-\mu)^{3+0} * H_{30}$, R-DQFT(H_{21}) is equivalent to $H_{21} * (-\mu)^{2+1}$, and T-DQFT(H_{12}) is equivalent to $H_{12} * (-\mu_1)^1(-\mu_2)^2$.

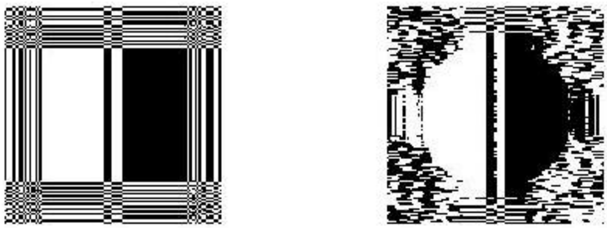
Case 4 to 7. GLCHF/L-DQFT, GLCHF/R-DQFT, MGLCHF I/T-DQFT, MGLCHF II/T-DQFT

It can be seen from Figure 7 to 10 that L-DQFT(L_{30}) is equivalent to $(-\mu)^{3+0} * L_{30}$, R-DQFT(L_{30}) is equivalent to $L_{30} * (-\mu)^{3+0}$, T-DQFT($L_{30}(\mu_1 + \mu_2)$) is equivalent to $L_{30}(\mu_1 + \mu_2) * (-\mu_2)^{3+0}$. T-DQFT($(\mu_1 + \mu_2)L_{30}$) is equivalent to $(-\mu_1)^{3+0} * L_{30}(\mu_1 + \mu_2)$.

Case 8 to 9. 2D-HGF/DRBQFT II, MGLCHF I&II/DRBQFT I

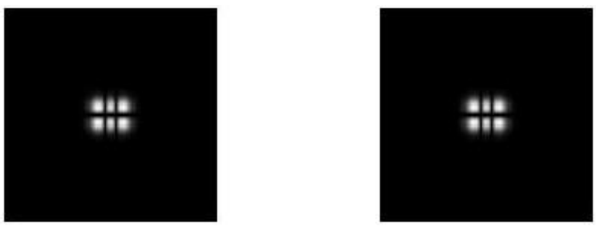
As depicted by Figure 11 and 12, DRBQFT I (H_{03}) is equivalent to $H_{03} * (-u_1)^{0+3}$, DRBQFT I ($L_{30}(u_1 + u_3)$) is equivalent to $L_{30} * (-u_1)^3(-u_3)^0$.Besides, Figure 13 demonstrate that ratio of spectrum to function multiplied by eigenvalue is equal to one (only middle part is shown). The phase errors shown in the border of Figure 4(b) to 12 (b) are due to finite-length effect of DQFT and DRBQFT and limited computational precision of Matlab.

(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)

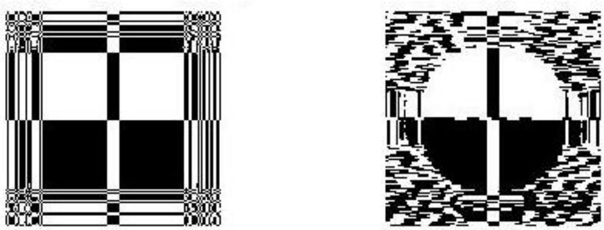


(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

Figure 4. $(-\mu)^{3+0} * H_{30}$andL-DQFT(H_{30}).



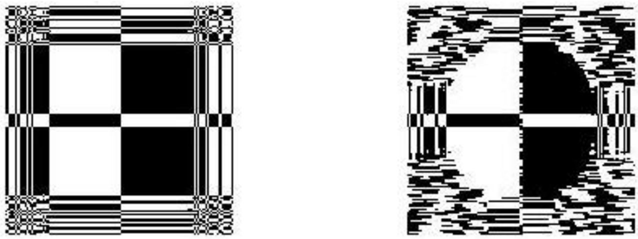
(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)



(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

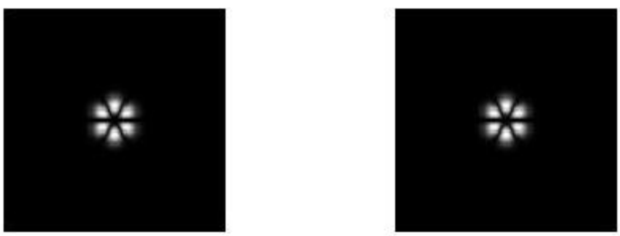
Figure 5. $H_{21} * (-\mu)^{2+1}$andR-DQFT(H_{21}).

(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)

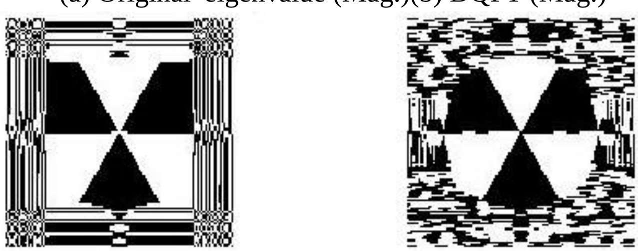


(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

Figure 6. $H_{12} * (-\mu_1)^1(-\mu_2)^2$ and T-DQFT(H_{12}).



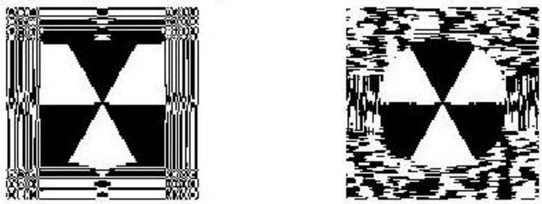
(a) Original*eigenvalue (Mag.)(b) DQFT (Mag.)



(c) Original*eigenvalue (Phase)(d) DQFT (Phase)

Figure 7. $(-\mu)^{3+0} * L_{30}$ andL-DQFT(L_{30}).

(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)



(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

Figure 8. $L_{30} * (-\mu)^{3+0}$ and R-DQFT(L_{30}).

(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)

(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

Figure 9. $L_{30}\,(\mu_1 + \mu_2) * (-\,\mu_2)^{3+0}$ and T-DQFT($L_{30}(\mu_1 + \mu_2)$).

4.2. Color Image Expansion and Partial Reconstruction Using GLCHFs

From the above experiments and discussions, we know that GLCHFs are the eigenfunctions of DQFT and DRBQFT. Therefore, GLCHFs possess some useful properties, such as shape invariance under DQFT and DRBQFT. We can represent GLCHFs as a 2D quaternion or 2D RB functions and use them as basis to expand quaternion encoded color images $I_Q(m,n)$, or RB encoded color images $I_{RB}\ (m,n)$, i.e.,

$$
\begin{aligned}
I_Q(m,n) &= i\cdot R(m,n) + j\cdot G(m,n) + k\cdot B(m,n) \\
&= \sum_{s=0}^{\infty}\sum_{t=0}^{s} h_{s-t,t}^{Q} L_{s-t,t}^{Q}(m,n) \\
I_{RB}(m,n) &= i'\cdot R(m,n) + j'\cdot G(m,n) + k'\cdot B(m,n) \\
&= \sum_{s=0}^{\infty}\sum_{t=0}^{s} h_{s-t,t}^{RB} L_{s-t,t}^{RB}(m,n)
\end{aligned}
\tag{40}
$$

where $R(\mathrm{m,n})$ is the red color channel, $G(\mathrm{m,n})$ is the green color channel, and $B(\mathrm{m,n})$ is the blue color channel of the color image, respectively. $\{i,j,k\}$ is the quaternion basis introduced in (2) and $\{i',j',k'\}$ is the RB basis mentioned in (5). $h_{s-t,t}^{Q}$ is the Gauss-Laguerre circular harmonic (GL-CH) expansion coefficient of order (*s*-*t*,*t*) for quaternion encoded color image and $L_{s-t,t}^{Q}(m,n)$ is quaternion encoded 2D GLCHF basis of order (*s*-*t*,*t*). On the other hand, $h_{s-t,t}^{RB}$ is the GL-CH expansion coefficient of order (*s*-*t*,*t*) for RB encoded color image and $L_{s-t,t}^{RB}(m,n)$ is RB encoded 2D GLCHF basis of order (*s*-*t*,*t*). For convenience, the representation of order (*s*-*t*,*t*) is replaced with (*a*,*b*). From (40), we see that quaternion/RB encoded color images can be expanded by using infinite number of expansion coefficients and GLCHFs. However, only some of the expansion coefficients and GLCHFs is useful and meaningful for image reconstruction task, only part of these coefficients and GLCHFs are retained and applied to partially reconstruct the original color image. As illustrated in Figure14, the square represents the domain of expansion coefficients $h_{a,b}^{Q}\ or\ h_{a,b}^{RB}$, and the shaded area is the coefficients that will be used. Two parameters *K* and *L* can be determined to do sifting of coefficients in the domain and define the shape of shaded area. In what follows, we perform several experiments to test the efficiency of partial reconstruction. The test color images are *N by N* pixels, where *N* is set to 64. Two methods are proposed to sift expansion coefficients of interests as follows:

Method 1.

For a fixed *K*, we increase *L* from zero to a predefined number (the increment step *d* can be verified). The sum of order (a+b) satisfy the following constraint,

$$(\mathrm{a}+\mathrm{b}) = L \leq 2N+2 \tag{41}$$

and the absolute difference $|a-b|$ satisfy another constraint,

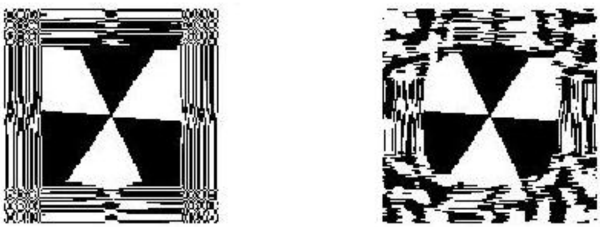
$$|a-b| \le N < \mathrm{K} \tag{42}$$

From Figure 15, the partially reconstructed color images by using the sifted expansion coefficients $h_{a,b}^{Q}$ or $h_{a,b}^{RB}$ with method 1reveal the details of image circularly and gradually. We cannot recognize the content of original image well when the order ($a + b = L$) is low, but when the order ($a + b = L$) is high, e.g., L = 100 ~ 130, we only use finite, small number of coefficients and GL-CH basis to approximate original images. It can be seen that the approximated images are visually pleasing with high fidelity.

Method 2.

For a fixed L, increasing K from zero to a predefined number (41) and (42) are also satisfied. From Figure 16, the partially reconstructed color images by using the sifted expansion coefficients $h_{a,b}^{Q}$ or $h_{a,b}^{RB}$ with method 2 approximate the support of the original image. We can recognize the content and support size of original image even when the order (K) is low. As for K is high, the approximations are slightly inferior to those of L is high, especially in the border of the reconstructed images. This is because the main purpose of method 2 is to obtain the support and roughly sketch of original images.

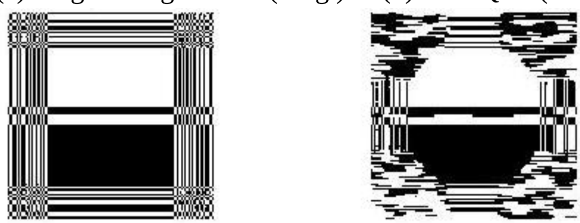
(a) Original*eigenvalue (Mag.) (b) DQFT (Mag.)



(c) Original*eigenvalue (Phase) (d) DQFT (Phase)

Figure 10. $(-\mu_1)^{3+0} * (\mu_1+\mu_2)L_{30}$ and T-DQFT($(\mu_1+\mu_2)L_{30}$).

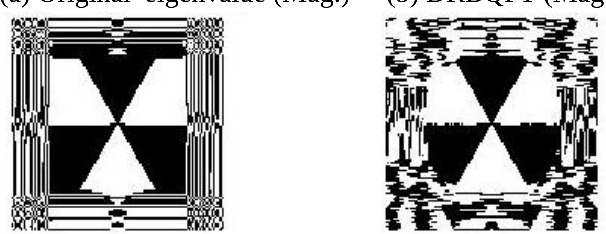
(a) Original*eigenvalue (Mag.) (b) DRBQFT (Mag.)



(c) Original*eigenvalue (Phase) (d) DRBQFT (Phase)

Figure 11. $H_{03} * (-u_1)^{0+3}$ and DRBQFT II (H_{03}).

(a) Original*eigenvalue (Mag.) (b) DRBQFT (Mag.)



(c) Original*eigenvalue (Phase) (d) DRBQFT (Phase)

Figure 12. $L_{30}(u_1 + u_3) * (-u_1)^3(-u_3)^0$ and DRBQFT I ($L_{30}(u_1 + u_3)$).

4.3. Expansion Coefficients $h_{a,b}^{Q}$ and $h_{a,b}^{RB}$ as Rotation Invariant Features

Eq. (40) is the image expansion by using GLCHFs, it can be represented as a matrix form:

$$I_Q = L_Q h_Q$$
$$I_{RB} = L_{RB} h_{RB} \tag{43}$$

where I_Q is a quaternion matrix and I_{RB} is a RB matrix formed by original color image. L_Q is quaternion encoded GLCHF basis matrix and L_{RB} is RB encoded GLCHF basis matrix. h_Q is quaternion encoded coefficient matrix and h_{RB} is RB encoded coefficient matrix. We multiply inverse matrix of L^Q or L^{RB} to the left-side of I^Q or I^{RB} in order to obtain expansion coefficients. That is:

$$L_Q^{-1} I_Q = h_Q$$
$$L_{RB}^{-1} I_{RB} = h_{RB} \tag{44}$$

V <121x121 double>

	41	42	43	44
40	1.0000	1.0000	1.0000	1.0000
41	1.0000	1.0000	1.0000	1.0000
42	1.0000	1.0000	1.0000	1.0000
43	1.0000	1.0000	1.0000	1.0000
44	1.0000	1.0000	1.0000	1.0000
45	1.0000	1.0000	1.0000	1.0000
46	1.0000	1.0000	1.0000	1.0000
47	1.0000	1.0000	1.0000	1.0000
48	1.0000	1.0000	1.0000	1.0000
49	1.0000	1.0000	1.0000	1.0000
50	1.0000	1.0000	1.0000	1.0000
51	1.0000	1.0000	1.0000	1.0000
52	1.0000	1.0000	1.0000	1.0000

Figure 13. Eigenvalue verification of Figure 4~12. The transformed spectrums of functions and the ratio of spectrum to original function multiplied by derived eigenvalue is 1.

Eq. (44) is Laguerre Gauss transform (LGT) of quaternion/RB encoded color images. $h_{a,b}^{Q}$ and $h_{a,b}^{RB}$ are LGT coefficients. Figure 17 and Table 5 demonstrate that $h_{a,b}^{Q}$ possess rotation invariant property, because the mean squared errors of h_Q^O (LGT coefficients of original color image) and h_Q^R (LGT coefficients of rotated color image) are very small under different rotation angles (the experiment of $h_{a,b}^{RB}$ is not shown here due to page limit).

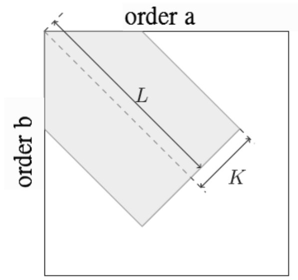
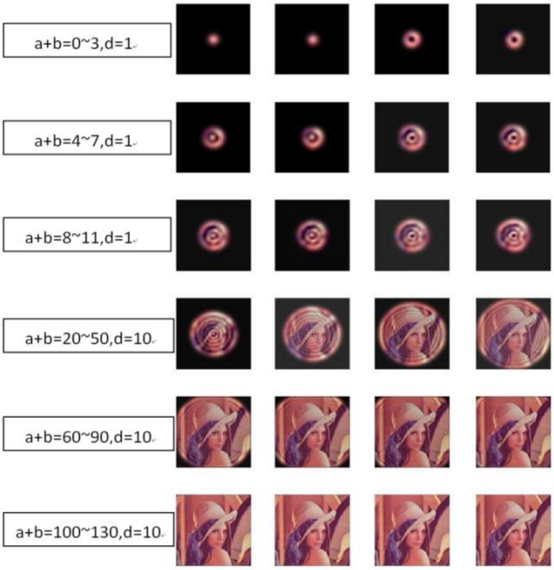


Figure 14. The illustration of partial GL-CH coefficients used to reconstruct color image.

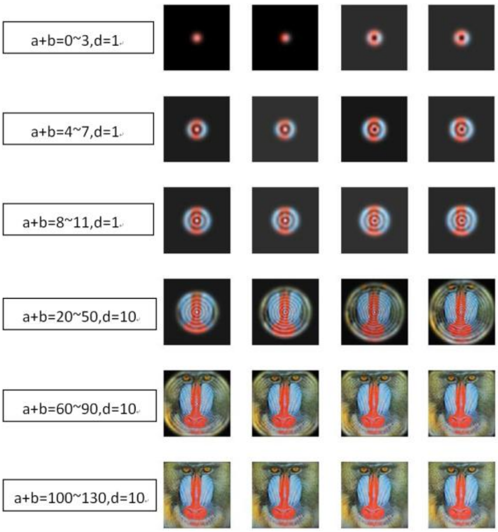
Table 5. Mean squared error of h_Q^O and h_Q^R under different rotation angles

Ang.	15	30	45	60	75	90	105	120	135	150	165	180
MSE	0.0077	0.0062	0.0058	0.0061	0.0068	9.43e-32	0.0077	0.0062	0.0058	0.0061	0.0068	7.46e-32



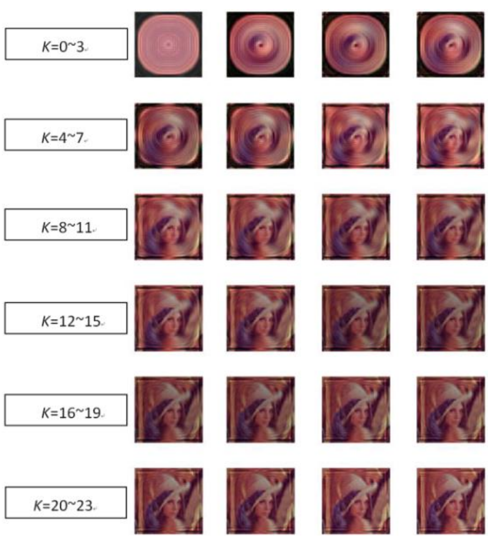
a

Figure 15. Continued on next page.



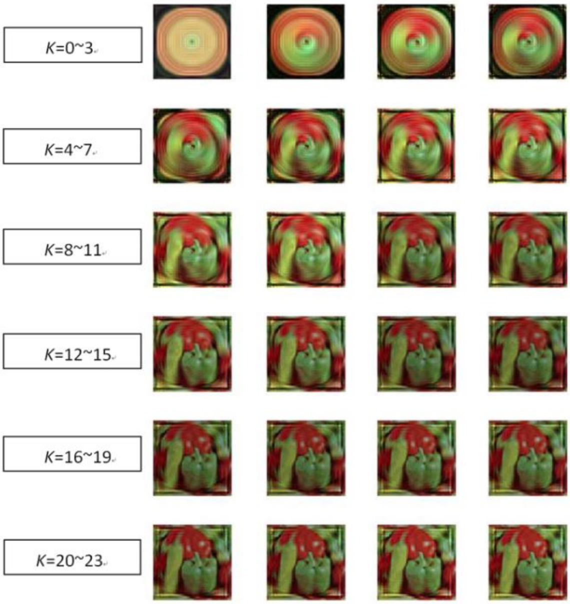
b

Figure 15. (a) Partially reconstructed quaternion encoded color image (Lena) by using $h_{a,b}^{Q}$ ((a,b) satisfies (41) and (42)) and $L_{a,b}^{Q}(m,n)$ with method 1. (b) Partially reconstructed RB encoded color image (Baboon) by using $h_{a,b}^{RB}$ and $L_{a,b}^{RB}(m,n)$ with method 1. d is the increment for $L=a+b=0\sim130, K=65>N$.



a

Figure 16. Continued on next page.



b

Figure 16. (a) Partially reconstructed quaternion encoded color image (Lena) by using $h^{Q}_{a,b}$ ((a,b) satisfies (41) and (42)) and $L^{Q}_{a,b}(m,n)$ with method 2. (b) Partially reconstructed RB encoded color image (Peppers) by using $h^{RB}_{a,b}$ and $L^{RB}_{a,b}(m,n)$ with method 2. $|a-b| < K$=0~23, L=2N+2=130.

4.4. Color Shape Matching by Using GLCHFs

In section 4.2, we proposed a method for color image decomposition. In this section, we further use the decomposition and quaternions/RBs algebra and propose novel color shape matching algorithm for jersey. We briefly summarize our color shape matching algorithms as follows and demonstrate some experimental results to verify our method by using GLCHFs and RBs. The test input images are depicted as follows (Figures 18-21).

Figure 17. (a)(c)(e)(g)(i)(k)(m)(o)(q)(s)(u)(w) Original Lena color images with zero padding and their rotated versions with different angles. (b)(d)(f)(h)(j)(l)(n)(p)(r)(t)(v)(x) Left: LGT coefficients $h_Q^O$of original Lena color images with zero padding. Middle: LGT coefficients $h_Q^R$of rotated Lena color images with different angles. Right: The difference between $h_Q^O$and h_Q^R, i.e., $\left|h_Q^R - h_Q^O\right|$. The white pixels in these figures are coefficients and differences with significant values.

Figure 18. Input image 1 for color matching.

Figure 19. Input image 2 for color matching.

Figure 20. Input image 3 for color matching.

Figure 21. Input image 4 for color matching.

Figure 22. Color patches that can be used to do color shape matching (Clipped From jersey in Figure 18-21).

4.4.1. Color Shape Matching Algorithm by Using GLCHFs and RBs for Jersey

1. Find the patches of colors so that we can do color shape matching by using them:

2. Use GLCHFs to approximate these patches and use those GLCHFs with circular shape (the fourth row, fourth column one, marked by black square) to perform the color matching task of human, below are examples of decomposition of patches:
3. Transform the interested color image $f(m_s, n_s)$ and GLCHFs approximated color patch $h(m_s, n_s)$ into I-H-S color space by:

$$\begin{bmatrix} f_I(m_s,n_s) \\ f_{V_1}(m_s,n_s) \\ f_{V_2}(m_s,n_s) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \cdot \begin{bmatrix} f_R(m_s,n_s) \\ f_G(m_s,n_s) \\ f_B(m_s,n_s) \end{bmatrix} \tag{45}$$

$$f_H(m_s,n_s) = \tan^{-1}\left(\frac{f_{V_2}(m_s,n_s)}{f_{V_1}(m_s,n_s)}\right) \qquad f_S(m_s,n_s) = \sqrt{f_{V_1}^2(m_s,n_s) + f_{V_2}^2(m_s,n_s)}$$

$$\begin{aligned} f_A(m_s,n_s) &= \sqrt{f_I^2(m_s,n_s) + f_S^2(m_s,n_s)} \\ &= \sqrt{f_R^2(m_s,n_s) + f_G^2(m_s,n_s) + f_B^2(m_s,n_s)} \\ f_\phi(m_s,n_s) &= \cos^{-1}\left(\frac{f_I(m_s,n_s)}{f_A(m_s,n_s)}\right) \\ &= \sin^{-1}\left(\frac{f_S(m_s,n_s)}{f_A(m_s,n_s)}\right) \\ -\pi \le f_H(m_s,n_s) < \pi, &-\pi/2 \le f_\phi(m_s,n_s) \le \pi/2 \end{aligned}$$

We can use RB polar form to represent the color image and patch as follows:

$$\begin{aligned} h(m_s,n_s) &\equiv A_h(m_s,n_s) \cdot e^{iH_h(m_s,n_s)} \cdot e^{k\phi_h(m_s,n_s)} \\ f(m_s,n_s) &\equiv A_f(m_s,n_s) \cdot e^{iH_f(m_s,n_s)} \cdot e^{k\phi_f(m_s,n_s)} \end{aligned} \tag{46}$$

4. Calculate the energy of color patch:

$$E_h = \sum_{m_s=0}^{M-1} \sum_{n_s=0}^{N-1} (|h(m_s,n_s)|)^2 = \sum_{m_s=0}^{M-1} \sum_{n_s=0}^{N-1} A_h^2(m_s,n_s) \tag{47}$$

and normalize the color patch $h(m_s,n_s)$ by E_h:

$$h_n(m_s,n_s) = h(m_s,n_s) / E_h \tag{48}$$

After normalizing by E_h, if input image matches the color patch, then the correlation values at the matching positions are nearly equal to 1 (1 + 0i + 0j + 0k).

5. Compute the RB Fourier transform of $h(m_s, n_s)$ and $f(m_s, n_s)$:

$$F(p,s) = DRBFT(f(m_s,n_s)) and \\ H_n(p,s) = DRBFT(h_n(m_s,n_s)) \tag{49}$$

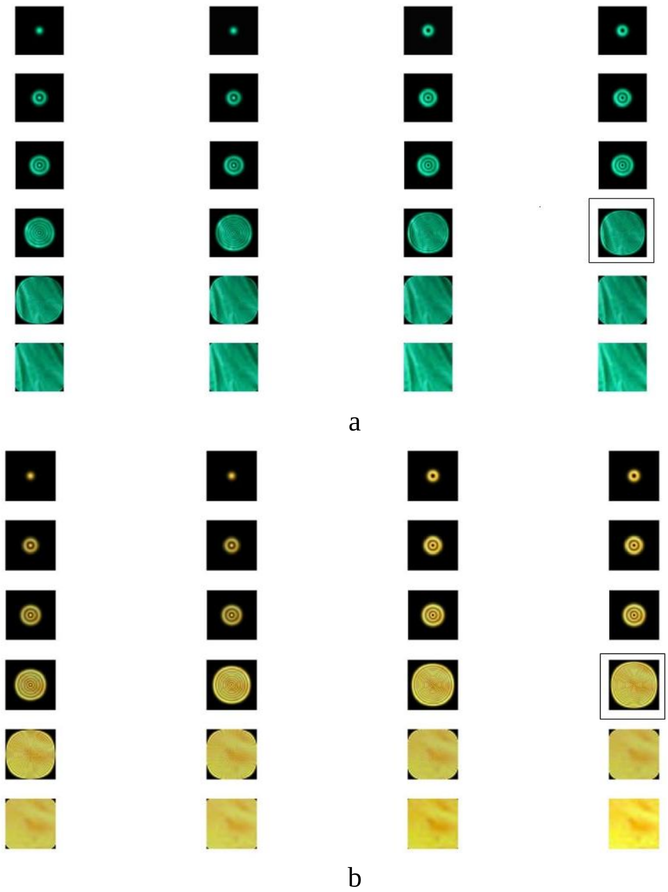


Figure 23. Decomposition of color patches by using GLCHFs and RBs, we use the one marked by black square to perform color shape matching (the one with circular shape).

6. Compute the RB correlation and phase-only correlation:

$$g(m_s,n_s) \equiv IDRBFT\left\{F_{(RB)}(p,s)H_{c-,n(RB)}(p,s)\right\}$$
$$\hat{g}(m_s,n_s) \equiv IDRBFT\left\{\frac{F_{(RB)}(p,s)H_{c-,n(RB)}(p,s)}{\left|F_{(RB)}(p,s)H_{c-,n(RB)}(p,s)\right|}\right\} \quad (50)$$

The definition of the RB correlation is:

$$\hat{g}(m_s,n_s) \equiv f(m_s,n_s) \otimes_{RB,PO} h(m_s,n_s)$$
$$\equiv IDRBFT\left(\frac{F(p,s)H_{c-}(p,s)}{\left|F(p,s)H_{c-}(p,s)\right|}\right) \quad (51)$$

Correlation can be viewed as a special case of convolution

$$f(m_s,n_s) \otimes_{RB} h(m_s,n_s) = f(m_s,n_s) *_{RB} \overline{h(-m_s,-n_s)} \quad (52)$$

so we just use the algorithms of convolution to implement correlation.

$$g(m_s,n_s) = IDRBFT(F(p,s)H_{c-}(p,s)) \quad (53)$$

where $F(p,s)$ and $H_{c-}(p,s)$ are the RB Fourier transform of $f(m_s,n_s)$ and $\overline{h(-m_s,-n_s)}$, respectively. (The subscript c- means conjugation and spatial reverse). The RB phase-only correlation can be defined as:

$$\hat{g}(m_s,n_s) \equiv f(m_s,n_s) \otimes_{RB,PO} h(m_s,n_s)$$
$$\equiv IDRBFT\left(\frac{F(p,s)H_{c-}(p,s)}{\left|F(p,s)H_{c-}(p,s)\right|}\right) \quad (54)$$

Using the result of the phase-only correlation we can find the positions of human object that have similar shape as color patch. Because the shape of human is nearly circular or ellipsoidal, therefore we can use GLCHFs circularly approximated color patch to do correlation and thus find out the locations of human or children in our test target images.

7. From the phase-only correlation result, we have found the candidate positions of matched objects.

Then, to determine if the average of brightness, chromaticity, hue, or saturation of these objects are the same as the one of the color patch, we define three parameters and their requirements as follows:

$$\rho_1 = \frac{|g_r(m_s,n_s)|}{|g_r(m_s,n_s) + |g_i(m_s,n_s)| + g_j(m_s,n_s) + g_k(m_s,n_s)|}$$

$$\rho_2 = \frac{|g_r(m_s,n_s) + g_k(m_s,n_s)|}{|g_r(m_s,n_s)| + |g_i(m_s,n_s)| + |g_j(m_s,n_s)| + |g_k(m_s,n_s)|}$$

$$\rho_3 = \frac{|g_r(m_s,n_s)| + |g_i(m_s,n_s)|}{|g_r(m_s,n_s) + g_i(m_s,n_s)| + |g_j(m_s,n_s)| + g_k(m_s,n_s)}$$

$$\begin{aligned}
&(for \quad average \quad brightness \quad match): \\
&c_1 < g(m_s,n_s) < c_2 \\
&c_1 < 1 < c_2 \\
&(for \quad average \quad chromaticity \quad match): \\
&\rho_1 \geq d_1 (d_1 \approx 1 \quad d_1 < 1) \\
&(for \quad average \quad hue \quad match): \\
&\rho_2 \geq d_2 (d_2 \approx 1 \quad d_2 < 1) \\
&(for \quad average \quad saturation \quad match): \\
&\rho_3 \geq d_3 (d_3 \approx 1 \quad d_3 < 1)
\end{aligned} \tag{55}$$

where *(r,i,j,k)* are real part (r),i imaginary part (i), j imaginary part (j), and k imaginary part (k) respectively.

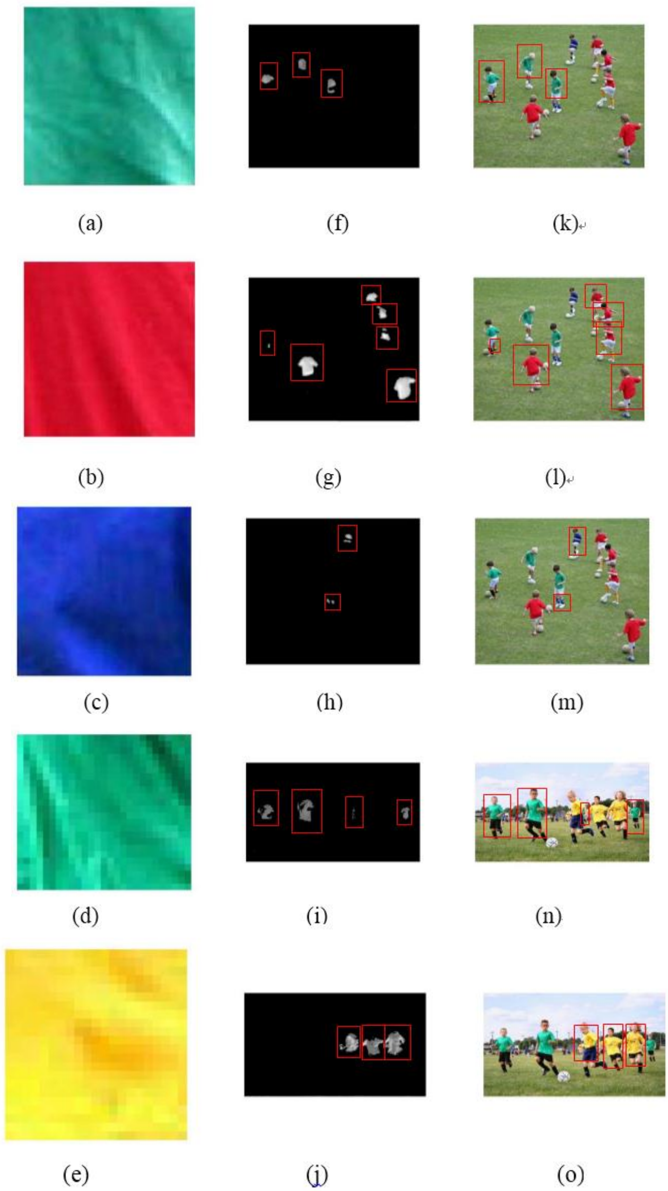


Figure 24. Color shape matching results of two test input images. (marked by red square) (a)-(e) color patches. (f)-(j) results of different color matching. (k)-(o) are original input color images. As can be seen from these results, the location of human can be found and we can separate human with different color clothes by performing the whole color shape matching task.

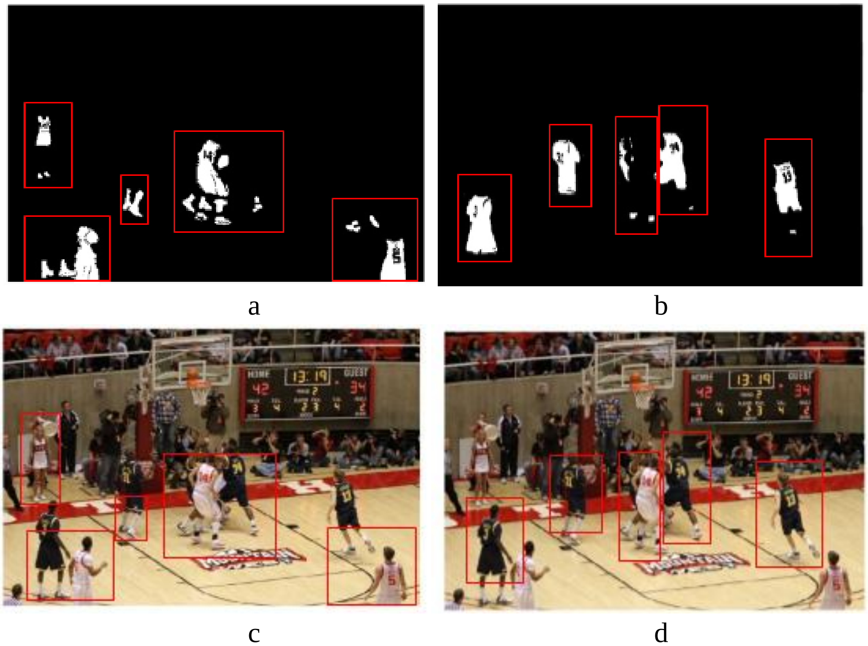


Figure 25. Color shape matching results. (marked by red square)(a) the matching result by using white color patch. (b) the matching result by using black patch (c)-(d) original test color image.

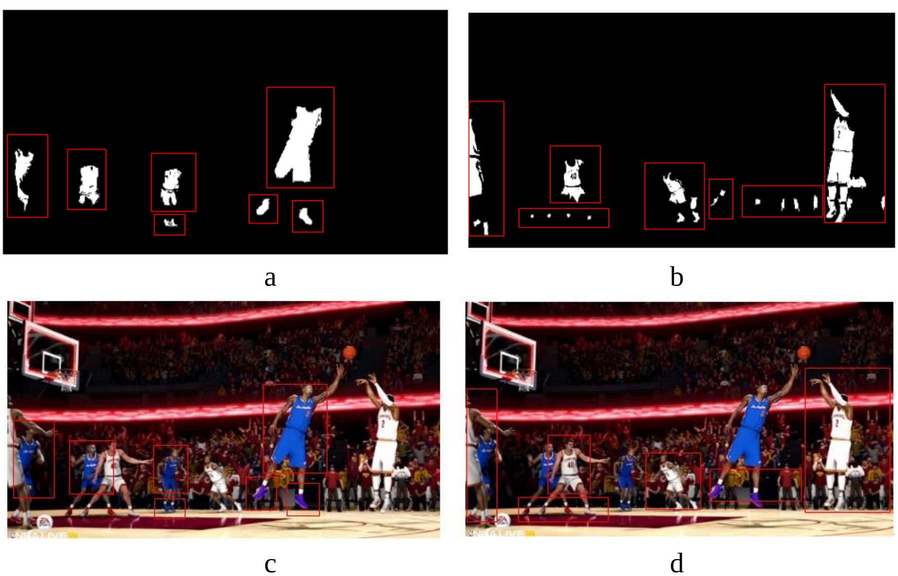


Figure 26. Color matching results. (marked by red square)(a) the matching result by using blue color patch. (b) the matching result by using white patch (c)-(d) original test color image.

8. Finally, we can find the location of human object and match the colors or their clothes by eq.(55).

Bellow (see Figure 32-34) are results of the color matching by applying above methods. We can see that the locations of human can be found and after thresholding, we can successfully accomplish the color matching mask of their clothes by using patches with different colors

Conclusion

In this work, we derive the eigenvalues of 2D-HGFs and GLCHFs for DQFT and DRBQFT. The experimental results verify our derivations. Two partial reconstruction methods of color image are proposed based on GLCHFs and the reconstructed results demonstrate the efficacy and usefulness of our methods. We also show that the expansion coefficients can be used as a rotation invariant feature and also propose a novel method to perform color shape matching of human jersey effectively.

References

[1] B.W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Trans. Acoust, Speech, Signal Process.*, vol. 30, pp. 25-31, 1982.

[2] F. A. Grunbaum, "The eigenvectors of the discrete Fourier transform: A version of the Hermite functions," *J. Math. Anal. Appl.*, vol. 88, pp.355-363, 1982.

[3] S. C. Pei, W. L. Hsue, and J. J. Ding, "Discrete fractional Fourier transform based on new nearly tridiagonal commuting matrices," *IEEETrans. Signal Process.*, vol. 54, no.10, pp.3815-3828, 2006.

[4] B. Santhanam and T. S. Santhanam, "On discrete Gauss– Hermite functions and eigenvectors of thediscrete Fourier transform," *J. Signal. Process.*, vol. 88, pp.2738-2746, Nov. 2008.

[5] S. C. Pei, J. J. Ding, K. W. Chang, "Eigenfunctions, eigenvalues, and fractionalization of the quaternion and biquternion Fourier transforms," *EUSIPCO 2010*, pp.1874-1878, Aug. 2010.

[6] S. Sangwine, "The discrete quaternion Fourier transform," *IEE Conf.*, vol. 2, pp. 790-793, 1997.

[7] S. C. Pei, J. H. Chang, and J. J. Ding, “Commutative reduced biquaterinions and their Fourier transform for signal and image processing applications,” *IEEETrans. Signal Processing*, vol. 52, no.7, pp.2012-2031, Jul. 2004.

[8] W. R. Hamilton, “Elements of quaternion,” *Longmans, Green and Co., London*, 1866.

[9] C. Moxey, S. Sangwine, T. A. Ell, “Hypercomplex correlation techniques for vector images,” *IEEE Trans. Signal Processing*, vol. 51, no.7, pp.1941-1953, Jul. 2003.

[10] S. C. Pei, J. J. Ding, and J. H. Chang, “Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D complex FFT,”*IEEE Trans. on Signal Processing*, vol. 49, no.11, pp.2783-2797, 2001.

[11] E. Bayro-Corrochano, “The theory and use of the quaternion wavelet transform,” *Journal of Mathematical Imaging and Vision*, vol. 24, pp.19-35, 2006.

[12] J. Zhou, Y. Xu, and X. Yang, “Quaternion wavelet phase based stereo matching for uncalibrated Images,” *Pattern Recognition Letters*, vol. 28, pp. 1509-1522, 2007.

[13] G. Metikas and S. Olhede, “Multiple multidimensional morse wavelets,” *IEEE Trans. on Signal Processing*, vol. 51, no.7, pp.1941-1953, Jul. 2003.

[14] W. Chan, H. Choi, and R. Baraniuk, “Coherent multiscale image processing using dual-tree quaternion wavelets,” *IEEE Trans. Image Processing*, vol. 17, no.7, pp.1069-1082, Jul. 2008.

[15] W. Chan, H. Choi, and R. Baraniuk, “Directional hypercomplex wavelets for multidimensional signal analysis and processing,” *Proc. IEEE int. Conf. Acoust, Speech, Signal Processing*, vol. 3, pp.III-996-III-999, May. 2004.

[16] T. Tsui, X. P. Zhang, and D. Androutsons, “Color image watermarking using multidimensional Fourier transforms,” *IEEE Trans. Inf. Forensics Security*, vol. 3, no.1, pp.16-28, Mar.2008.

[17] W. R. Hamilton, “Lectures on quaternions,” *Article 669*, 1853.

[18] H. D. Schtte, J. Wenzel, “Hypercomplex numbers in digital signal processing,” *IEEE International Symposium on Circuits and Systems*, vol. 2, pp.1557-1560, 1990.

[19] X. F. Gong, Z. W. Liu, and Y. G. Xu, “Quad-quaternion MUSIC for DOA estimation using electromagnetic vector sensors,” *EURASIP Journal on Advances in Signal Processing*, vol. 2008.

[20] S. C. Pei, J. H. Chang, J. J. Ding, M. Y. Chen, “Eigenvalues and singular value decompositions of reduced biquaternion matrices,” *IEEE Trans. Cirt. and Sys. I*, vol. 55, no.9, pp.2673-2685, 2008.

[21] E. D. Di Claudio, G. Jacovitti, and A. Laurenti, "Maximum likelihood orientation estimation of 1-D Pattern in Laguerre-Gauss Subspaces," *IEEE Trans. Image Processing*, vol. 19, no.5, pp.1113-1125, May. 2010.

[22] C. Segre, "Le rappresentazioni reali delle forme complesse a gli enti i per algebrici," *Math. Ann.*, vol. 40, no.3, pp. 413-467, Sep, 1892.

[23] C. M. Davenport, "Commutative hypercomplex mathematics," Available online athttp://home.comcast.net/~cmdaven/hyprcplx.htm.

[24] R. M. Yamaleev, "Multicomplex algebras on polynomials and generalized Hamilton dynamics," *J. Math. Anal. Appl.*, vol. 322, no.2, pp. 815-824, Oct. 2006.

[25] C. M. Davenport, "A communitative hypercomplex algebra with associated function theory," In: R. Ablamowicz, editor, Clifford Algebra with Numeric and Symbolic Computations, Birkhauser, Boston, pp.213-227, 1996.

[26] D. V. Sorokin, M. M. Mizotin, and A. S. Krylov, "Gauss-Laguerre keypoints extraction using fast Hermite projection method," *in lm. Anal. And Rec., ser. LNCS*, A. Campilho and M. Kamel, Eds. vol. 6753, pp. 284-293, *Springer*, 2011.

[27] Quaternion toolbox for Matlab (QTFM toolbox). Available online at http://qtfm.sourceforge.net/.

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Chapter 6

THE QUATERNIONS WITH AN APPLICATION OF QUADROTORS TEAM FORMATION

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Abstract

The unit quaternion system was invented in 1843 by William Rowan Hamilton as an extension to the complex number to find an answer to the question (how to multiply triplets?). Yet, quaternions are extensively used to represent the attitude of a rigid body such as quadrotors, which is able to alleviate the singularity problem caused by the Euler angles representation. The singularity is in general a point at which a given mathematical object is not defined and it outcome of the so called gimbal lock. The singularity is occur when the pitch angles rotation is $\theta = \pm 90^o$. In this chapter, a leader-follower formation control problem of quadrotors is investigated. The quadrotor dynamic model is represented by unit quaternion with the consideration of external disturbance. Three different control techniques are proposed for both the leader and the follower robots. First, a nonlinear H_∞ design approach is derived by solving a Hamilton-Jacobi inequality following from a result for general nonlinear

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affine systems. Second, integral backstepping (IBS) controllers are also addressed for the leader-follower formation control problem. Then, an iterative Linear Quadratic Regulator (iLQR) is derived to solve the problem of leader-follower formation. The simulation results from all types of controllers are compared and robustness performance of the H_∞ controllers, fast convergence and small tracking errors of iLQR controllers over the IBS controllers are demonstrated.

Keywords: Quaternion, Quadrotor UAVs, Leader-follower formation control

1. Introduction

In the last decade, the focus on control single unit quadrotors has expanded to controlling a team of quadrotors for these to be able to achieve their tasks in variable weather and complicated environments. Team formation flight also provides advantages over the use of an individual quadrotor in both civil and military applications, such as inspection of an inaccessible area, disaster management, and search and rescue in risky circumstances, etc. Most of these applications demand more than one quadrotor to accomplish the desired objective [1, 2]. The leader-follower approach is one of the main approaches of formation control design.

Distributed and decentralised control techniques were used in the literature to solve the leader-follower control problem. The distributed control technique assumes that not all followers receive the leader's information and there is a kind of cooperation among them [3–14], while the decentralised control technique proposes that all followers are able to receive the leader's information [15–24]. Different controllers have been implemented with both distributed and decentralised control techniques.

1.1. Distributed Control Technique

A robust LQR controller was proposed for individual quadrotors and team formation as well in [6]. The controller was designed for a linearised system around the hovering point. The simulation results indicated the ability of the controller to overcome the changes in communication topology among the robots with no dynamic effects. A NNs controller was presented in [3] for addressing the leader-follower problem. These two studies used Lyapunov theory to analyse the controller stability.

A BS controller was discussed in [8] based on graph theory to maintain the distance among the robots and in [5] with balanced graph and strong connection among the robots. The quadrotors' dynamic systems were linearised around the hovering point and a good performance was obtained in normal circumstances. A distributed cohesive motion control scheme was presented in [9] for 3D motion to maintain the distance among robots. This technique was developed to become a decentralised technique and significant attempts to deal with decentralised control techniques have been made. Three time scale controllers based on the sliding mode controller were proposed in [4] for dealing with the quadrotor formation problem. The controllers were used for the path tracking, attitude tracking and velocity in order to keep the formation and maintain the distance among the robots with the presence of external disturbance affecting the leader robot only. The simulation results proved the effectiveness of the proposed scheme.

A nonlinear control theory was presented to ensure the stability of quadrotors team formation in [7]. The wireless communication among the team was obtained via medium access control protocols. Experimental tests verified the proposed algorithm with time delay consideration. In [10] the problem of the leader-follower consensus of a swarm of rigid body space crafts system was analysed based on quaternion representation using a distributed control technique. They assumed that the communication between two neighbouring followers is bidirectional and that all followers can receive the leader information. Stability analysis was obtained via Lyapunov theory and the simulation results proved the attitude and angular velocity tracking stability. In [11] a MPC technique with integrated trajectory planning was analysed with a planning horizon for both team formation and obstacle avoidance. The method showed good simulation results. A distributed coordinated control scheme was proposed by [12] to solve the problem of time-delay in leader-follower team formation communication of quadrotors and the simulation results under sufficient conditions demonstrated the validity of the presented control technique. Xiwang et al. [13] proposed a consensus-based approach for the time varying formation control problem. The simulation and the practical test of five quadrotors demonstrated the validation of the proposed control approach. A vision-based servoing distributed control approach was presented in [14], where the quadrotors equipped cameras to track a moving target which provided the position information to be used for controllers.

1.2. Decentralised Control Technique

Abdessameud and Tayebi [15] proposed a procedure which depends on a quaternion representation and is split up into translational and rotational control design under the upper bounded translational control input. Analysis of the closed-loop system stability was achieved using Lyapunov theory. The proposed strategy took 8 seconds to catch the desired formation shape. A hybrid supervisory control based on a polar partitioning approach was suggested in [16] for the team formation problem and for collision avoidance as well. The combination of discrete quadrotors dynamic system and the supervisor was achieved using the parallel composition and the simulation results displayed that this method allows the supervisors to achieve a free collision in normal environments. A MPC technique was proposed in [21], where its hierarchical control effectiveness was compared with the potential field technique. The stability of the feedback controller based on fluid dynamic models in [17] was obtained based on smoothed-particle hydrodynamic. The simulation results of the above methods validated the proposed approaches.

Authors in [18] proposed the trajectory planners and feedback controllers for following the planned trajectory. Next they proposed a nonlinear decentralised controller for an aggressive formation problem in the micro quadrotors team in [19]. Communication failures and network time delays impact on team formation efficiency were considered. Local information of neighbour robots in the team was used for individual trajectory planning. Preserving the required form was based on the status estimation of neighbour robots. Then the authors presented two approaches to overcome the problem of concurrent assignment and planning of trajectories (CAPT) for the quadrotors team, a decentralised D-CAPT and centralised C-CAPT in [20]. The decentralised D-CAPT and centralised C-CAPT results were compared in simulation and practice and the experimental results demonstrated a good performance in indoor application.

In [25] a human user for teleoperation with a haptic device was proposed for the quadrotor team formation control problem with the cooperation of a BS controller. The simulation results revealed the ability of the human user to teleoperate in order to perform the formation. A triangle formation control of three quadrotors using optimal control techniques via the Pontryagin maximum principle was presented in [26] and the simulation results showed the effectiveness of using team formation rather than using an individual quadrotor in terms of fuel consumption. In [27] a consensus problem of swarm systems was discussed to obtain the time-varying formation based on double-integrator system modelling. The experimental results of the three quadrotors in formation verified the effectiveness of the proposed approach in dynamic-free conditions.

A new developed framework gathering with a nonlinear MPC technique was presented in [28] to solve the problem of coalition formation. The simulation results showed a zero steady state error in free disturbance and dynamic circumstances. Koksal et al. [22] presented an adaptive formation scheme for quadrotors leader-follower formation. They proposed a distributed control scheme for the kinematic part, an adaptive LQ controller for pitch and roll angles, proportional control for yaw angle and a PID controller for altitude. Several scenarios were implemented in simulation and experiment to validate the algorithm. In [23] a combination of LQR and SM controllers were proposed for a 2D quadrotors leader-follower formation, where the LQR controller was used for position control while two SM controllers were used for the attitude and for maintaining the distance between the robots. The simulation results demonstrated the successfulness of combining the two control techniques. A BS control approach with nonlinear controllers was introduced for handling the team formation problem in [24] and the simulation results proved the effectiveness of the proposed controllers.

The results in most of the previous papers on leader-follower formation control of multi-quadrotor system did not consider the effect of external disturbances, such as payload changes (or mass changes), wind disturbance, inaccurate model parameters, etc., which often affected the quadrotors' control performance. Therefore, a quadrotor controller must be robust enough in order to reject the effect of disturbances and cover the change in model parameter uncertainties and external disturbances. Robust state feedback controllers are very demanding when dealing with the quadrotor control problem. The H_∞ control approach was able to attenuate the disturbance energy by measuring the ratio between the energy of cost vector and the energy of disturbance signal vector [29].

The rest of this chapter is organized as follows: Section 2 presents the quadrotor dynamical model derivation based on quaternion representation. Section 3 introduces the leader-follower formation control problem with one leader and one follower in a distributed way. Section 4 provides a review on H_∞ optimal control approach. The main result of this approach is given as well, including the details of the designed state feedback controller for the formation problem. In section 5, presents the integral backstepping concept and the formation controllers with its stability analysis. In section 6, presents the derivation of iLQR controller for the leader and the follower. Section 7 shows the performance of the presented controllers while the conclusions of this study are indicated in section 8.

2. Mathematical Model

To control the motion and rotation of the quadrotor UAV, the mathematical dynamic model should be achieved. The quadrotor UAV system has a nonlinear dynamic system and complicated structure; therefore, it is difficult to represent its motion and rotation in a simple model. The dynamic model of the quadrotor UAV depends on some assumptions [30]:

- The structure of the quadrotor is rigid and symmetrical;
- The propellers are rigid;
- The centre of mass and body fixed frame are coincides;
- Thrust and drugs are proportional to the square of the propellers; and
- The difference of gravity by altitude or the spin of the earth is minor.

According to these assumptions, the mathematical model can be derived to perform the quadrotor UAV fuselage dynamics in space, where it will be easy to add to it the effects of aerodynamic forces generated by the rotation of the propellers. The coordinate reference system of the quadrotor includes two frames of reference, the inertial (earth fixed) frame mentioned $\mathcal{I}(x_{\mathcal{I}}, y_{\mathcal{I}}, z_{\mathcal{I}})$ and the rigid (body fixed) frame mentioned $\mathcal{B}(x_{\mathcal{B}}, y_{\mathcal{B}}, z_{\mathcal{B}})$. Several techniques can be used to perform the rigid body rotation in space such as Euler angles, Quaternions and Tait-Bryan angles [8]. The method used to describe the position and orientation of the quadrotor is the quaternion method. It is a hyper complex number of 4-tuple $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ which can be written in many ways as $Q = q_0 + q_1 i + q_2 j + q_3 k$ and $Q = [q_0, \mathbf{q}^T]^T$ [31] [32] [33].

The north east down (NED) coordinate system is used to parametrise the dynamic model of the quadrotor with an angle of one-axis rotation α around the Euler axis of unit vector $\mathrm{k} \in \mathbb{R}^3$ which has a direct physical connection and can be written as:

$$Q = \begin{bmatrix} \cos\frac{\alpha}{2} \\ \mathrm{k}\sin\frac{\alpha}{2} \end{bmatrix} \quad (1)$$

where $\mathrm{k} = \frac{\mathbf{q}}{\|\mathbf{q}\|}$ and $\alpha = 2\arccos q_0$. Moreover, as any complex number the norm, complex conjugate and inverse of the quaternion can be defined as:

$$\|Q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{2}$$

$$\bar{Q} = \begin{bmatrix} q_0 \\ -q_1 \\ -q_2 \\ -q_3 \end{bmatrix} \tag{3}$$

$$Q^{-1} = \frac{\bar{Q}}{\|Q\|}. \tag{4}$$

The unit quaternion can be used to represent the coordinate transformation between the inertial frame $\mathcal{I}$ and the body frame $\mathcal{B}$ by defining the multiplication and the inverse quaternion. The multiplication of two quaternions $Q = [q_0, \mathbf{q}^T]^T$ and $Q' = [{q'}_0, \mathbf{q'}^T]^T$ is defined as:

$$\begin{aligned} Q \otimes Q' &= \begin{bmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0 I + S(\mathbf{q}) \end{bmatrix} \begin{bmatrix} {q'}_0 \\ \mathbf{q}' \end{bmatrix} \\ &= \begin{bmatrix} q_0 {q'}_0 - \mathbf{q}^T \mathbf{q}' \\ {q'}_0 \mathbf{q} + q_0 \mathbf{q}' + S(\mathbf{q})\mathbf{q}' \end{bmatrix}. \end{aligned}$$

The inverse unit quaternion is defined as $Q^{-1} = [q_0, -\mathbf{q}^T]^T$ for $Q = [q_0, \mathbf{q}^T]^T$. A vector $\mathrm{x}_{\mathcal{I}} \in \mathbb{R}^3$ in the inertial frame can be expressed as a vector $\mathrm{x}_{\mathcal{B}} \in \mathbb{R}^3$ in the body frame via $\mathrm{x}_{\mathcal{B}} = R^T \mathrm{x}_{\mathcal{I}}$. Using $\bar{\mathrm{x}} = [0, \mathrm{x}^T]^T$, the transformation from the inertial frame to the body frame is expressed as $\bar{\mathrm{x}}_{\mathcal{B}} = Q^{-1} \otimes \bar{\mathrm{x}}_{\mathcal{I}} \otimes Q$.

And if the norm of the quaternion is equal to one $\|Q\| = 1$, it means that the inverse is the same as the conjugate, which is the case used to represent the coordinate transformation between the inertial frame $\mathcal{I}$ and the body frame $\mathcal{B}$ by defining the multiplication and the inverse quaternion. The multiplication of two quaternions $Q = [q_0, \mathbf{q}^T]^T$ and $Q' = [{q'}_0, \mathbf{q'}^T]^T$ is defined as:

$$\begin{aligned} Q \otimes Q' &= \begin{bmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0 I + S(\mathbf{q}) \end{bmatrix} \begin{bmatrix} {q'}_0 \\ \mathbf{q}' \end{bmatrix} \\ &= \begin{bmatrix} q_0 {q'}_0 - \mathbf{q}^T \mathbf{q}' \\ {q'}_0 \mathbf{q} + q_0 \mathbf{q}' + S(\mathbf{q})\mathbf{q}' \end{bmatrix} \end{aligned} \tag{5}$$

where $S : \mathbb{R}^4 \to \mathbb{R}^{3\times 3}$ is the skew-symmetric cross product matrix, and $Q_S : \mathbb{R}^4 \to \mathbb{R}^{4\times 4}$ is the quaternion skew-symmetric cross matrix and they are defined as:

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{6}$$

$$Q_S(Q) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \tag{7}$$

$$\bar{Q}_S(Q) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}. \tag{8}$$

The derivative of the quaternion Q is linked with the quadrotor angular velocity as follows:

$$\dot{Q}'_\omega(Q, \omega') = \frac{1}{2} \begin{bmatrix} 0 \\ \omega' \end{bmatrix} \otimes Q = \frac{1}{2} \bar{Q}_S(Q) \begin{bmatrix} 0 \\ \omega' \end{bmatrix} \tag{9}$$

$$\dot{Q}_\omega(Q, \omega) = \frac{1}{2} Q \otimes \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \frac{1}{2} Q_S(Q) \begin{bmatrix} 0 \\ \omega \end{bmatrix}. \tag{10}$$

However, as mentioned above, the quaternion is a unit vector which is utilised as a rotation operator. Then the rotation from the fixed frame to the body frame requires a rotational matrix which is the same as in the Euler angles method but it does not contain trigonometric functions can be evaluated by rotating a vector from the fixed frame to the body frame as follows:

$$\begin{aligned} \begin{bmatrix} 0 \\ k' \end{bmatrix} &= Q \otimes \begin{bmatrix} 0 \\ k \end{bmatrix} \otimes Q^{-1} = Q \otimes \begin{bmatrix} 0 \\ k \end{bmatrix} \otimes \bar{Q} \\ &= \bar{Q}_S(Q)^T Q_S(Q) \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0^T \\ 0 & R_q \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix} \end{aligned} \tag{11}$$

where $k \in \mathbb{R}^3$ is a vector to be rotated from the fixed frame to the body frame and

$$R_q = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}; \tag{12}$$

that is $k' = R_q k$ and $k = R_q^T k'$.

Computing the quaternion parameters from Euler angles or computing the Euler angles from the quaternion parameters can be presented using the relationships [34]:

$$Q = \begin{bmatrix} \cos(\frac{\varphi}{2})\cos(\frac{\theta}{2})\cos(\frac{\psi}{2}) + \sin(\frac{\varphi}{2})\sin(\frac{\theta}{2})\sin(\frac{\psi}{2}) \\ \sin(\frac{\varphi}{2})\cos(\frac{\theta}{2})\cos(\frac{\psi}{2}) - \cos(\frac{\varphi}{2})\sin(\frac{\theta}{2})\sin(\frac{\psi}{2}) \\ \cos(\frac{\varphi}{2})\sin(\frac{\theta}{2})\cos(\frac{\psi}{2}) + \sin(\frac{\varphi}{2})\cos(\frac{\theta}{2})\sin(\frac{\psi}{2}) \\ \cos(\frac{\varphi}{2})\cos(\frac{\theta}{2})\sin(\frac{\psi}{2}) - \sin(\frac{\varphi}{2})\sin(\frac{\theta}{2})\cos(\frac{\psi}{2}) \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan 2(2(q_0 q_1 + q_2 q_3), q_0^2 - q_1^2 - q_2^2 + q_3^2) \\ \arcsin(2(q_0 q_2 - q_1 q_3)) \\ \arctan 2(2(q_0 q_3 + q_1 q_2), q_0^2 + q_1^2 - q_2^2 - q_3^2) \end{bmatrix}. \tag{14}$$

2.1. Quaternion Kinematics

The kinematic equations of the movements of a unit quaternion $Q(t)$ can be driven by rotating the quadrotor with its angular velocity vector ω in the three directions to make a slight change in the movement of the quadrotor Δt and the change will be as follows [35]:

$$Q(t + \Delta t) = \left[\cos(\tfrac{\Delta\alpha}{2}) I + \sin(\tfrac{\Delta\alpha}{2}) \begin{bmatrix} 0 & n_3 & -n_2 & n_1 \\ -n_3 & 0 & n_1 & n_2 \\ n_2 & -n_1 & 0 & n_3 \\ -n_1 & -n_2 & -n_3 & 0 \end{bmatrix} \right] Q(t) \tag{15}$$

where $\Delta\alpha = \omega\Delta t$. Then if Δt is considered small, these expressions hold, $\cos(\frac{\Delta\alpha}{2}) \cong 1$, $\sin(\frac{\alpha}{2}) \cong \frac{1}{2}\omega\Delta t$. According to these assumptions, Equation (15) can be written as:

$$Q(t + \Delta t) = \left[\ 1 + \tfrac{1}{2} S(\omega) \Delta t \ \right] Q(t). \tag{16}$$

Thus the kinematic quaternion movement is

$$\dot{Q} = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t) - Q(t)}{\Delta t} = \frac{1}{2} S(\omega) Q \tag{17}$$

where

$$S_S(\omega) = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}. \tag{18}$$

Then the time derivative of the quaternion kinematics can be written in the following two forms:

$$\dot{Q} = \frac{1}{2} Q \otimes \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ \omega \end{bmatrix} \otimes Q. \tag{19}$$

2.2. Quadrotor Kinematics and Dynamics

The quaternion formula of the dynamics of a solid shape under the effect of external forces applied to the centre mass which is distinct in the body fixed frame can be separated into translational and rotational motions and it can be defined as:

2.2.1. For Translational Motion

$$m \frac{d\mathbf{v}}{dt_{\mathcal{I}}} = f. \tag{20}$$

Applying the Coriolis equation to (20) we have

$$m \frac{d\mathbf{v}}{dt_{\mathcal{I}}} = m\left(\frac{d\mathbf{v}}{dt_{\mathcal{I}}} + \omega_{\mathcal{B}/\mathcal{I}} \times \mathbf{v}\right) = f. \tag{21}$$

Applying Equation (21) in body coordinates with $\mathbf{v}^{\mathcal{B}} = (u, v, w)^T$ and $\omega^{\mathcal{B}}_{\mathcal{B}/\mathcal{I}} = (\omega_x, \omega_y, \omega_z)^T$ it will be:

$$m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = m(0 + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}) = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \tag{22}$$

or

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \omega_z v - \omega_y w \\ \omega_x w - \omega_z u \\ \omega_y u - \omega_x v \end{bmatrix} + \frac{1}{m} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}. \tag{23}$$

2.2.2. For Rotational Motion

From Newtons second law

$$\frac{dh^{\mathcal{B}}}{dt_{\mathcal{I}}} = m. \tag{24}$$

Applying the equation of Coriolis to Equation (24) we get

$$\frac{dh}{dt_{\mathcal{I}}} = \frac{dh}{dt_{\mathcal{B}}} + \omega_{\mathcal{B}/\mathcal{I}} \times h = m. \tag{25}$$

From the body coordinate we have $h^{\mathcal{B}} = J\omega^{\mathcal{B}}_{\mathcal{B}/\mathcal{I}}$, then Equation (25) can be resolved in the body coordinate frame. The equations of motion of the quadrotor UAVs depend on the two frames which can be written as in [36].

$$\begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = 0 + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \tau_\varphi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \tag{26}$$

or

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \frac{1}{J_x} & 0 & 0 \\ 0 & \frac{1}{J_y} & 0 \\ 0 & 0 & \frac{1}{J_z} \end{bmatrix} \left(\begin{bmatrix} \omega_y \omega_z (J_y - J_z) \\ \omega_x \omega_z (J_x - J_z) \\ \omega_x \omega_y (J_x - J_y) \end{bmatrix} + \begin{bmatrix} \tau_\varphi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \right). \tag{27}$$

or

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \frac{\omega_y \omega_z (J_y - J_z)}{J_x} \\ \frac{\omega_x \omega_z (J_x - J_z)}{J_y} \\ \frac{\omega_x \omega_y (J_x - J_y)}{J_z} \end{bmatrix} + \begin{bmatrix} \frac{\tau_\varphi}{J_x} \\ \frac{\tau_\theta}{J_y} \\ \frac{\tau_\psi}{J_z} \end{bmatrix}. \tag{28}$$

The relationship between position and velocities is given by

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\theta^T \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{29}$$

The total force applied to the quadrotor is given by $f = f_1 + f_2 + f_3 + f_4$ and the torque applied on the UAVs body which is created by the propellers τ and is equal to the difference between each pair of opposite propellers is

$$\begin{bmatrix} \tau_\varphi \\ \tau_\theta \\ \tau\psi \end{bmatrix} = \begin{bmatrix} l(f_4 - f_2) \\ l(f_1 - f_3) \\ f_2 + f_4 - f_1 - f_3 \end{bmatrix}. \tag{30}$$

hence, the effect of gravity can be written as:

$$f_g = R_q \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} = \begin{bmatrix} -2mg(q_1q_3 + q_0q_2) \\ -2mg(q_2q_3 - q_0q_1) \\ -mg(q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix}. \tag{31}$$

Then the translational equations can be written as:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \omega_z v - \omega_y w \\ \omega_x w - \omega_z u \\ \omega_y u - \omega_x v \end{bmatrix} + \begin{bmatrix} -2mg(q_1q_3 + q_0q_2) \\ -2mg(q_2q_3 - q_0q_1) \\ -mg(q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix} + \frac{1}{m} \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \begin{bmatrix} 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \frac{f}{m}. \tag{33}$$

In the rotational motion part, two differential equations hold: the quaternion and the angular velocity differential equation. The quaternion rate equation can be rewritten as:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \tag{34}$$

Then the full model for the quadrotor kinematics and dynamics can be summarised as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \begin{bmatrix} 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \frac{f}{m} \tag{35}$$

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \frac{\omega_y \omega_z (J_y - J_z)}{J_x} \\ \frac{\omega_x \omega_z (J_x - J_z)}{J_y} \\ \frac{\omega_x \omega_y (J_x - J_y)}{J_z} \end{bmatrix} + \begin{bmatrix} \frac{\tau_{q1}}{J_x} \\ \frac{\tau_{q2}}{J_y} \\ \frac{\tau_{q3}}{J_z} \end{bmatrix}. \tag{37}$$

The full mathematical model is

$$\begin{cases} \ddot{x} = 2(q_1 q_3 + q_0 q_2)\frac{f}{m} \\ \ddot{y} = 2(q_2 q_3 - q_0 q_1)\frac{f}{m} \\ \ddot{z} = -g + (q_0^2 - q_1^2 - q_2^2 + q_3^2)\frac{f}{m} \\ \dot{q}_0 = \frac{1}{2}(-q_1\omega_x - q_2\omega_y - q_3\omega_z) \\ \dot{q}_1 = \frac{1}{2}(q_0\omega_x - q_3\omega_y + q_2\omega_z) \\ \dot{q}_2 = \frac{1}{2}(q_3\omega_x + q_0\omega_y - q_1\omega_z) \\ \dot{q}_3 = \frac{1}{2}(-q_2\omega_x + q_1\omega_y + q_0\omega_z) \\ \dot{\omega}_x = \omega_y\omega_z\frac{J_y - J_z}{J_x} - \frac{J_r}{J_x}\omega_y\Omega + \frac{l}{J_x}\tau_{q1} \\ \dot{\omega}_y = \omega_z\omega_x\frac{J_y - J_z}{J_x} + \frac{J_r}{J_x}\omega_x\Omega + \frac{l}{J_x}\tau_{q2} \\ \dot{\omega}_z = \omega_x\omega_y\frac{J_y - J_z}{J_x} + \frac{l}{J_x}\tau_{q3} \end{cases}. \tag{38}$$

3. Leader-Follower Formation Problem for Quadrotors

3.1. Quadrotor Model

To describe the orientation of a quadrotor, the quaternion representation is used, which is able to alleviate the singularity problem caused by the Euler angles representation . The full dynamic model of a quadrotor can be written as:

$$\begin{cases} \dot{\mathbf{p}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = -g\mathbf{e} + \frac{f_i}{m_i}R_i\mathbf{e} \\ \begin{bmatrix} \dot{q}_{i0} \\ \dot{\mathbf{q}}_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\mathbf{q}_i^T\omega_i \\ (q_{i0}I + S(\mathbf{q}_i))\,\omega_i \end{bmatrix} \\ J_i\dot{\omega}_i = -S(\omega_i)J_i\omega_i - G(\omega_i) + \tau_i \end{cases} \tag{39}$$

where i is L for the leader and F for the follower, m_i is the quadrotor mass, $\omega_i = [\omega_{ix}, \omega_{iy}, \omega_{iz}]^T$ is the angular velocity in the body frame, J_i is the 3×3 diagonal matrix representing three inertial moments in the body frame, $G(\omega_i)$ represents the gyroscopic effect, τ_i is the torque vector applied on the quadrotor, the unit quaternion $[q_{i0}, q_{i1}, q_{i2}, q_{i3}]^T = [q_{i0}, \mathbf{q}_i^T]^T$ where $\mathbf{q}_i = [q_{i1}, q_{i2}, q_{i3}]^T$ is the vector part and q_{i0} is the scalar part of the quaternion, $\mathbf{v}_i = [v_{ix}, v_{iy}, v_{iz}]^T$ is the linear velocity, $\mathbf{p}_i = [x_i, y_i, z_i]^T$ is the position vector, the vector $\mathbf{e} = [0, 0, 1]^T$, and I is the 3×3 unit matrix. The rotation matrix R_i is related to the unit quaternion through the Rodrigues formula:

$$R_i = ({q_{i0}}^2 - \mathbf{q}_i^T \mathbf{q}_i)I + 2\mathbf{q}_i \mathbf{q}_i^T + 2q_{i0} S(\mathbf{q}_i)$$

and S is the skew-symmetric cross product matrix:

$$S(\mathbf{q}_i) = \begin{bmatrix} 0 & -q_{i3} & q_{i2} \\ q_{i3} & 0 & -q_{i1} \\ -q_{i2} & q_{i1} & 0 \end{bmatrix}$$

3.2. Leader-Follower Formation Control Problem

One leader and one follower are considered in the leader-follower formation control problem to be solved in this work. The leader control problem is formulated as a trajectory tracking problem, and the follower control problem is also formulated as a tracking problem, but with a different tracking target.

The follower will keep its yaw angle (q_{F0}, q_{F3}) as the same as the leader when it maintains the formation pattern. It will moves to a desired position $\mathbf{p}_{Fd}$, which is determined by a desired distance d, a desired incidence angle ρ, and a desired bearing angle σ. A new frame F' is defined by the translation of the leader frame L to the frame with the desired follower position $\mathbf{p}_{Fd}$ as the origin. As shown in figure 1, the desired incidence angle is measured between the desired distance d and $x - y$ plane in the new frame F', and the desired bearing angle is measured between x axis and the projection of d in $x - y$ plane in the new frame F'. The desired position $\mathbf{p}_{Fd}$ is

$$\mathbf{p}_{Fd} = \mathbf{p}_L - R_L^T d \begin{bmatrix} \cos\rho\cos\sigma \\ \cos\rho\sin\sigma \\ \sin\rho \end{bmatrix}$$

where $\mathbf{p}_L$ is the leader position.

Now, the formation control problem for the follower is to satisfy the following conditions:

$$\begin{cases} \lim_{t\to\infty}(\mathbf{p}_{Fd} - \mathbf{p}_F) = 0 \\ \lim_{t\to\infty}(q_{L0} - q_{F0}) = 0 \\ \lim_{t\to\infty}(q_{L3} - q_{F3}) = 0 \end{cases} \quad (40)$$

The leader just tracks a desired trajectory represented by $(\mathbf{p}_{Ld}, q_{L0d}, q_{L3d})$. So, the formation control problem for the leader is to satisfy the following conditions:

$$\begin{cases} \lim_{t\to\infty}(\mathbf{p}_{Ld} - \mathbf{p}_L) = 0 \\ \lim_{t\to\infty}(q_{L0d} - q_{L0}) = 0 \\ \lim_{t\to\infty}(q_{L3d} - q_{L3}) = 0 \end{cases} \quad (41)$$

In summary, the leader-follower formation control problem to be solved in this work is a distributed control scheme, i.e. the leader and the follower have their own individual controllers without the need for a centralised unit. Assume both the leader and the follower are able to obtain their own pose information and the follower is able to obtain the leader's pose information via wireless communication. The design goal of the controllers is to find the state feedback control law for the thrust and torque inputs for both the leader and the follower. The leader-follower formation control problem is solved if both the conditions (40) and (41) are satisfied.

The communication among the agents is assumed to be available. The position $\mathbf{p}_L$, quaternion components q_{L0} and q_{L3} of the leader L and its first and second derivatives $\dot{q}_{L0}$, $\ddot{q}_{L0}$, $\dot{q}_{L3}$ and $\ddot{q}_{L3}$ are assumed available and measurable. The linear velocity of the leader L and its derivative $\mathbf{v}_L$ and $\dot{\mathbf{v}}_L$ are assumed bounded and available for the follower.

4. Formation H_∞ Controllers

The controller design for the leader and the follower is based on H_∞ suboptimal control. The follower H_∞ controller is designed by following the introduction of an error state model, and the introduction of a H_∞ control theorem for general affine systems. Then the leader H_∞ controller is briefly presented later.

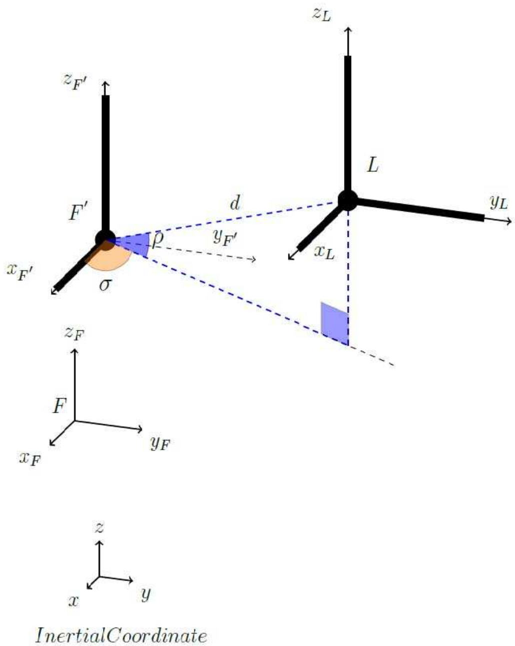

Figure 1. Body frames in formation.

4.1. Follower State Error Model

The control strategy for the follower is to track the desired position $\mathbf{p}_{Fd}$. The tracking errors for the follower according to the nonlinear dynamic system (39) can be written as:

$$\tilde{\mathbf{p}}_F = \mathbf{p}_{Fd} - \mathbf{p}_F$$
$$\tilde{\mathbf{v}}_F = \mathbf{v}_{Fd} - \mathbf{v}_F$$
$$\begin{bmatrix} \tilde{q}_{F0} \\ \tilde{\mathbf{q}}_F \end{bmatrix} = \begin{bmatrix} q_{F0d} - q_{F0} \\ \mathbf{q}_{Fd} - \mathbf{q}_F \end{bmatrix}$$
$$\tilde{\omega}_F = \omega_{Fd} - \omega_F$$

where $\mathbf{v}_{Fd} = \dot{\mathbf{p}}_{Fd}$ is the desired linear velocity, $[q_{F0d}, \mathbf{q}_{Fd}]^T = [q_{L0}, 0, 0, q_{L3}]^T$ is the desired quaternion, and $[\omega_{Fd}] = [0, 0, 0]^T$ is the desired angular velocity.

Then equation (39) can be rewritten in an error form as:

$$\begin{cases} \dot{\tilde{\mathbf{p}}}_F = \tilde{\mathbf{v}}_F \\ \dot{\tilde{\mathbf{v}}}_F = \dot{\mathbf{v}}_{Fd} + g\mathbf{e} - \frac{f_F}{m_F} R_F \mathbf{e} \\ \begin{bmatrix} \dot{\tilde{q}}_{F0} \\ \dot{\tilde{\mathbf{q}}}_F \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \tilde{\mathbf{q}}_F^T \tilde{\omega}_F \\ -\left(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F)\right) \tilde{\omega}_F \end{bmatrix} \\ \dot{\tilde{\omega}}_F = J_F^{-1} S(\tilde{\omega}_F) J_F \tilde{\omega}_F + J_F^{-1} G(\tilde{\omega}_F) - J_F^{-1} \tau_F \end{cases} \tag{42}$$

Consider the external disturbances $\mathbf{d}_F = [\mathbf{d}_{\mathbf{v}_F}^T, \mathbf{d}_{\omega_F}^T]^T$ applied to the nonlinear system (42), where $\mathbf{d}_{\mathbf{v}_F} = [d_{v_{Fx}}, d_{v_{Fy}}, d_{v_{Fz}}]^T$, $\mathbf{d}_{\omega_F} = [d_{\omega_{Fx}}, d_{\omega_{Fy}}, d_{\omega_{Fz}}]^T$ are the disturbance vectors applied to $\tilde{\mathbf{p}}_F$ and $\tilde{\omega}_F$, respectively. Those disturbances are used here to model the changes of mass and moment, and the wind disturbances.

Let

$$\mathbf{x}_F = \begin{bmatrix} \tilde{\mathbf{p}}_F \\ \tilde{q}_{F0} \\ \tilde{\mathbf{q}}_F \\ \tilde{\mathbf{v}}_F \\ \tilde{\omega}_F \end{bmatrix}$$

$$\mathbf{u}_F = \begin{bmatrix} \dot{\mathbf{v}}_{Fd} + g\mathbf{e} - \frac{f_F}{m_F} R_F \mathbf{e} \\ G(\tilde{\omega}_F) - \tau_F \end{bmatrix}$$

The nonlinear dynamic system (42) with the disturbance vector $\mathbf{d}_F$ can be written into an affine nonlinear form:

$$\dot{\mathbf{x}}_F = f(\mathbf{x}_F) + g(\mathbf{x}_F)\mathbf{u}_F + k(\mathbf{x}_F)\mathbf{d}_F \tag{43}$$

where

$$f(\mathbf{x}_F) = \begin{bmatrix} \tilde{\mathbf{v}}_F \\ \frac{1}{2}\tilde{\mathbf{q}}_F^T \tilde{\omega}_F \\ -\frac{1}{2}\left(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F)\right) \tilde{\omega}_F \\ 0_{3\times 1} \\ J_F^{-1} S(\tilde{\omega}_F) J_F \tilde{\omega}_F \end{bmatrix}$$

$$g(\mathbf{x}_F) = k(\mathbf{x}_F) = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{1\times 3} & 0_{1\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \\ I & 0_{3\times 3} \\ 0_{3\times 3} & J_F^{-1} \end{bmatrix}$$

4.2. H_∞ Suboptimal Control Approach

In this section, a brief overview on H_∞ suboptimal control approach is summarized for systems of the form:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} + k(\mathbf{x})\mathbf{d} \\ \mathbf{y} = h(\mathbf{x}) \end{cases} \tag{44}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a state vector, $\mathbf{u} \in \mathbb{R}^m$ is an input vector, $\mathbf{y} \in \mathbb{R}^p$ is an output vector, and $\mathbf{d} \in \mathbb{R}^q$ is a disturbance vector. Detailed information on H_∞ control approach can be found in [29].

We assume the existence of an equilibrium $\mathbf{x}_*$, i.e. $f(\mathbf{x}_*) = 0$, and we also assume $h(\mathbf{x}_*) = 0$. Given a smooth state feedback controller:

$$\begin{cases} \mathbf{u} = l(\mathbf{x}) \\ l(\mathbf{x}_*) = 0 \end{cases} \tag{45}$$

The H_∞ suboptimal control problem considers the L_2-gain from the disturbance $\mathbf{d}$ to the vector of $\mathbf{z} = [\mathbf{y}^T, \mathbf{u}^T]^T$. This problem is defined below.

Problem 1. *Let γ be a fixed nonnegative constant. The closed loop system consisting of the nonlinear system (44) and the state feedback controller* (45) *is said to have L_2-gain less than or equal to γ from $\mathbf{d}$ to $\mathbf{z}$ if*

$$\int_0^T \|\mathbf{z}(t)\|^2 dt \leq \gamma^2 \int_0^T \|\mathbf{d}(t)\|^2 dt + K(\mathbf{x}(0)) \tag{46}$$

for all $T \geq 0$ and all $\mathbf{d} \in L_2(0, T)$ with initial condition $\mathbf{x}(0)$, where $0 \leq K(\mathbf{x}) < \infty$ and $K(\mathbf{x}_*) = 0$.

For the nonlinear system (44) and $\gamma > 0$, define the Hamiltonian $H_\gamma(\mathbf{x}, V(\mathbf{x}))$ as below:

$$H_\gamma(\mathrm{x}, V(\mathrm{x})) = \frac{\partial V(\mathrm{x})}{\partial \mathrm{x}} f(\mathrm{x}) + \frac{1}{2}\frac{\partial V(\mathrm{x})}{\partial \mathrm{x}} \left[\frac{1}{\gamma^2} k(\mathrm{x})k^T(\mathrm{x}) - g(\mathrm{x})g^T(\mathrm{x})\right] \frac{\partial^T V(\mathrm{x})}{\partial \mathrm{x}} + \frac{1}{2} h^T(\mathrm{x})h(\mathrm{x}) \tag{47}$$

Theorem 1. *[29] If there exists a smooth solution $V \geq 0$ to the Hamilton-Jacobi inequality:*

$$H_\gamma(\mathbf{x}, V(\mathbf{x})) \leq 0;\ V(\mathbf{x}_*) = 0 \tag{48}$$

then the closed-loop system for the state feedback controller:

$$\mathbf{u} = -g^T(\mathbf{x})\frac{\partial^T V(\mathbf{x})}{\partial \mathbf{x}} \tag{49}$$

has L_2-gain less than or equal to γ, and $K(\mathbf{x}) = 2V(\mathbf{x})$.

The nonlinear system (44) is called zero-state observable if for any trajectory $\mathbf{x}(t)$ such that $\mathbf{y}(t) = 0, \mathbf{u}(t) = 0, \mathbf{d}(t) = 0$ implies $\mathbf{x}(t) = \mathbf{x}_*$.

Proposition 1. *[29] If the nonlinear system* (44) *is zero-state observable and there exists a proper solution $V \geq 0$ to the Hamilton-Jacobi inequality, then $V(\mathbf{x}) > 0$ for $\mathbf{x}(t) \neq \mathbf{x}_*$ and the closed loop system* (44), (49) *with $\mathbf{d} = 0$ is globally asymptotically stable.*

4.3. Follower H_∞ Controller

The H_∞ suboptimal control approach will be used to design the follower controller in this section. The following form of energy function V is suggested for the dynamic model (43):

$$V(\mathbf{x}_F) = \frac{1}{2}\begin{bmatrix} \tilde{\mathbf{p}}_F^T & \tilde{\mathbf{q}}_F^T & \tilde{\mathbf{v}}_F^T & \tilde{\omega}_F^T \end{bmatrix} \begin{bmatrix} C_{Fp}I & 0_{3\times3} & K_{Fp} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_F K_{Fq} \\ K_{Fp} & 0_{3\times3} & K_{Fv} & 0_{3\times3} \\ 0_{3\times3} & J_F K_{Fq} & 0_{3\times3} & J_F K_{F\omega} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}}_F \\ \tilde{\mathbf{q}}_F \\ \tilde{\mathbf{v}}_F \\ \tilde{\omega}_F \end{bmatrix} + 2C_{Fq}(1 - \tilde{q}_{F0}) \tag{50}$$

where diagonal matrices $K_{Fp} > 0, K_{Fq} > 0, K_{Fv} > 0, K_{F\omega} > 0$ are the proportional and derivative gains for translational and rotational parts. $C_{Fp} > 0, C_{Fq} > 0$ are constants. We have:

$$\frac{\partial V(\mathbf{x}_F)}{\partial \mathbf{x}_F} = [C_{Fp}\tilde{\mathbf{p}}_F + K_{Fp}\tilde{\mathbf{v}}_F \quad -2C_{Fq} \quad J_F K_{Fq}\tilde{\omega}_F \quad K_{Fp}\tilde{\mathbf{p}}_F + K_{Fv}\tilde{\mathbf{v}}_F \quad J_F K_{Fq}\tilde{\mathbf{q}}_F + J_F K_{F\omega}\tilde{\omega}_F]$$

Accordingly the controller is

$$\begin{aligned} \mathbf{u}_F &= -g^T(\mathbf{x}_F)\frac{\partial^T V(\mathbf{x}_F)}{\partial \mathbf{x}_F} \\ &= -\begin{bmatrix} K_{Fp}\tilde{\mathbf{p}}_F + K_{Fv}\tilde{\mathbf{v}}_F \\ K_{Fq}\tilde{\mathbf{q}}_F + K_{F\omega}\tilde{\omega}_F \end{bmatrix} \end{aligned} \tag{51}$$

The following weighting matrices are chosen with diagonal matrices $W_{F1} > 0$, $W_{F2} > 0$, $W_{F3} > 0$ and $W_{F4} > 0$.

$$h(\mathbf{x}_F) = [\sqrt{W_{F1}}\tilde{\mathbf{p}}_F^T \quad \sqrt{W_{F2}}\tilde{\mathbf{q}}_F^T \quad \sqrt{W_{F3}}\tilde{\mathbf{v}}_F^T \quad \sqrt{W_{F4}}\tilde{\omega}_F^T]^T$$

which satisfies $h(\mathbf{x}_{*F}) = 0$, where the equilibrium point $\mathbf{x}_{*F} = [0_{1\times 3}, 1, 0_{1\times 3}, 0_{1\times 3}, 0_{1\times 3}]^T$. And we know

$$V(\mathbf{x}_{*F}) = 0 \tag{52}$$

Now the team formation problem of the quadrotors under the disturbance $\mathbf{d}_F$ is defined below.

Problem 2. *Given the equilibrium point $\mathbf{x}_{*F}$, find the parameters K_{Fp}, K_{Fq}, K_{Fv}, $K_{F\omega}$, C_{Fp}, C_{Fq} in order to enable the closed-loop system* (43) *with the above controller $\mathbf{u}_F$* (51) *to have L_2-gain less than or equal to γ_F.*

Next we want to show our main result in the following theorem.

Theorem 2. *If the following conditions are satisfied, the closed-loop system* (43) *with the above controller $\mathbf{u}_F$* (51) *has L_2-gain less than or equal to γ_F. And the closed loop system* (43), (51) *with $\mathbf{d}_F = 0$ is asymptotically locally stable for the equilibrium point $\mathbf{x}_{*F}$.*

$$C_{Fp}C_{Fq} \geq 0$$

$$C_{Fp}K_{Fv} \geq K_{Fp}^2$$

$$C_{Fp}C_{Fq}K_{Fv}K_{F\omega} \geq C_{Fp}J_F K_{Fq}^2 K_{Fv} - J_F K_{Fq}^2 K_{Fp}^2 + C_{Fq}K_{Fp}^2 K_{F\omega}$$

$$C_{Fp} = K_{Fp}K_{Fv}\left(1 - \frac{1}{\gamma_F^2}\right)$$

$$C_{Fq} = K_{Fq}K_{F\omega}\left(\frac{1}{\gamma_F^2} - 1\right)$$

$$\|K_{Fp}\|^2 \geq \frac{\gamma_F^2\|W_{F1}\|}{\gamma_F^2 - 1} \tag{53}$$

$$\|K_{Fq}\|^2 \geq \frac{\gamma_F^2\|W_{F2}\|}{\gamma_F^2 - 1} \tag{54}$$

$$\|K_{Fv}\|^2 \geq \frac{\gamma_F^2(\|W_{F3}\| + 2\|K_{Fp}\|)}{\gamma_F^2 - 1} \tag{55}$$

$$\|K_{F\omega}\|^2 \geq \frac{\gamma_F^2(\|W_{F4}\| - \sqrt{3}\|J_F\|\|K_{Fq}\|)}{\gamma_F^2 - 1} \tag{56}$$

$$\|W_{F1}\| > 0; \ \|W_{F2}\| > 0; \ \|W_{F3}\| > 0; \ \|W_{F4}\| > 0$$

Proof. With the given conditions, we need to show (1) $V(\mathbf{x}_F) \geq 0$ and (2) the Hamiltonian $H_{\gamma_F}(\mathbf{x}_F, V(\mathbf{x}_F)) \leq 0$. Then the first part of the theorem can be proved by using Theorem 1.

(1) Since

$$\begin{aligned} 2(1-\tilde{q}_{F0}) &= (1-\tilde{q}_{F0})^2 + \tilde{\mathbf{q}}_F^T\tilde{\mathbf{q}}_F \\ &\geq \tilde{\mathbf{q}}_F^T\tilde{\mathbf{q}}_F \end{aligned}$$

then

$$V(\mathbf{x}_F) \geq \frac{1}{2}\begin{bmatrix} \tilde{\mathbf{p}}_F^T & \tilde{\mathbf{q}}_F^T & \tilde{\mathbf{v}}_F^T & \tilde{\omega}_F^T \end{bmatrix} \begin{bmatrix} C_{Fp}I & 0_{3\times3} & K_{Fp} & 0_{3\times3} \\ 0_{3\times3} & C_{Fq}I & 0_{3\times3} & J_F K_{Fq} \\ K_{Fp} & 0_{3\times3} & K_{Fv} & 0_{3\times3} \\ 0_{3\times3} & J_F K_{Fq} & 0_{3\times3} & J_F K_{F\omega} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}}_F \\ \tilde{\mathbf{q}}_F \\ \tilde{\mathbf{v}}_F \\ \tilde{\omega}_F \end{bmatrix}$$

Thus the condition for $V(\mathbf{x}_F) \geq 0$ are

$$\begin{aligned} C_{Fp}C_{Fq} &\geq 0 \\ C_{Fp}K_{Fv} &\geq K_{Fp}^2 \\ C_{Fp}C_{Fq}K_{Fv}K_{F\omega} &\geq C_{Fp}J_F K_{Fq}^2 K_{Fv} - J_F K_{Fq}^2 K_{Fp}^2 + C_{Fq}K_{Fp}^2 K_{F\omega} \end{aligned}$$

(2)

$$\begin{aligned} H_{\gamma_F}(\mathbf{x}_F, V(\mathbf{x}_F)) &= \tilde{\mathbf{p}}_F^T C_{Fp}\tilde{\mathbf{v}}_F - \tilde{\mathbf{q}}_F^T C_{Fq}\tilde{\omega}_F + \tilde{\mathbf{v}}_F^T K_{Fp}\tilde{\mathbf{v}}_F - \frac{1}{2}\tilde{\omega}_F^T J_F K_{Fq}(\tilde{q}_{F0}I \\ &+ S(\tilde{\mathbf{q}}_F))\tilde{\omega}_F + \tilde{\mathbf{q}}_F^T K_{Fq} S(\tilde{\omega}_F) J_F \tilde{\omega}_F + \tilde{\omega}_F^T K_{F\omega} S(\tilde{\omega}_F) J_F \tilde{\omega}_F \\ &+ \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{Fp}\tilde{\mathbf{p}}_F + K_{Fv}\tilde{\mathbf{v}}_F\|^2 + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{Fq}\tilde{\mathbf{q}}_F \\ &+ K_{F\omega}\tilde{\omega}_F\|^2 + \frac{1}{2}\|W_{F1}\|\|\tilde{\mathbf{p}}_F\|^2 + \frac{1}{2}\|W_{F2}\|\|\tilde{\mathbf{q}}_F\|^2 \\ &+ \frac{1}{2}\|W_{F3}\|\|\tilde{\mathbf{v}}_F\|^2 + \frac{1}{2}\|W_{F4}\|\|\tilde{\omega}_F\|^2 \end{aligned}$$

By choosing

$$\begin{aligned} C_{Fp} &= K_{Fp}K_{Fv}\left(1 - \frac{1}{\gamma_F^2}\right) \\ C_{Fq} &= K_{Fq}K_{F\omega}\left(\frac{1}{\gamma_F^2} - 1\right) \end{aligned}$$

then

$$H_{\gamma_F}(\mathbf{x}_F, V(\mathbf{x}_F)) = \tilde{\mathbf{v}}_F^T K_{Fp}\tilde{\mathbf{v}}_F + \tilde{\mathbf{q}}_F^T K_{Fq} S(\tilde{\omega}_F) J_F \tilde{\omega}_F - \frac{1}{2}\tilde{\omega}_F^T J_F K_{Fq}(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F))\tilde{\omega}_F + \tilde{\omega}_F^T K_{F\omega} S(\tilde{\omega}_F) J_F \tilde{\omega}_F + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right) (\|K_{Fp}\|^2\|\tilde{\mathbf{p}}_F\|^2 + \|K_{Fv}\|^2\|\tilde{\mathbf{v}}_F\|^2) + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right) (\|K_{Fq}\|^2\|\tilde{\mathbf{q}}_F\|^2 + \|K_{F\omega}\|^2\|\tilde{\omega}_F\|^2) + \frac{1}{2}\|W_{F1}\|\|\tilde{\mathbf{p}}_F\|^2 + \frac{1}{2}\|W_{F2}\|\|\tilde{\mathbf{q}}_F\|^2 + \frac{1}{2}\|W_{F3}\|\|\tilde{\mathbf{v}}_F\|^2 + \frac{1}{2}\|W_{F4}\|\|\tilde{\omega}_F\|^2$$

By using $\|S(\tilde{\omega}_F))\| = \|\tilde{\omega}_F\|$, $|\tilde{\mathbf{v}_F}^T K_{Fp}\tilde{\mathbf{v}}_F| \leq \|K_{Fp}\|\|\tilde{\mathbf{v}}_F\|^2$, $\|(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F))\|$
$\leq \sqrt{3}$, $|\tilde{\omega}_F^T J_F K_{Fq}(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F))\tilde{\omega}_F| \leq \|K_{Fq}\|\|J_F\|\|\tilde{\omega}_F\|^2\|(\tilde{q}_{F0} I + S(\tilde{\mathbf{q}}_F))\|$,
$\tilde{\mathbf{q}}_F^T K_{Fq} S(\tilde{\omega}_F) J_F \tilde{\omega}_F = 0$ and $\tilde{\omega}_F^T K_{F\omega} S(\tilde{\omega}_F) J_F \tilde{\omega}_F = 0$ we have

$$H_{\gamma_F}(\mathbf{x}_F, V(\mathbf{x}_F)) \leq \frac{-\sqrt{3}}{2}\|K_{Fq}\|\|J_F\|\|\tilde{\omega}_F\|^2 + \|K_{Fp}\|\|\tilde{\mathbf{v}}_F\|^2 + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right) (\|K_{Fp}\|^2\|\tilde{\mathbf{p}}_F\|^2 + \|K_{Fv}\|^2\|\tilde{\mathbf{v}}_F\|^2) + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)(\|K_{Fq}\|^2\|\tilde{\mathbf{q}}_F\|^2 + \|K_{F\omega}\|^2\|\tilde{\omega}_F\|^2) + \frac{1}{2}\|W_{F1}\|\|\tilde{\mathbf{p}}_F\|^2 + \frac{1}{2}\|W_{F2}\|\|\tilde{\mathbf{q}}_F\|^2 + \frac{1}{2}\|W_{F3}\|\|\tilde{\mathbf{v}}_F\|^2 + \frac{1}{2}\|W_{F4}\|\|\tilde{\omega}_F\|^2$$

Thus, the conditions for $H_{\gamma_F}(\mathbf{x}_F, V(\mathbf{x}_F)) \leq 0$ are

$$\frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{Fp}\|^2 + \frac{1}{2}\|W_{F1}\| \leq 0$$
$$\frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{Fq}\|^2 + \frac{1}{2}\|W_{F2}\| \leq 0$$
$$\|K_{Fp}\| + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{Fv}\|^2 + \frac{1}{2}\|W_{F3}\| \leq 0$$
$$\frac{-\sqrt{3}}{2}\|J_F\|\|K_{Fq}\| + \frac{1}{2}\left(\frac{1}{\gamma_F^2} - 1\right)\|K_{F\omega}\|^2 + \frac{1}{2}\|W_{F4}\| \leq 0$$

i.e.

$$\|K_{Fp}\|^2 \geq \frac{\gamma_F^2 \|W_{F1}\|}{\gamma_F^2 - 1}$$
$$\|K_{Fq}\|^2 \geq \frac{\gamma_F^2 \|W_{F2}\|}{\gamma_F^2 - 1}$$
$$\|K_{Fv}\|^2 \geq \frac{\gamma_F^2 (\|W_{F3}\| + 2\|K_{Fp}\|)}{\gamma_F^2 - 1}$$
$$\|K_{F\omega}\|^2 \geq \frac{\gamma_F^2 (\|W_{F4}\| - \sqrt{3}\|J_F\|\|K_{Fp}\|)}{\gamma_F^2 - 1}$$

It is trivial to show that the nonlinear system (43) is zero-state observable for the equilibrium point $\mathbf{x}_{*F}$. Further due to the fact that $V(\mathbf{x}_F) \geq 0$ and it is a proper function (i.e. for each $\beta > 0$ the set $\{x_F : 0 \leq V(x_F) \leq \beta\}$ is compact), the closed-loop system (43), (51) with $\mathbf{d}_F = 0$ is asymptotically locally stable for the equilibrium point $\mathbf{x}_{*F}$ according to Proposition 1. This proves the second part of the theorem. □

Remark 1. *It should be noted that the proof of Theorem 2,* $\lim_{t\to\infty} \tilde{\mathbf{p}}_F = 0$*,* $\lim_{t\to\infty} \tilde{\mathbf{q}}_F = 0$*,* $\lim_{t\to\infty} \tilde{\mathbf{v}}_F = 0$ *and* $\lim_{t\to\infty} \tilde{\omega}_F = 0$*, meets the conditions of* (40).

Finally from $\mathbf{u}_F$, we can have

$$\mathbf{u}_F = \begin{bmatrix} \dot{\mathbf{v}}_{Fd} + g\mathbf{e} - \frac{f_F}{m_F} R_F \mathbf{e} \\ G(\tilde{\omega}_F) - \tau_F \end{bmatrix} = -\begin{bmatrix} K_{Fp}\tilde{\mathbf{p}}_F + K_{Fv}\tilde{\mathbf{v}}_F \\ K_{Fq}\tilde{\mathbf{q}}_F + K_{F\omega}\tilde{\omega}_F \end{bmatrix}$$

Then the total force and the torque vector applied to the follower, f_F and τ_F are obtained,

$$\begin{aligned} f_F =& (k_{Fz}\tilde{z}_F + k_{Fv_z}\tilde{v}_{Fz} + \dot{v}_{Lz} - d(R_{31}\cos\rho\cos\sigma + R_{32}\cos\rho\sin\sigma + R_{33}\sin\rho) \\ &+ g)\frac{m_F}{q_{F0}^2 - q_{F1}^2 - q_{F2}^2 + q_{F3}^2} \\ \tau_F =& K_{Fq}\tilde{\mathbf{q}}_F + K_{F\omega}\tilde{\omega}_F + G(\tilde{\omega}_F) \end{aligned}$$

where

$$\ddot{R}_L^T = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

4.4. Leader H_∞ Controller

The control strategy for the leader is to track a desired trajectory $(\mathbf{p}_{Ld}, q_{L0d}, q_{L3d})$. The tracking errors for the leader according to the nonlinear dynamic system (39) can be written as:

$$\begin{gathered} \tilde{\mathbf{p}}_L = \mathbf{p}_{Ld} - \mathbf{p}_L \\ \tilde{\mathbf{v}}_L = \mathbf{v}_{Ld} - \mathbf{v}_L \\ \begin{bmatrix} \tilde{q}_{L0} \\ \tilde{\mathbf{q}}_L \end{bmatrix} = \begin{bmatrix} q_{L0d} - q_{L0} \\ \mathbf{q}_{Ld} - \mathbf{q}_L \end{bmatrix} \\ \tilde{\omega}_L = \omega_{Ld} - \omega_L \end{gathered}$$

where $q_{L0d}, \mathbf{q}_{Ld}, \mathbf{v}_{Ld}, \omega_{Ld}$ are assumed to be constant for the desired tracking trajectory. Then equation (39) can be rewritten in an error form as:

$$\begin{cases} \dot{\tilde{\mathbf{p}}}_L = \tilde{\mathbf{v}}_L \\ \dot{\tilde{\mathbf{v}}}_L = g\mathbf{e} - \frac{f_L}{m_L} R_L \mathbf{e} \\ \begin{bmatrix} \dot{\tilde{q}}_{L0} \\ \dot{\tilde{\mathbf{q}}}_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \tilde{\mathbf{q}}_L^T \tilde{\omega}_L \\ -\left(\tilde{q}_{L0} I + S(\tilde{\mathbf{q}}_L)\right) \tilde{\omega}_L \end{bmatrix} \\ \dot{\tilde{\omega}}_L = J_L^{-1} S(\tilde{\omega}_L) J_L \tilde{\omega}_L + J_L^{-1} G(\tilde{\omega}_L) - J_L^{-1} \tau_L \end{cases} \quad (57)$$

Let

$$\mathbf{x}_L = \begin{bmatrix} \tilde{\mathbf{p}}_L \\ \tilde{q}_{L0} \\ \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{v}}_L \\ \tilde{\omega}_L \end{bmatrix}$$

$$\mathbf{u}_L = \begin{bmatrix} g\mathbf{e} - \frac{f_L}{m_L} R_L \mathbf{e} \\ G(\tilde{\omega}_L) - \tau_L \end{bmatrix}$$

The nonlinear dynamic system (57) with the disturbance vector $\mathbf{d}_L$ can be written into an affine nonlinear form:

$$\dot{\mathbf{x}}_L = f(\mathbf{x}_L) + g(\mathbf{x}_L)\mathbf{u}_L + k(\mathbf{x}_L)\mathbf{d}_L \tag{58}$$

where

$$f(\mathbf{x}_L) = \begin{bmatrix} \tilde{\mathbf{v}}_L \\ \frac{1}{2}\tilde{\mathbf{q}}_L^T\tilde{\omega}_L \\ -\frac{1}{2}\left(\tilde{q}_{L0}I + S(\tilde{\mathbf{q}}_L)\right)\tilde{\omega}_L \\ 0_{3\times 1} \\ J_L^{-1}S(\tilde{\omega}_L)J_L\tilde{\omega}_L \end{bmatrix}$$

$$g(\mathbf{x}_L) = k(\mathbf{x}_L) = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{1\times 3} & 0_{1\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \\ I & 0_{3\times 3} \\ 0_{3\times 3} & J_L^{-1} \end{bmatrix}$$

The H_∞ suboptimal control approach is used to design the leader controller. By defining an energy function, the leader controller is obtained as below by following a similar procedure for stability analysis.

$$\begin{cases} f_L = (k_{Lz}\tilde{z}_L + k_{Lv_z}\tilde{v}_{Lz} + g)\,\frac{m_L}{q_{L0}^2 - q_{L1}^2 - q_{L2}^2 + q_{L3}^2} \\ \tau_L = K_{Lq}\tilde{\mathbf{q}}_L + K_{L\omega}\tilde{\omega}_L + G(\tilde{\omega}_L) \end{cases}$$

where $\tilde{\mathbf{p}}_L = [\tilde{x}_L, \tilde{y}_L,\ \tilde{z}_L]^T$ is the position tracking error vector and $\tilde{\mathbf{v}}_L = [\tilde{v}_{Lx}, \tilde{v}_{Ly}, \tilde{v}_{Lz}]^T$ is the linear velocity error vector. The diagonal matrices $K_{Lp} = diag(k_{Lx}, k_{Ly},\ k_{Lz})$, $K_{Lv} = diag(k_{Lv_x}, k_{Lv_y}, k_{Lv_z})$, $K_{Lq} = diag[k_{Lq_1}, k_{Lq_2}, k_{Lq_3}]$, $K_{L\omega} = diag\ (k_{L\omega_x}, k_{L\omega_y}, k_{L\omega_z})$ are selected to satisfy the stability conditions, which have been presented in [37] [38].

5. Integral Backstepping Follower Formation Control

Integral backstepping control is one of popular control approaches for both individual and multiple quadrotors. In this section, it will be applied for the leader-follower formation problem. The leader and the follower desired quaternions are assumed to be $q_{L1d} = q_{L2d} = 0$ and $q_{F1d} = q_{L1}$ and $q_{F2d} = q_{L2}$. An IBS controller for the follower is developed first. The IBS controller for the leader is from our previous work [39] and its main result is just presented in this section. They will be used in the simulation later for evaluating the robustness of H_∞ controllers.

5.1. Follower Integral Backstepping Controller

The IBS controller for the follower is to track the leader and maintain a desired distance between them with desired incidence and bearing angles. We start with the follower's translational part, which can be rewritten from the dynamic model (39) as:

$$\ddot{\mathbf{p}}_F = f(\mathbf{p}_F) + g(\mathbf{p}_F) f_F \tag{59}$$

where f_F is the total thrust control input and

$$f(\mathbf{p}_F) = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$$

$$g(\mathbf{p}_F) = \begin{bmatrix} u_{Fx}/m_F \\ u_{Fy}/m_F \\ ({q_{F0}}^2 - {q_{F1}}^2 - {q_{F2}}^2 + {q_{F3}}^2)/m_F \end{bmatrix}$$

with

$$\begin{cases} u_{Fx} = 2\left(q_{F1}q_{F3} + q_{F0}q_{F2}\right) \\ u_{Fy} = 2\left(q_{F2}q_{F3} - q_{F0}q_{F1}\right) \end{cases}$$

Then the position tracking error between the leader and the follower can be calculated as

$$\tilde{\mathbf{p}}_F = \mathbf{p}_{Fd} - \mathbf{p}_F = \mathbf{p}_L - R_L^T d \begin{bmatrix} \cos\rho\cos\sigma \\ \cos\rho\sin\sigma \\ \sin\rho \end{bmatrix} - \mathbf{p}_F \tag{60}$$

and its derivative

$$\dot{\tilde{\mathbf{p}}}_F = \dot{\mathbf{p}}_{Fd} - \dot{\mathbf{p}}_F = \dot{\mathbf{p}}_{Fd} - \mathbf{v}_F \tag{61}$$

where $\mathbf{v}_F$ is a virtual control, and its desirable value can be described as:

$$\mathbf{v}_F^d = \dot{\mathbf{p}}_{Fd} + b_F \tilde{\mathbf{p}}_F + k_F \bar{\mathbf{p}}_F \tag{62}$$

where b_F and k_F are two positive matrices, $\bar{\mathbf{p}}_F = \int \tilde{\mathbf{p}}_F dt$ is the integral of the follower position error and added to minimize the steady-state error.

Now, consider the linear velocity error between the leader and the follower as:

$$\tilde{\mathbf{v}}_F = \mathbf{v}_F^d - \dot{\mathbf{p}}_F \tag{63}$$

By substituting (62) into (63) we obtain:

$$\tilde{\mathbf{v}}_F = \dot{\mathbf{p}}_{Fd} + b_F\tilde{\mathbf{p}}_F + k_F\bar{\mathbf{p}}_F - \dot{\mathbf{p}}_F \tag{64}$$

and its time derivative

$$\dot{\tilde{\mathbf{v}}}_F = \ddot{\mathbf{p}}_{Fd} + b_F\dot{\tilde{\mathbf{p}}}_F + k_F\tilde{\mathbf{p}}_F - \ddot{\mathbf{p}}_F \tag{65}$$

Then from (62) and (63) we can rewrite (61) in terms of linear velocity error as:

$$\dot{\tilde{\mathbf{p}}}_F = \tilde{\mathbf{v}}_F - b_F\tilde{\mathbf{p}}_F - k_F\bar{\mathbf{p}}_F \tag{66}$$

By substituting (59) and (66) into (65), the time derivative of linear velocity error can be rewritten as:

$$\dot{\tilde{\mathbf{v}}}_F = \ddot{\mathbf{p}}_{Fd} + b_F\tilde{\mathbf{v}}_F - b_F^2\dot{\tilde{\mathbf{p}}}_F - b_Fk_F\bar{\mathbf{p}}_F + k_F\tilde{\mathbf{p}}_F - f(\mathbf{p}_F) - g(\mathbf{p}_F)f_F \tag{67}$$

The desirable time derivative of the linear velocity error is supposed to be:

$$\dot{\tilde{\mathbf{v}}}_F = -c_F\tilde{\mathbf{v}}_F - \tilde{\mathbf{p}}_F \tag{68}$$

where c_F is a positive diagonal matrix. Now, the total thrust f_F, the longitudinal u_{Fx} and lateral u_{Fy} motion control can be found by subtracting (67) from (68) as follows:

$$\begin{aligned} f_F =&(g + \dot{v}_{Lz} + (1 - b_{Fz}^2 + k_{Fz})\tilde{z}_F + (b_{Fz} + c_{Fz})\tilde{v}_{Fz} - b_{Fz}k_{Fz}\bar{z}_F - d(R_{31}\cos\rho\cos\sigma \\ &+ R_{32}\cos\rho\sin\sigma + R_{33}\sin\rho))\frac{m_F}{({q_{F0}}^2 - {q_{F1}}^2 - {q_{F2}}^2 + {q_{F3}}^2)} \end{aligned} \tag{69}$$

$$\begin{aligned} u_{Fx} =&(\dot{v}_{Lx} + (1 - b_{Fx}^2 + k_{Fx})\tilde{x}_F + (b_{Fx} + c_{Fx})\tilde{v}_{Fx} - b_{Fx}k_{Fx}\bar{x}_F - d(R_{11}\cos\rho\cos\sigma \\ &+ R_{12}\cos\rho\sin\sigma + R_{13}\sin\rho))\frac{m_F}{f_F} \end{aligned} \tag{70}$$

$$\begin{aligned} u_{Fy} =&(\dot{v}_{Ly} + (1 - b_{Fy}^2 + k_{Fy})\tilde{y}_F + (b_{Fy} + c_{Fy})\tilde{v}_{Fy} - b_{Fy}k_{Fy}\bar{y}_F - d(R_{21}\cos\rho\cos\sigma \\ &+ R_{22}\cos\rho\sin\sigma + R_{23}\sin\rho))\frac{m_F}{f_F} \end{aligned} \tag{71}$$

For the attitude stability, a nonlinear controller from [39] is used.

$$\tau_F = K_{Fq}\tilde{\mathbf{q}}_F + K_{F\omega}\tilde{\omega}_F + G(\tilde{\omega}_F)$$

The attitude stability for the follower was demonstrated in [39] . Next, we show the stability of follower's translational part.

5.2. Stability Analysis for Follower

The following candidate Lyapunov function is chosen for the stability analysis of follower's translational part with the integral backstepping controller:

$$V = \frac{1}{2}(\tilde{\mathbf{p}}_F^T\tilde{\mathbf{p}}_F + \tilde{\mathbf{v}}_F^T\tilde{\mathbf{v}}_F + k_F\bar{\mathbf{p}}_F^T\bar{\mathbf{p}}_F) \tag{72}$$

and its time derivative

$$\dot{V} = \tilde{\mathbf{p}}_F^T\dot{\tilde{\mathbf{p}}}_F + \tilde{\mathbf{v}}_F^T\dot{\tilde{\mathbf{v}}}_F + k_F\bar{\mathbf{p}}_F^T\dot{\bar{\mathbf{p}}}_F \tag{73}$$

By substituting $\dot{\bar{\mathbf{p}}}_F = \tilde{\mathbf{p}}_F$, and equations (66) and (68) into (73), equation (73) becomes:

$$\dot{V} = -b_F\tilde{\mathbf{p}}_F^T\tilde{\mathbf{p}}_F - c_F\tilde{\mathbf{v}}_F^T\tilde{\mathbf{v}}_F \leq 0 \tag{74}$$

Finally, (74) is less than zero provided b_F and c_F are positive diagonal matrices, i.e. $\dot{V} < 0, \forall(\tilde{\mathbf{p}}_F, \tilde{\mathbf{v}}_F) \neq 0$ and $\dot{V}(0) = 0$. It can be concluded from the positive definition of V and applying LaSalle theorem that a global asymptotic stability is guaranteed. This leads to the conclude that $\lim_{t\to\infty}\tilde{\mathbf{p}}_F = 0$ and $\lim_{t\to\infty}\tilde{\mathbf{v}}_F = 0$, which meets the position condition of (40).

5.3. Leader Integral Backstepping Controller

The leader is to track a desired trajectory $\mathbf{p}_{Ld}$. Its integral backstepping controller was developed in [39]. The result is that the total force and horizontal position control laws f_L, u_{Lx} and u_{Ly} can be written as

$$f_L = \frac{(\ddot{z}_{Ld} + g + (1 - {b_{Lz}}^2 + k_{Lz})\tilde{z}_L + (b_{Lz} + c_{Lz})\tilde{v}_{Lz} - b_{Lz}k_{Lz}\bar{z}_L)\, m_L}{q_{L0}^2 - q_{L1}^2 - q_{L2}^2 + q_{L3}^2} \tag{75}$$

$$u_{Lx} = (\ddot{x}_{Ld} + (1 - {b_{Lx}}^2 + k_{Lx})\tilde{x}_L + (b_{Lx} + c_{Lx})\tilde{v}_{Lx} - b_{Lx}k_{Lx}\bar{x}_L)\frac{m_L}{f_L} \tag{76}$$

$$u_{Ly} = (\ddot{y}_{Ld} + (1 - {b_{Ly}}^2 + k_{Ly})\tilde{y}_L + (b_{Ly} + c_{Ly})\tilde{v}_{Ly} - b_{Ly}k_{Ly}\bar{y}_L)\frac{m_L}{f_L} \tag{77}$$

where the linear velocity tracking errors $\tilde{v}_{Lx}, \tilde{v}_{Ly}$ and $\tilde{v}_{Lz}$ are defined as:

$$\begin{cases} \tilde{v}_{Lx} = b_{Lx}\tilde{x}_L + \dot{x}_{Ld} + k_{Lx}\bar{x}_L - \dot{x}_L \\ \tilde{v}_{Ly} = b_{Ly}\tilde{y}_L + \dot{y}_{Ld} + k_{Ly}\bar{y}_L - \dot{y}_L \\ \tilde{v}_{Lz} = b_{Lz}\tilde{z}_L + \dot{z}_{Ld} + k_{Lz}\bar{z}_L - \dot{z}_L \end{cases}$$

And the torque vector applied to the leader quadrotor $\tau_L \in \mathbb{R}^3$ is designed as

$$\tau_L = K_{Lq}\tilde{\mathbf{q}}_L + K_{L\omega}\tilde{\omega}_L + G(\tilde{\omega}_L)$$

6. Formation iLQR Controllers

The controller design for the leader and the follower is based on iLQR optimal control algorithm. The follower iLQR controller is designed by following the introduction of an error state model. Then the leader iLQR controller is briefly presented later.

6.1. iLQR Optimal Control Approach

iLQR is one of the optimal control strategies that is formulated to obtain the control signals that minimises a performance criterion to satisfy the physical model constraints. The iLQR strategy is utilised based on LQR technique to design the full state quadrotor's controller. Linearising the nonlinear dynamic model (39), we obtain

$$\mathbf{x}_{ik+1} = f(\mathbf{x}_{ik}, \mathbf{u}_{ik}) \tag{78}$$

with a quadratic cost function of the form

$$\begin{aligned} \mathbf{J}_i =& \frac{1}{2}(\mathbf{x}_{iN} - \mathbf{x}_i^*)^T \mathbf{Q}_{iN}(\mathbf{x}_{iN} - \mathbf{x_i}^*) + \frac{1}{2}\sum_{k=0}^{N-1}(\mathbf{x}_{ik}^T \mathbf{Q}_i \mathbf{x}_{ik} \\ &+ \mathbf{u}_{ik}^T \mathbf{R}_i \mathbf{u}_{ik}) \end{aligned} \tag{79}$$

where $\mathbf{x}_i = [x_i, \dot{x}_i, y_i, \dot{y}_i, z_i, \dot{z}_i, q_{i0}, q_{i1}, q_{i2}, q_{i3}, \omega_{ix}, \omega_{iy}, \omega_{iz}]^T$ and the quadrotor is controlled by its altitude force f_i and attitude torque vector τ_i. The control vector can be defined as $\mathbf{u}_i = [f_i, \tau_{iq_1}, \tau_{iq_2}, \tau_{iq_3}]^T$. We will not use the notation i in the coming equations for simplicity.

The proposed strategy starts with initial control signals $k = 0$, and the linearised nonlinear system around the control signal $\mathbf{u}_k$ and state $\mathbf{x}_k$, then solves the LQR problem. Then these steps are repeated (iterated) until a good performance is achieved. Let the deviations from $\mathbf{u}_k$ and $\mathbf{x}_k$ be $\delta\mathbf{u}_k$ and $\delta\mathbf{x}_k$ respectively. The linearisation model is

$$\delta\mathbf{x}_{k+1} = A_k\delta\mathbf{x}_k + B_k\delta\mathbf{u}_k \tag{80}$$

Where the matrices $A_k = \mathbf{J}_\mathbf{x} f(\mathbf{x}_k, \mathbf{u}_k)$ and $B_k = \mathbf{J}_\mathbf{u} f(\mathbf{x}_k, \mathbf{u}_k)$ are denoted by the Jacobians. These are evaluated along $\mathbf{x}_k$ and $\mathbf{u}_k$ respectively. Based on the linear model (80), the cost function (79) can be written as:

$$\begin{aligned}\mathbf{J} =& \frac{1}{2}(\mathbf{x}_N + \delta\mathbf{x}_N - \mathbf{x}^*)^T\mathbf{Q}_N(\mathbf{x}_N + \delta\mathbf{x}_N - \mathbf{x}^*) \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}((\mathbf{x}_k + \delta\mathbf{x})^T\mathbf{Q}(\mathbf{x}_k + \delta\mathbf{x}) \\ &+ (\mathbf{u}_k + \delta\mathbf{u})^T\mathbf{R}(\mathbf{u}_k + \delta\mathbf{u})).\end{aligned} \tag{81}$$

Adding a constraint to the cost function (81), the value function is

$$\begin{aligned}\mathbf{V} =& \frac{1}{2}(\mathbf{x}_N + \delta\mathbf{x}_N - \mathbf{x}^*)^T\mathbf{Q}_N(\mathbf{x}_N + \delta\mathbf{x}_N - \mathbf{x}^*) \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}((\mathbf{x}_k + \delta\mathbf{x})^T\mathbf{Q}(\mathbf{x}_k + \delta\mathbf{x}) + (\mathbf{u}_k + \delta\mathbf{u})^T \\ &\mathbf{R}(\mathbf{u}_k + \delta\mathbf{u}) + \delta\lambda_{k+1}^T(A_k\delta\mathbf{x}_k + B_k\delta\mathbf{u}_k - \delta\mathbf{x}_{k+1})).\end{aligned} \tag{82}$$

The following Hamiltonian function is a first step to proceed towards the optimal control

$$\begin{aligned}\mathbf{H}_k =& (\mathbf{x}_k + \delta\mathbf{x}_k)^T\mathbf{Q}(\mathbf{x}_k + \delta\mathbf{x}_k) + (\mathbf{u}_k + \delta\mathbf{u}_k)^T\mathbf{R}(\mathbf{u}_k + \delta\mathbf{u}_k) \\ &+ \delta\lambda_{k+1}^T(A_k\delta\mathbf{x}_k + B_k\delta\mathbf{u}_k)\end{aligned} \tag{83}$$

and its derivatives with respect to $\delta\mathbf{x}_k$, $\delta\mathbf{u}_k$ and $\delta\mathbf{x}_N$ are

$$\begin{cases} \dfrac{\partial \mathbf{H}_k}{\partial(\delta\mathbf{x}_k)} = \delta\lambda_k \\ \dfrac{\partial \mathbf{H}_k}{\partial(\delta\mathbf{u}_k)} = 0 \\ \dfrac{\partial \mathbf{H}_k}{\partial(\delta\mathbf{x}_N)} = \delta\lambda_N, \end{cases}$$

which leads to the following conditions:

$$\delta\lambda_k = A_k^T \delta\lambda_{k+1} + \mathbf{Q}(\delta\mathbf{x}_k + \mathbf{x}_k) \tag{84}$$

$$0 = \mathbf{R}(\mathbf{u}_k + \delta\mathbf{u}_k) + B_k^T \delta\lambda_{k+1} \tag{85}$$

$$\delta\lambda_N = \mathbf{Q}_f(\mathbf{x}_N + \delta\mathbf{x}_N - \mathbf{x}^*). \tag{86}$$

Based on the boundary condition (86), $\delta\lambda_k$ is assumed to be

$$\delta\lambda_k = S_k \delta\mathbf{x}_k + \nu_k \tag{87}$$

for some unknown sequences S_k and ν_k. The boundary conditions for S_k and ν_k are

$$\begin{cases} S_N = \mathbf{Q}_N \\ \nu_N = \mathbf{Q}_N(\mathbf{x}_N - \mathbf{x}_*) \end{cases} \tag{88}$$

and from the boundary condition (85), $\delta\mathbf{u}_k$ is obtained as:

$$\delta\mathbf{u}_k = -\mathbf{R}^{-1} B_k^T \delta\lambda_{k+1} - \mathbf{u}_k. \tag{89}$$

By solving equations (80), (85) and (87), we obtain

$$\delta\mathbf{u}_k = -K\delta\mathbf{x}_k - K_\nu \nu_{k+1} - K_\mathbf{u}\mathbf{u}_k \tag{90}$$

where

$$K = (B_k^T S_{k+1} B_k + \mathbf{R})^{-1} B_k^T S_{k+1} A_k \tag{91}$$

$$K_\nu = (B_k^T S_{k+1} B_k + \mathbf{R})^{-1} B_k^T \tag{92}$$

$$K_\mathbf{u} = (B_k^T S_{k+1} B_k + \mathbf{R})^{-1} \mathbf{R}. \tag{93}$$

Backward recursion equations are used to solve the entire sequences S_k and ν_k as:

$$S_k = A_k^T S_{k+1}(A_k - B_k K) + \mathbf{Q} \tag{94}$$

$$\nu_k = (A_k - B_k K)^T \nu_{k+1} - K^T \mathbf{R}\mathbf{u}_k + \mathbf{Q}\mathbf{x}_k \tag{95}$$

where the gains K and $K_{\mathbf{u}}$ are built on the Riccati equation while the gain K_ν is reliant on auxiliary sequence (95).

The entire sequences of S_k and ν_k can be solved by the backward recursion (94) and (95) respectively, with the final state weighting matrix boundary condition S_N stated in the cost function (81). The control law 90 includes three terms. The gains of the first and the third terms depend on the solution of Riccati equation, while the second term gain depends on the auxiliary sequence ν_k. In the first term, $\delta\mathbf{x}_k$ represents the error between the actual quadrotor state and the desired state, and in the third term, $\mathbf{u}_k$ represents the nominal control action. Once the modified LQR problem is solved, an improved nominal control sequence can be obtained: $\mathbf{u}_k^* = \mathbf{u}_k + \delta\mathbf{u}_k$, where $\mathbf{u}_k$ is the nominal control and $\mathbf{u}_k^*$ is the improved control. Then the total control laws are concluded as follows:

$$\begin{cases} \delta\mathbf{u}_{ik} = -K_i \delta\mathbf{x}_{ik} - K_{i\nu}\nu_{ik+1} - K_{i\mathbf{u}}\mathbf{u}_{ik} \\ K_i = (B_{ik}^T S_{ik+1} B_{ik} + \mathbf{R}_i)^{-1} B_{ik}^T S_{ik+1} A_{ik} \\ K_{i\nu} = (B_{ik}^T S_{ik+1} B_{ik} + \mathbf{R}_i)^{-1} B_{ik}^T \\ K_{i\mathbf{u}} = (B_{ik}^T S_{ik+1} B_{ik} + \mathbf{R}_i)^{-1} \mathbf{R}_i \\ S_{ik} = A_{ik}^T S_{ik+1}(A_{ik} - B_{ik}K_i) + \mathbf{Q}_i \\ \nu_{ik} = (A_{ik} - B_{ik}K_i)^T \nu_{ik+1} - K_i^T \mathbf{R}_i \mathbf{u}_{ik} + \mathbf{Q}_i \mathbf{x}_{ik} \\ \mathbf{u}_{ik}^* = \mathbf{u}_{ik} + \delta\mathbf{u}_{ik} \end{cases} \tag{96}$$

6.2. Leader and Follower iLQR Controllers

By following the leader-follower formation control problem described in Subsection 3.2, the leader control law set is

$$\begin{cases} \delta f_{Lk} = -K_{Lz}\delta z_{Lk} - K_{Lz\nu}\nu_{zL_{k+1}} - K_{fL} f_{Lk} \\ \delta\tau_{Lqk} = -K_{Lq}\delta\mathbf{q}_{Lk} - K_{L\mathbf{q}\nu}\nu_{\mathbf{q}L_{k+1}} - K_{\tau qL}\tau_{Lqk} \\ f_{Lk} = \frac{m_L g}{q_{L0k}^2 - q_{L1k}^2 - q_{L2k}^2 + q_{L3k}^2} \\ f_{Lk}^* = f_{Lk} + \delta f_{Lk} \\ \tau_{Lqk}^* = \tau_{Lqk} + \delta\tau_{Lqk} \end{cases}$$

where

$$\delta z_{Lk} = \begin{bmatrix} z_{Lkd} - z_{Lk} \\ v_{Lzkd} - v_{Lzk} \end{bmatrix}$$

$$\delta \mathbf{q}_{Lk} = \begin{bmatrix} \mathbf{q}_{Lkd} - \mathbf{q}_{Lk} \\ \omega_{Lkd} - \omega_{Lk} \end{bmatrix}$$

and the follower control law set is

$$\begin{cases} \delta f_{Fk} = -K_{Fz}\delta z_{Fk} - K_{Fz\nu}\nu_{zF_{k+1}} - K_{fF}f_{Fk} \\ \delta \tau_{Fqk} = -K_{Fq}\delta \mathbf{q}_{Fk} - K_{F\mathbf{q}\nu}\nu_{\mathbf{q}F_{k+1}} - K_{\tau qF}\tau_{Fqk} \\ f_{Fk} = (g + \dot{v}_{Lz} - d(R_{31}\cos\rho\cos\sigma + R_{32}\cos\rho\sin\sigma \\ +R_{33}\sin\rho))\frac{m_F}{q_{F0k}^2 - q_{F1k}^2 - q_{F2k}^2 + q_{F3k}^2} \\ f_{Fk}^* = f_{Fk} + \delta f_{Fk} \\ \tau_{Fqk}^* = \tau_{Fqk} + \delta \tau_{Fqk} \end{cases}$$

where

$$\delta z_{Fk} = \begin{bmatrix} z_{Fkd} - z_{Fk} \\ v_{Fzkd} - v_{Fzk} \end{bmatrix}$$

$$\delta \mathbf{q}_{Fk} = \begin{bmatrix} \mathbf{q}_{Lk} - \mathbf{q}_{Fk} \\ \omega_{Fkd} - \omega_{Fk} \end{bmatrix}$$

7. Simulations

The proposed H_∞, IBS and iLQR controllers were tested in a MATLAB simulator of two quadrotors, one leader and one follower. The quadrotor parameters used in the simulation are described in Table 1. Same path was presented in the simulation to show the performance of using the proposed three controllers. The desired path to be tracked by the leader was

$$\begin{cases} x_{Ld} = 2\cos(t\pi/80) \quad ; \quad y_{Ld} = 2\sin(t\pi/80) \\ z_{Ld} = 1 + 0.1t \quad ; \quad q_{L3d} = 0 \end{cases}$$

with the initial conditions $\mathbf{p}_L = [2,0,0]^T$ metres and $[q_{L0}, \mathbf{q}_L^T]^T = [-1,0,0,0]^T$. The follower tried to maintain the desired distance with the leader $d = 2$ metres, the desired incidence angle $\rho = 0$ and the desired bearing angle $\sigma = -\pi/12$. The initial condition of the follower was $\mathbf{p}_F = [0.1, 0.5, 0]^T$ metres and $[q_{F0}, \mathbf{q}_F^T]^T = [-1,0,0,0]^T$.

7.1. H_∞ Controller

To test the robustness of the proposed H_∞ controller, the model parameter uncertainties (mass and inertia) were increased and decreased by $\pm 20\%$ and a

Table 1. Quadrotor Parameters

Symbol	Definition	Value	Units
J_x	Roll Inertia	4.4×10^-3	$kg.m^2$
J_y	Pitch Inertia	4.4×10^-3	$kg.m^2$
J_z	Yaw Inertia	8.8×10^-3	$kg.m^2$
m	Mass	0.5	kg
g	Gravity	9.81	m/s^2
l	Arm Length	0.17	m
J_r	Rotor Inertia	4.4×10^-5	$kg.m^2$

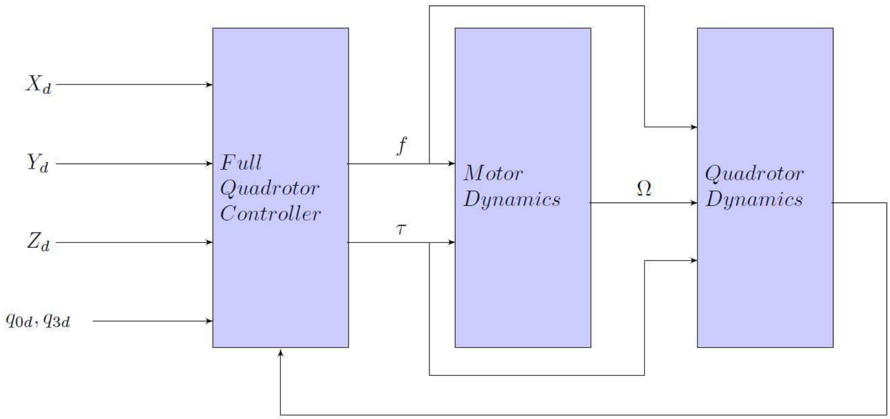

Figure 2. One Loop Control Block Diagram.

force disturbance of 2N was added in different operation times to the positions for 0.25 seconds duration, while the disturbance added to the attitude was of the form

$$\mathbf{d}_1 = 0.01 + 0.01\sin(0.024\pi t) + 0.05\sin(1.32\pi t). \tag{97}$$

The constant γ was chosen to be $\gamma_L = \gamma_F = 1.05$ and the weighting matrices were chosen to be $W_{L1z} = 1150$, $W_{F1z} = 1575$, $W_{L2} = W_{F2} = diag(0.0235, 0.0235, 0.0009)$, $W_{L3z} = 10$, $W_{F3z} = 675$, and $W_{L4} = W_{F4} = diag(0.0043, 0.0043, 0.00156)$. Under these parameters, the feedback control matrices were obtained to be $k_{Lz} = 111$, $k_{Lv_z} = 50$,

$k_{Fz} = 130$, $k_{Fv_z} = 100$, $K_{Lq} = K_{Fq} = diag(0.5, 0.5, 0.095)$ and $K_{L\omega} = K_{F\omega} = diag(0.07, 0.07, 0.025)$.

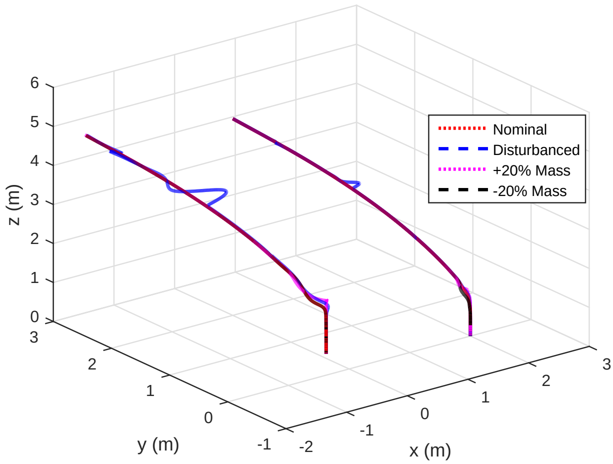

Figure 3. Leader-Follower Formation in First Path under H_∞ Controller Based on Quaternion Representation.

The obtained results are shown in Figures 3 - 6 with the conditions (1) no disturbance, (2) force disturbance $d_{v_{iz}} = -2$Nm at $10 \leq t \geq 10.25$ seconds, $d_{v_{ix}} = 2$Nm at $20 \leq t \geq 20.25$ seconds, $d_{v_{iy}} = 2$Nm at $30 \leq t \geq 30.25$ seconds and the attitude part for the leader and the follower is disturbed using (97), (3) $+20\%$ model parameter uncertainty, and (4) -20% model parameter uncertainty. The above conditions were applied for the leader and the follower at the same time.

Figure 3 shows the formation trajectories of two quadrotors obtained using the H_∞ controllers when they tracked the desired path. From this figure we can see that the H_∞ controllers produced good formation performances with small acceptable errors, fast rejection of the external disturbances, and quick recovery of the model parameter uncertainties. The quaternions of the leader and the follower are shown in Figures 4 and 5 with small oscillations. The distances between the leader and the follower for are shown in Figure 6. Again, less oscillation in disturbance rejection was observed from the result.

Figure 7 shows the performance of the two quadrotors when only the leader

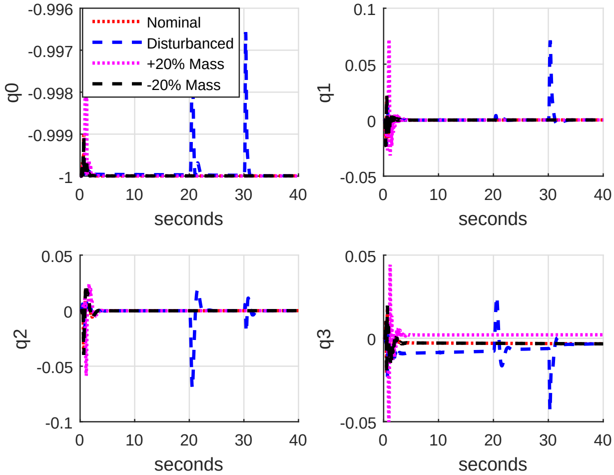

Figure 4. Leader Quaternions in First Path under H_∞ Controller Based on Quaternion Representation.

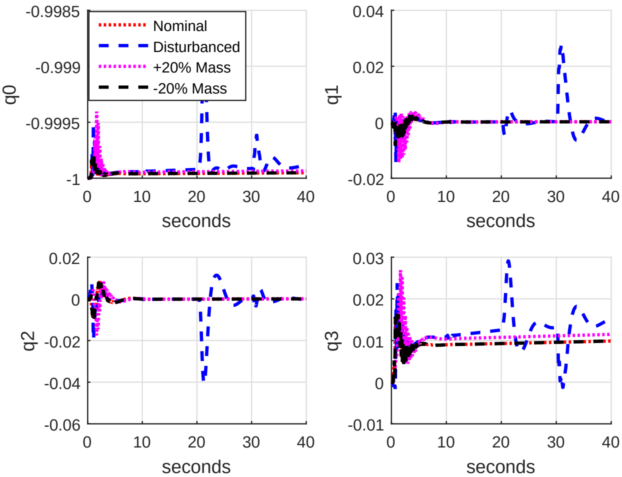

Figure 5. Follower Quaternions in First Path under H_∞ Controller Based on Quaternion Representation.

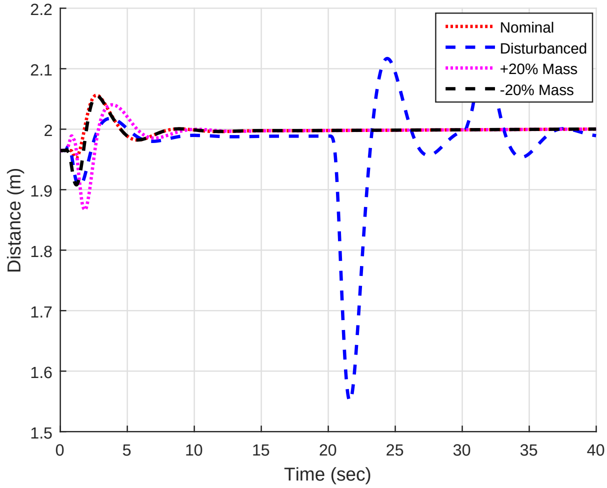

Figure 6. The Distance Between the Leader and the Follower under H_∞ Controller.

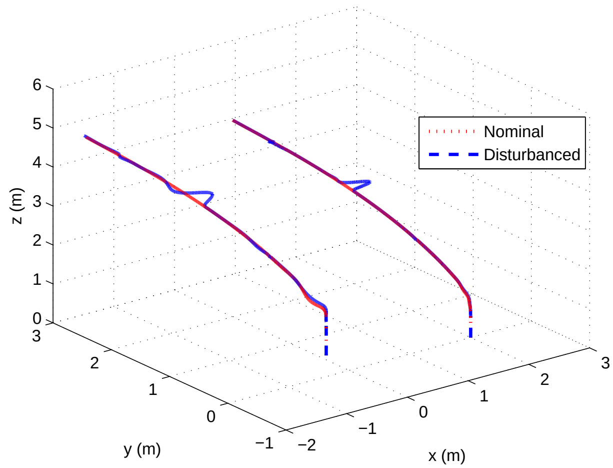

Figure 7. Leader-Follower Formation in First Path under H_∞ Controller Based on Quaternion Representation with Leader Disturbance Only.

was affected by force disturbance $d_{v_{Lz}} = -4$Nm during $10 \leq t \geq 10.25$ seconds, $d_{v_{Lx}} = 4$Nm during $20 \leq t \geq 20.25$ seconds, $d_{v_{Ly}} = 4$Nm during $30 \leq t \geq 30.25$ seconds, and the leader attitude part is disturbed using (97).

It is clear that the follower tracked the leader and maintained the distance with very small errors in all circumstances.

7.2. IBS Controller

The IBS controllers were tested in simulation to track a desired path by the leader and maintain the desired distance, desired incidence angle and desired bearing angle between them for the follower. The parameters chosen were $b_L = diag(180, 0.34, 0.34)$, $c_L = diag(0.7, 0.02, 0.02)$, $k_L = diag(0.0516, 0.0081, 0.0081)$, $b_F = diag(12, 0.7, 0.7)$, $c_F = diag(1.4, 0.02, 0.02)$ and $k_F = diag(0.01, 0.001, 0.001)$.

Figure 8 shows the formation trajectories of two quadrotors obtained by using the IBS controller. From this figure we can see that the IBS controller performed with high error, large oscillation in disturbance rejection and model parameter uncertainty recovery.

The quaternions of the leader and the follower are shown in Figures 9 and 10, respectively. High oscillation is observed in these two figures. The distances between the leader and the follower are shown in Figure 11. Again, high oscillation can be observed from the result of this figure.

Figure 12 shows the performance using the IBS controller when only the leader was affected by force disturbance $d_{v_{Lz}} = -4$Nm during $10 \leq t \geq 10.25$ seconds, $d_{v_{Lx}} = 4$Nm during $20 \leq t \geq 20.25$ seconds, $d_{v_{Ly}} = 4$Nm during $30 \leq t \geq 30.25$ seconds, and the attitude part is disturbance using (97).

It is clear that the follower tracked the leader and maintained the distance with high error and oscillation in all circumstances.

7.3. iLQR Controller

To validate the iLQR control strategy, it was tested in the simulation of two quadrotors in the leader-follower formation problem. The desired path was also used to test the LQR control for comparison purposes.

Figure 13 shows the response of the leader while tracking the predefined path and the follower maintaining the desired distance, the bearing angle and the incidence angle. The quaternion components responses of the leader and

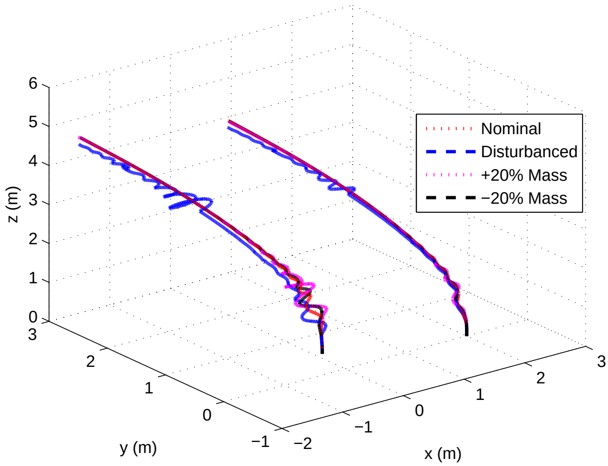

Figure 8. Leader-Follower Formation in First Path under IBS Controller Based on Quaternion Representation.

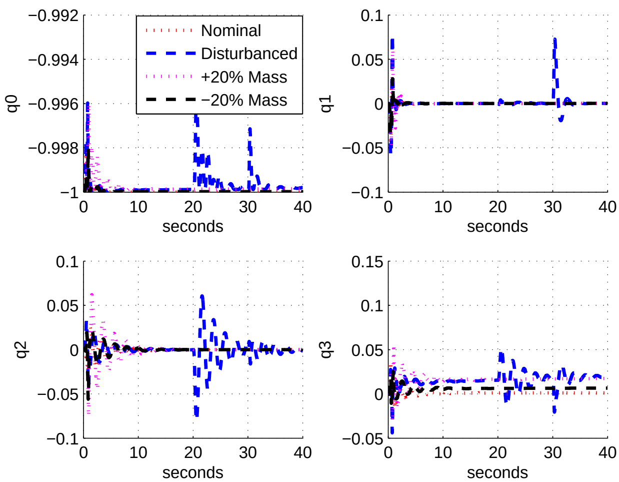

Figure 9. Leader Quaternions in First Path under IBS Controller Based on Quaternion Representation.

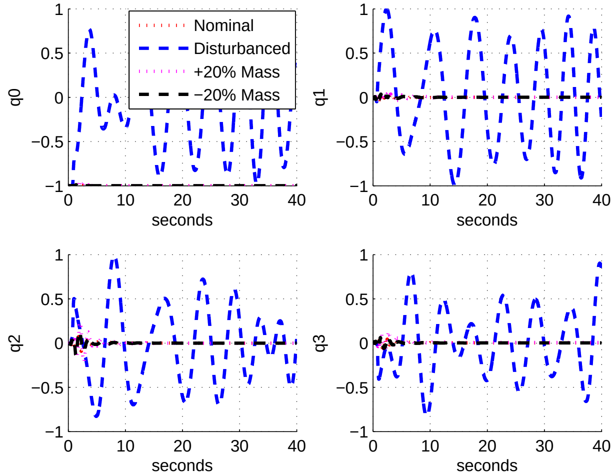

Figure 10. Follower Quaternions in First Path under IBS Controller Based on Quaternion Representation.

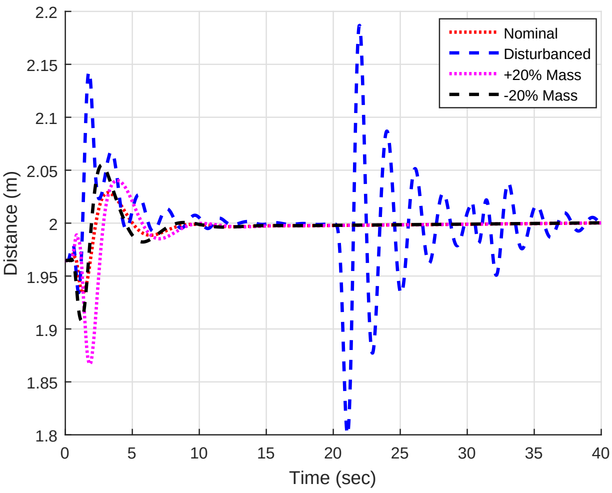

Figure 11. The Distance between the Leader and the Follower under IBS Controller.

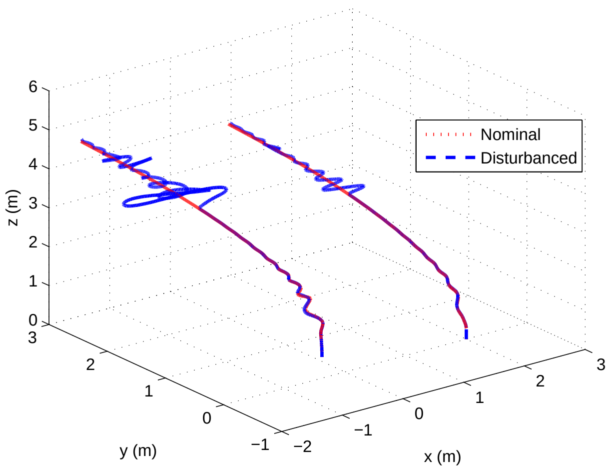

Figure 12. Leader-Follower Formation in First Path under IBS Controller Based on Quaternion Representation with Leader Disturbance Only.

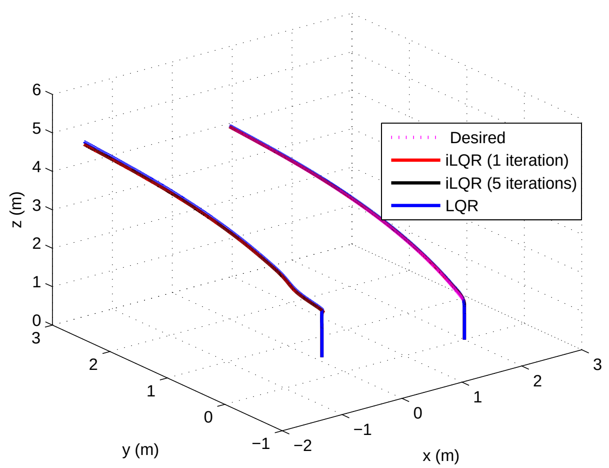

Figure 13. Leader-Follower Formation in First Path under iLQR and LQR Controllers.

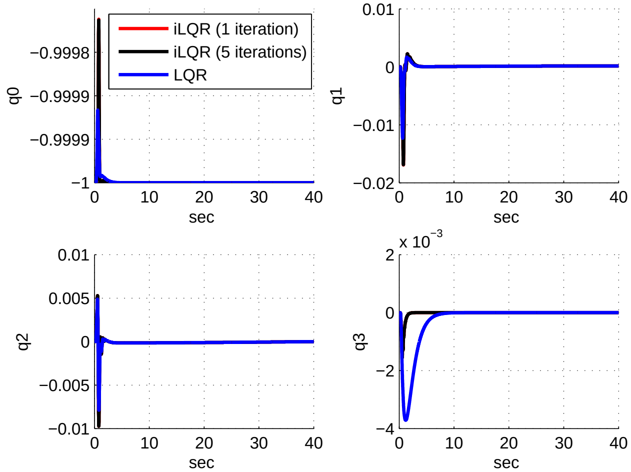

Figure 14. Leader Quaternions in First Path under iLQR and LQR Controllers.

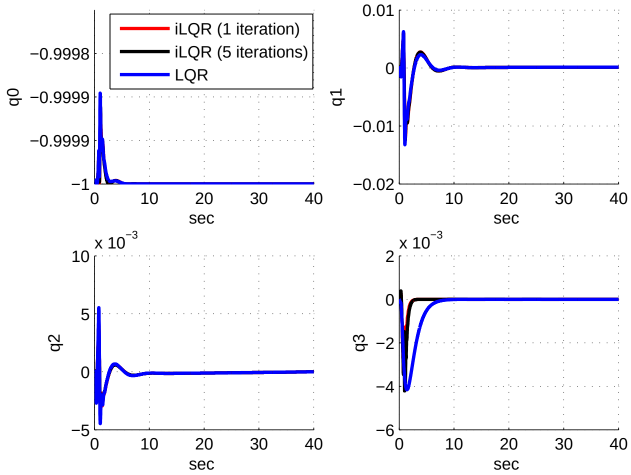

Figure 15. Follower Quaternions in First Path under iLQR and LQR Controllers.

the follower in tracking the desired path are shown in Figures 14 and 15, respectively. Figure 16 shows the distances between the leader and the follower.

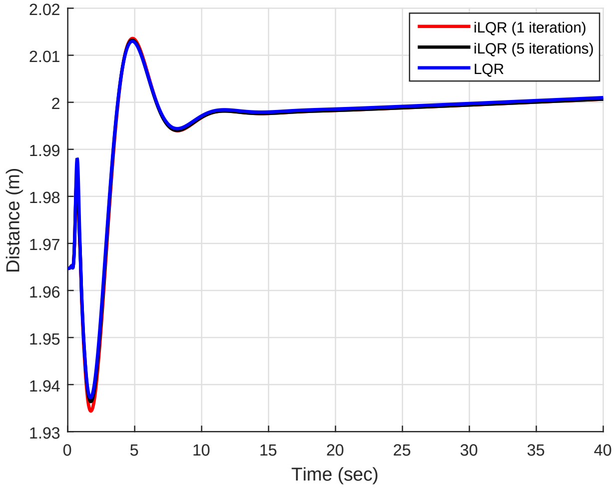

Figure 16. The Distance between the Leader and the Follower under iLQR and LQR Controllers.

The error in using the iLQR controller was smaller than that in using LQR. However, when the iLQR controller ran for five iterations, the response was slightly improved.

In conclusion, it is obvious that the proposed iLQR controller maintained the distance between the leader and the follower faster than LQR controller.

Conclusion

This chapter has presented the performance of applying the H_∞, IBS and iLQR controllers to the leader-follower formation control problem of quadrotors when its dynamic model was derived based on unit quaternion. The H_∞ controller was developed to reject the external disturbances and recover the model parameter uncertainties change. Then, its stability and robustness were analysed and a set of corresponding conditions were given.

The IBS controller was developed based on BS control theory with adding an integral action to minimise the steady state error which appeared when the BS controller was used for leader-follower formation problem. The main drawback of the IBS controller is that its stability is guaranteed but the performance is not, and it has three coupling parameters to be tuned compared with a guaranteed stability and performance of the H_∞ controller. Another noteworthy drawback of the IBS controller that was noticed in the current study is the considerable overshoot in its response due to the effect of the integral parameter and high oscillations when external disturbances were applied to the system dynamics in leader-follower formation.

The iLQR controller is essentially based on the LQR controller with an iteration technique. It has a set of gains equal to the number of operating samples by linearising the system in each sample of operation.

The controllers were tested in the MATLAB simulater. The simulation results show that the proposed H_∞ controller achieved excellent performance compared with those of IBS controller.

The proposed iLQR controller was based on finding a linearised system at each time step of the operation, while the LQR controller was based on obtaining a linearised system at the operating point (hovering point). The solutions of the two controllers establish the potential of the proposed iLQR law by improving the tracking accuracy and the speed of catching the desired path and maintaining the distances between the leader and the follower compared with the LQR controller. The iLQR controller performed better than the LQR controller, especially in quaternion components performance.

As a result, the proposed H_∞ controller indeed produced better control performance than the IBS controller in all circumstances, and the iLQR controller perform faster than LQR controller with less errors.

References

[1] S. Islam, P. Liu, and A. El Saddik, "Robust control of four-rotor unmanned aerial vehicle with disturbance uncertainty," *Industrial Electronics, IEEE Transactions on*, vol. 62, pp. 1563–1571, March 2015.

[2] J. Ferruz, V. M. Vega, A. Ollero, and V. Blanco, "Reconfigurable control architecture for distributed systems in the hero autonomous helicopter," *Industrial Electronics, IEEE Transactions on*, vol. 58, no. 12, pp. 5311–5318, December 2011.

[3] T. Dierks and S. Jagannathan, "Neural network control of quadrotor UAV formations," in American Control Conference ACC '09, (Hyatt Regency Riverfront, St. Louis, MO, USA), pp. 2990–2996, 2009.

[4] G. Damien, “Unmanned aerial vehicle formation flight using sliding mode disturbance observers aerial vehicles,” in Thanh Mung Lam (Ed.),ISBN: 978-953-7619-41-1, InTech, 2009.

[5] D. Lee, “Distributed backstepping control of multiple thrust-propelled vehicles on balanced graph,” in 18th IFAC Congress, (Milano, Italy), pp. 8872–8877, 2011.

[6] U. Pilz, A. Popov, and H. Werner, “Robust controller design for formation flight of quad-rotor helicopters,” in Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, (Shanghai, P.R. China), pp. 8322–8327, 2009.

[7] J. A. Guerrero, P. Castillo, and Y. Challal, “Quadrotors formation control a wireless medium access aware approach,” *Journal of Intelligent and Robotic Systems*, vol. 4, pp. 221–231, 2013.

[8] G. Regula and B. Lantos, “Formation control of quadrotor helicopters with guaranteed collision avoidance via safe path,” *Electrical Engineering and Computer Science*, vol. 4, no. 2012, pp. 113–124, 2013.

[9] I. Bayezit and B. Fidan, “Distributed cohesive motion control of flight vehicle formations,” *Industrial Electronics, IEEE Transactions on*, vol. 60, no. 12, pp. 5763–5772, December 2013.

[10] H. Cai and J. Huang, “The leader-following consensus of multiple rigid spacecraft systems,” in American Control Conference ACC ’13, (Washington, DC, USA), pp. 824–829, 2013.

[11] M. Saska, Z. Kasl, and L. Preucil, “Motion planing and control of formations of micro aerial vehicles,” in Preprints of the 19th World Congress, The International Federation of Automatic Control IFAC, (Cape Town, South Africa), pp. 1228–1233, 2014.

[12] A. Abdessameud, A. Tayebi, and I. G. Polushin, “Motion coordination of thrust-propelled underactuated vehicles in the presence of communication delays,” in Preprints of the 19th World Congress, The International Federation of Automatic Control IFAC, (Cape Town, South Africa), pp. 3170–3175, 2014.

[13] X. Dong, B. Yu, Z. Shi, and Y. Zhong, “Time-varying formation control for unmanned aerial vehicles: Theories and applications,” *Control Systems Technology, IEEE Transactions on*, vol. 23, pp. 340–348, Jan 2015.

[14] F. Poiesi and A. Cavallaro, “Distributed vision-based flying cameras to film a moving target,” in Intelligent Robots and Systems (IROS), 2015 IEEE/RSJ International Conference on, pp. 2453–2459, Sept 2015.

[15] A. Abdessameud and A. Tayebi, “Formation control of vtol-uavs,” in Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on, pp. 3454–3459, Dec 2009.

[16] A. Karimoddini, M. Karimadini, and H. Lin, “Decentralized hybrid formation control of unmanned aerial vehicles,” arXiv preprint arXiv:1403.0258, no. 1, 2014.

[17] L. C. A. Pimenta, G. A. S. Pereira, N. Michael, R. Mesquita, M. M. Bosque, L. Chaimowics, and V. Kumer, “Swarm coordination based on smoothed particle hydrodynamics technique,” *IEEE Transaction on Robotics*, vol. 29, no. 2, pp. 383–399, 2013.

[18] M. Turpin, N. Michael, and V. Kumar, “Decentralized formation control with variable shapes for aerial robots,” in 2012 IEEE International Conference on Robotics and Automation, (RiverCentre, Saint Paul, Minnesota, USA), pp. 23–30, 2012.

[19] M. Turpin, N. Michael, and V. Kumar, “Trajectory design and control for aggressive formation flight with quadrotors,” *Autonomous Robots*, vol. 33, no. 1-2, pp. 143–156, 2012.

[20] M. Turpin, N. Michael, and V. Kumar, “Capt: Concurrent assignment and planning of trajectories for multiple robots,” *The International Journal of Robotics Research*, vol. 33, no. 1, pp. 98–112, 2014.

[21] A. Bemporad and C. Rocchi, “Decentralized hybrid model predictive control of a formation of unmanned aerial vehicles,” in 18th IFAC World Congress, (Milano, Italy), 2011.

[22] N. Koksal, B. Fidan, and K. Buyukkabasakal, "Real-time implementation of decentralized adaptive formation control on multi-quadrotor systems," in Control Conference (ECC), 2015 European, pp. 3162–3167, July 2015.

[23] K. Ghamry and Y. Zhang, "Formation control of multiple quadrotors based on leader-follower method," in Unmanned Aircraft Systems (ICUAS), 2015 International Conference on, pp. 1037–1042, June 2015.

[24] Y. X. Min, K.-C. Cao, H. H. Sheng, and Z. Tao, "Formation tracking control of multiple quadrotors based on backstepping," in Control Conference (CCC), 2015 34th Chinese, pp. 4424–4430, July 2015.

[25] F. B. Choi and D. Lee, "Teleoperation control of formation among multiple under-actuated quadrotor UAVs," 2013 IEEE RO-MAN, The 22nd IEEE International Symposium on Robot and Human Interactive Communication, no. 3, pp. 306–307, 2013.

[26] R. Tjahjana and P. Sasongko, "Multi flying vehicle control for saving fuel," *Applied Mathematical Sciences*, vol. 8, no. 15, pp. 733–742, 2014.

[27] B. Yu, X. Dong, Z. Shi, and Y. Zhong, "Formation control for quadrotor swarm systems: Algorithms and experiments," in 32nd Chines Control Conference CCC, (Xi'an, China), pp. 7099–7104, 2013.

[28] A. Mancini, A. Benini, E. Frontoni, P. Zingaretti, and S. Longhi, "Coalition formation for unmanned quadrotors," in Volume 3: 2011 ASME/IEEE International Conference Mechatroncs and Embedded Systems and Applications, Parts A and B, (Washington, DC, USA), pp. 745–752, 2011.

[29] A. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[30] B. C. Min, C. H. Cho, K. M. Choi, and D. H. Kim, "Development of a micro quad-rotor UAV for monitoring an indoor environment," *Advances in Robotics*, Springer-Verlag Berlin Heidelberg, vol. 5744, pp. 262–271, 2009.

[31] J. Kuipers, "Quaternions and rotation sequences," in *Geometry, Integrability and Quantization*, Ivailo M. Mladenov and Gregory L. Naber, Editors, Coral Press, Sofia 2000, (Vama, Bulgaria), pp. 127–143, 1999.

[32] J. Diebel, "Representing attitude: Euler angles, unit quaternions, and rotation vectors," Stanford University Report, 2006.

[33] M. Schreier, "Modeling and adaptive control of a quadrotor," in Proceeding of 2012 IEEE International Conference on Mechatronics and Automation ICMA, (Chengdu, China), pp. 383–390, 2012.

[34] E. Fresk and G. Nikolakopoulos, "Full quaternion based attitude control for a quadrotor," in 2013 European Control Conference (ECC), pp. 3864–3869, 2013.

[35] E. Reyes-Valeria, R. Enriquez-Caldera, S. Camacho-Lara, and J. Guichard, "Lqr control for a quadrotor using unit quaternions: Modeling and simulation," in 2013 IEEE International Conference on Electronics, Communications and Computing (CONIELECOMP), (Cholula), pp. 172–178, 2013.

[36] R. Beard, "Quadrotor dynamics and control," Brigham Young University Report, 2008.

[37] W. Jasim and D. Gu, "H_∞ control for quadrotor attitude stabilization," in UKACC 10th International Conference on Control (CONTROL 2014), (Loughborough, UK), pp. 25–30, 2014.

[38] W. Jasim and D. Gu, "H_∞ path tracking control for quadrotors based on quaternion representation," in 15th Towards Autonomous Robotic Systems (TAROS 2014), (Birmingham, UK), pp. 72–84, 2014.

[39] W. Jasim and D. Gu, "Integral backstepping controller for quadrotor path tracking," in Advanced Robotics, The 17th International Conference on, pp. 50–55, July 2015.

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Chapter 7

DETERMINANTAL REPRESENTATIONS OF THE DRAZIN AND W-WEIGHTED DRAZIN INVERSES OVER THE QUATERNION SKEW FIELD WITH APPLICATIONS

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Keywords: quaternion matrix, generalized inverse, Drazin inverse, weighted Drazin inverse, Moore-Penrose inverse, weighted Moore-Penrose inverse, system of linear equations, Cramer's rule, quaternion matrix equation, generalized inverse solution, least squares solution, Drazin inverse solution, differential matrix equation

AMS Subject Classification: 15A09, 15A24, 11R52

Abstract

A generalized inverse of a given quaternion matrix (similarly, as for complex matrices) exists for a larger class of matrices than the invertible matrices. It has some of the properties of the usual inverse, and agrees with the inverse when a given matrix happens to be invertible. There

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exist many different generalized inverses. In this chapter, we consider determinantal representations of the Drazin and weighted Drazin inverses over the quaternion skew field.

Due to the theory of column-row determinants recently introduced by the author, we derive determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field. Using obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ in both cases when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian and arbitrary, where $\mathbf{A}$, $\mathbf{B}$ can be noninvertible matrices of appropriate sizes. We obtain determinantal representations of solutions of the differential quaternionic matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where $\mathbf{A}$ is noninvertible as well.

Also, we obtain new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. We give determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, we derive explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \,|\, i^2 = j^2 = k^2 = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, and by $\mathbb{H}^{m\times n}_r$ the set of all $m \times n$ matrices over $\mathbb{H}$ with a rank r. Let $\mathrm{M}\,(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices and $\mathbf{I}$ be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, we denote by $\mathbf{A}^*$, $\operatorname{rank} \mathbf{A}$ the conjugate transpose (Hermitian adjoint) matrix and the rank of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices can be extended to quaternion matrices as follows.

Definition 1.1. *The Moore-Penrose inverse of* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, denoted by* $\mathbf{A}^{\dagger}$*, is the unique matrix* $\mathbf{X} \in \mathbb{H}^{n\times m}$ *satisfying the following equations,*

$$\mathbf{AXA} = \mathbf{A}; \tag{1.1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{1.2}$$

$$(\mathbf{AX})^{*} = \mathbf{AX}; \tag{1.3}$$

$$(\mathbf{XA})^{*} = \mathbf{XA}. \tag{1.4}$$

Definition 1.2. *For* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = Ind\,\mathbf{A}$ *the smallest positive number such that* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k}$*, the Drazin inverse of* $\mathbf{A}$ *is defined to be the unique matrix* $\mathbf{X}$ *that satisfying (1..2) and the following equations,*

$$\mathbf{AX} = \mathbf{XA}; \tag{1.5}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^{k}. \tag{1.6}$$

It is denoted by $\mathbf{X} = \mathbf{A}^{D}$*. In particular, when* $Ind\,\mathbf{A} = 1$*, then the matrix* $\mathbf{X}$ *is called the group inverse and is denoted by* $\mathbf{X} = \mathbf{A}^{g}$*.*

If $Ind\mathbf{A} = 0$, then $\mathbf{A}$ is invertible, and $\mathbf{A}^{D} \equiv \mathbf{A}^{\dagger} = \mathbf{A}^{-1}$.

Cline and Greville [1] extended the Drazin inverse of a square matrix to a rectangular matrix, which can be generalized to the quaternion algebra as follows.

Definition 1.3. *For* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}^{n\times m}$*, the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *is the unique solution to equations,*

$$(\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^{k}; \tag{1.7}$$

$$\mathbf{XWAWX} = \mathbf{X}; \tag{1.8}$$

$$\mathbf{AWX} = \mathbf{XWA}, \tag{1.9}$$

where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$*. It is denoted by* $\mathbf{X} = \mathbf{A}_{d,\mathbf{W}}$.

The problem of determinantal representation of generalized inverse matrices only recently begun to be decided through the theory of column-row determinants introduced in [2, 3]. The theory of row and column determinants develops the classical approach to a definition of a determinant as an alternating sum of products of elements of a quadratic matrix but with a predetermined order of factors in each summand of a determinant. A determinant of a matrix with noncommutative elements is often called the noncommutative determinant. Unlike other known noncommutative determinants such as determinants of Dieudonné [4], Study [5], Moore [6,7], Chen [8], quasideterminants of Gelfand-Retakh [9], the double determinant built on the theory of the column-row determinants has properties similar to the usual determinant, in particular, it can be expand along arbitrary rows and columns. This property is necessary for determinantal representations of the inverse and generalized inverse matrices. Determinantal representations of the Moore-Penrose inverse, the minimum norm least squares solutions of some quaternion matrix equations over the quaternion skew-field have been obtained in [10, 11]. Determinantal representations of an outer inverse $\mathbf{A}^{(2)}_{T,S}$ has introduced in [12,13] using column-row determinants as well. Recall that an outer inverse of a matrix $\mathbf{A}$ over complex field with prescribed range space T and null space S is a solution of (1..2) with restrictions,

$$\mathcal{R}(\mathbf{X}) = T, \ \mathcal{N}(\mathbf{X}) = S.$$

Within the framework of the theory of column-row determinants Song [14] also has gave a determinantal representation of the W-weighted Drazin inverse over the quaternion skew-field using its characterization by an outer inverse $\mathbf{A}^{(2)}_{T,S}$. But, in obtaining of this determinantal representation, auxiliary matrices that different from $\mathbf{A}$ or its powers are needed. In this chapter, we shall obtain new determinantal representations of the Drazin inverse and the W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ by using only their entries. These determinantal representations of the Drazin and W-weighted Drazin inverse will be used for explicit determinantal representation formulas of the Drazin and W-weighted Drazin inverse solutions of some quaternion matrix equations.

The chapter is organized as follows. We start with some basic concepts and results from the theory of row-column determinants and the theory of quaternion matrices in Section 2. In Section 3, we give the determinantal representations of the Drazin inverse of a Hermitian quaternion matrix in Subsection 3.1 and an arbitrary quaternion matrix in Subsection 3.2. In Section 4, we obtain explicit representation formulas for the Drazin inverse solutions of quaternion matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, and $\mathbf{XB} = \mathbf{D}$. In Subsection 4.1, we consider the case when $\mathbf{A}$, $\mathbf{B}$ are Hermitian, and they are arbitrary in Subsection 4.1. In Section 4.3, we show numerical examples to illustrate the main results. In Section 5, we apply the obtained determinantal representations of the Drazin inverse to solutions of differential matrix equations. In Subsection 5.1, we give a background for quaternion-valued differential equations. In Subsection 5.2, determinantal representations of solutions of the differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$ are derived, where $\mathbf{A} \in \mathbb{H}^{n\times n}$ is noninvertible. It is demonstrated in an example in Subsection 5.3. In Section 6, we obtain determinantal representations of the W-weighted Drazin inverse by using introduced above determinantal representations of the Drazin inverse in Subsection 6.1, the Moore-Penrose inverse in Subsection 6.2, and the limit representations of the W-weighted Drazin inverse in some special case in Subsection 6.3. In Subsection 6.4, we show a numerical example to illustrate the main result. By using determinantal representations of the W-weighted Drazin inverse obtained in the previous section, we get explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of some quaternion matrix equations in Section 7. In Subsection 7.1, we consider the background of the problem of Cramer's rule for the W-weighted Drazin inverse solution. In Subsection 7.2, we obtain explicit representation formulas of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of the quaternion matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$. In Subsection 7.3, we give numerical examples to illustrate the main result.

Facts set forth in Sections 3 and 4 were partly published in [15], in Section 6 were published in [16] and in Section 7 were partly published in [17].

2. Preliminaries. Elements of the Theory of the Column and Row Determinants

Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$. Through the chapter, we denote $i = 1, \ldots, n$ by $i = \overline{1, n}$.

Definition 2.1. *The i-th row determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is defined* *for all* $i = \overline{1, n}$ *by putting*

$$\begin{array}{l}\mathrm{rdet}_i \mathbf{A} = \\ \sum\limits_{\sigma \in S_n} (-1)^{n-r} (a_{i\, i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}) \ldots (a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}), \\ \sigma = (i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}),\end{array}$$

with conditions $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ *and* $i_{k_t} < i_{k_t+s}$ *for* $t = \overline{2, r}$ *and* $s = \overline{1, l_t}$.

Definition 2.2. *The* j*-th column determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is defined for all* $j = \overline{1, n}$ *by putting*

$$\begin{array}{l}\mathrm{cdet}_j \mathbf{A} = \\ \sum\limits_{\tau \in S_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} i_{k_r}}) \ldots (a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}), \\ \tau = (j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}) \ldots (j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}) \ (j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j),\end{array}$$

with conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ *and* $j_{k_t} < j_{k_t+s}$ *for* $t = \overline{2, r}$ *and* $s = \overline{1, l_t}$.

Suppose $\mathbf{A}^{ij}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting both the i-th row and the j-th column. Let $\mathbf{a}_{.j}$ be the j-th column and $\mathbf{a}_{i.}$ be the i-th row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its j-th column with the column-vector $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its i-th row with the row-vector $\mathbf{b}$.

We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$. These properties completely have been proved in [2, 3].

Proposition 2.1. *If* $b \in \mathbb{H}$*, then* $\mathrm{rdet}_i \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i \mathbf{A}$ *for all* $i = \overline{1, n}$.

Proposition 2.2. *If* $b \in \mathbb{H}$*, then* $\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cdet}_j \mathbf{A} \cdot b$ *for all* $j = \overline{1, n}$.

Proposition 2.3. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in I_n$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = \overline{1, n}$*, then*

$$\begin{array}{c}\mathrm{rdet}_i \mathbf{A} = \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{c}), \\ \mathrm{cdet}_i \mathbf{A} = \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{c}),\end{array}$$

where $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{H}^{1\times n}$, $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{H}^{1\times n}$, $i = \overline{1, n}$.

Proposition 2.4. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in J_n$ *such that* $a_{i\,t} = b_i + c_i$ *for all* $i = \overline{1, n}$*, then*

$$\mathrm{rdet}_j\, \mathbf{A} = \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{b}) + \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{c})\,,$$
$$\mathrm{cdet}_j\, \mathbf{A} = \mathrm{cdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{b}) + \mathrm{cdet}_j \mathbf{A}_{\,.\,t}\,(\mathbf{c})\,,$$

where $\mathbf{b} = (b_1, \ldots, b_n)^T \in \mathbb{H}^{n\times 1}$*,* $\mathbf{c} = (c_1, \ldots, c_n)^T \in \mathbb{H}^{n\times 1}$*,* $j = \overline{1, n}$*.*

Proposition 2.5. *If* $\mathbf{A}^*$ *is the Hermitian adjoint matrix of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, then* $\mathrm{rdet}_i\, \mathbf{A}^* = \overline{\mathrm{cdet}_i\, \mathbf{A}}$ *for all* $i = \overline{1, n}$*.*

The following lemmas enable us to expand $\mathrm{rdet}_i\, \mathbf{A}$ by cofactors along the i-th row and $\mathrm{cdet}_j \mathbf{A}$ along the j-th column respectively for all $i, j = \overline{1, n}$.

Lemma 2.3. *Let* $R_{i\,j}$ *be the* ij*-th right cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, that is,* $\mathrm{rdet}_i\, \mathbf{A} = \sum\limits_{j=1}^{n} a_{i\,j} \cdot R_{i\,j}$ *for all* $i = \overline{1, n}$*. Then*

$$R_{i\,j} = \begin{cases} -\mathrm{rdet}_k\, \mathbf{A}^{ii}_{.j}\,(\mathbf{a}_{.i})\,, & i \neq j, \quad k = \begin{cases} j, & if \quad i > j; \\ j-1, & if \quad i < j; \end{cases} \\ \mathrm{rdet}_k\, \mathbf{A}^{ii}, & i = j, \qquad k = \min\{I_n \setminus i\}, \end{cases} \tag{2.1}$$

where $\mathbf{A}^{i\,i}_{.\,j}\,(\mathbf{a}_{.\,i})$ *is obtained from* $\mathbf{A}$ *by replacing the* j*-th column with the* i*-th column, and then by deleting both the* i*-th row and column.*

Lemma 2.4. *Let* $L_{i\,j}$ *be the* ij*-th left cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, that is,* $\mathrm{cdet}_j\, \mathbf{A} = \sum\limits_{i=1}^{n} L_{i\,j} \cdot a_{i\,j}$ *for all* $j = \overline{1, n}$*. Then*

$$L_{i\,j} = \begin{cases} -\mathrm{cdet}_k\, \mathbf{A}^{jj}_{i.}\,(\mathbf{a}_{j.})\,, & i \neq j, \quad k = \begin{cases} i, & if \quad j > i; \\ i-1, & if \quad j < i; \end{cases} \\ \mathrm{cdet}_k\, \mathbf{A}^{ii}, & i = j, \quad k = \min\{J_n \setminus j\}, \end{cases} \tag{2.2}$$

where $\mathbf{A}^{jj}_{i.}\,(\mathbf{a}_{j\,.})$ *is obtained from* $\mathbf{A}$ *by replacing the* i*-th row with the* j*-th row, and then by deleting both the* j*-th row and column.*

The following theorem has a key value in the theory of column-row determinants.

Theorem 2.5. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $\mathrm{rdet}_1 \mathbf{A} = \cdots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \cdots = \mathrm{cdet}_n \mathbf{A} \in \mathbb{R}$.

Remark 2.6. *Since all column and row determinants of a Hermitian matrix over* $\mathbb{H}$ *are equal, we can define the determinant of Hermitian* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *by putting for all* $i = \overline{1, n}$,

$$\det \mathbf{A} := \mathrm{rdet}_i \, \mathbf{A} = \mathrm{cdet}_i \, \mathbf{A}.$$

Properties of the determinant of a Hermitian matrix is completely explored in [3] by its row and column determinants. They can be summarized by the following theorems.

Theorem 2.7. *If the* i*-th row of a Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is replaced with a left linear combination of its other rows, i.e.* $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.}$, *where* $c_l \in \mathbb{H}$ *for all* $l = \overline{1, k}$ *and* $i, i_l \in I_n$, *then*

$$\mathrm{rdet}_i \, \mathbf{A}_{i.} \left(c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.} \right) = \mathrm{cdet}_i \, \mathbf{A}_{i.} \left(c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.} \right) = 0.$$

Theorem 2.8. *If the* j*-th column of a Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is replaced with a right linear combination of its other columns, i.e.* $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k$, *where* $c_l \in \mathbb{H}$ *for all* $l = \overline{1, k}$ *and* $j, j_l \in J_n$, *then*

$$\mathrm{cdet}_j \, \mathbf{A}_{.j} \left(\mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k \right) = \mathrm{rdet}_j \, \mathbf{A}_{.j} \left(\mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k \right) = 0.$$

The following theorem on determinantal representations of an inverse matrix of Hermitian follows directly from these properties.

Theorem 2.9. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, and* $\det \mathbf{A} \neq 0$, *then there exists a unique right inverse matrix* $(R\mathbf{A})^{-1}$ *and a unique left inverse matrix* $(L\mathbf{A})^{-1}$ *of* $\mathbf{A}$, *where* $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$, *and they possess the following determinantal representations, respectively,*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \tag{2.3}$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \tag{2.4}$$

where R_{ij}, L_{ij} *are the right (2.1) and left (2.2)* ij*-th cofactors of* $\mathbf{A}$ *for all* $i, j = \overline{1, n}$.

Remark 2.10. *If* $\det \mathbf{A} = 0$, *we say that a Hermitian quaternion matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is singular because, in this case,* $\mathbf{A}$ *is noninvertible.*

Since principal submatrices of a Hermitian matrix are Hermitian, the principal minor can be defined as the determinant of its principal submatrix by analogy to the commutative case. In [3], we have introduced the rank by principle minors that is the maximal order of a nonzero principal minor of a Hermitian matrix. The following theorem determines a relationship between it and the column rank of a matrix defining as ceiling amount of right-linearly independent columns, and the row rank defining as ceiling amount of left-linearly independent rows.

Theorem 2.11. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then its rank by principal minors are equal to its column and row ranks.*

Due to the non-commutativity of quaternions, there are two types of eigenvalues. A quaternion λ is said to be a right eigenvalue of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ if $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda$, and λ is a left eigenvalue if $\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$ for some nonzero quaternion column-vector $\mathbf{x} \in \mathbb{H}^n$.

The theory on the left eigenvalues of quaternion matrices has been investigated, in particular, in [18,19,20]. The theory on the right eigenvalues of quaternion matrices is more developed. In particular, we note [21,23,24,25,26,27].

Proposition 2.6. *[25] Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian. Then* $\mathbf{A}$ *has exactly* n *real right eigenvalues.*

Right and left eigenvalues are in general unrelated [27] but it is not for Hermitian matrices. Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda \mathbf{I} - \mathbf{A}$ is Hermitian.

Definition 2.12. *If* $\lambda \in \mathbb{R}$, *then for a Hermitian matrix* $\mathbf{A}$ *the polynomial* $p_{\mathbf{A}}(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A})$ *is said to be the characteristic polynomial of* $\mathbf{A}$.

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues which are its right eigenvalues as well. We can prove the following theorem by analogy to the commutative case (see, e.g. [28]).

Theorem 2.13. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $p_{\mathbf{A}}(\lambda) = \lambda^n - d_1 \lambda^{n-1} + d_2 \lambda^{n-2} - \ldots + (-1)^n d_n$, *where* d_k *is the sum of principle minors of* $\mathbf{A}$ *of order* k, $1 \leq k < n$, *and* $d_n = \det \mathbf{A}$.

3. Determinantal Representations of the Drazin Inverse

As one of the important types of generalized inverses of matrices, the Drazin inverses and their applications have well been examined in the literature (see, e.g., [29, 30, 31, 32, 33, 34]). In [35], Stanimirovič and Djordjević have introduced a determinantal representation of the Drazin inverse of a complex matrix based on its full-rank representation. In [36], we obtain determinantal representations of the Drazin inverse of a complex matrix used its limit representation. It allowed to obtain the analogs of Cramer's rule for the Drazin inverse solutions of some matrix equations. In this chapter we extend studies conducted in [36] from the complex field to the quaternion skew field.

3.1. Analogues of the Classical Adjoint Matrix for the Drazin Inverse of a Hermitian Matrix

For Hermitian matrices, we apply the method which consists of the theorem on the limit representation of the Drazin inverse, lemmas on rank of matrices and on characteristic polynomial. This method at first has been used in [36], afterwards in [37, 38]. By analogy to [39] the following theorem on the limit representation of the quaternion Drazin inverse can be proved.

Theorem 3.1. *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$*, then*

$$\mathbf{A}^D = \lim_{\lambda\to 0}\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k = \lim_{\lambda\to 0}\mathbf{A}^k\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1},$$

where $\lambda \in \mathbb{R}_+$*, and* $\mathbb{R}_+$ *is a set of the real positive numbers.*

Denote by $\mathbf{a}_{.j}^{(m)}$ and $\mathbf{a}_{i.}^{(m)}$ the j-th column and the i-th row of $\mathbf{A}^m$, respectively.

Lemma 3.2. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$*, then*

$$\operatorname{rank}\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{A}^{k+1}\right). \tag{3.1}$$

Proof. The matrix $\mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)$ can be represented as follows

$$\mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)=\begin{pmatrix} \sum_{s=1}^{n} a_{1s}a_{s1}^{(k)} & \dots & \sum_{s=1}^{n} a_{1s}a_{sn}^{(k)} \\ \dots & \dots & \dots \\ a_{j1}^{(k)} & \dots & a_{j\,n}^{(k)} \\ \dots & \dots & \dots \\ \sum_{s=1}^{n} a_{ns}a_{s1}^{(k)} & \dots & \sum_{s=1}^{n} a_{ns}a_{sn}^{(k)} \end{pmatrix}$$

Let $\mathbf{P}_{l\,i}\left(-a_{l\,j}\right) \in \mathbb{H}^{n\times n}$, $(l \neq i)$, be a matrix with $-a_{l\,j}$ in the (l,i)-entry, 1 in all diagonal entries, and 0 in others. This is a matrix of an elementary transformation. It follows that

$$\tilde{\mathbf{A}} := \mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)\cdot\prod_{l\neq i}\mathbf{P}_{l\,i}\left(-a_{l\,j}\right)=\begin{pmatrix} \sum_{s\neq j} a_{1s}a_{s1}^{(k)} & \dots & \sum_{s\neq j} a_{1s}a_{sn}^{(k)} \\ \dots & \dots & \dots \\ a_{j1}^{(k)} & \dots & a_{j\,n}^{(k)} \\ \dots & \dots & \dots \\ \sum_{s\neq j} a_{ns}a_{s1}^{(k)} & \dots & \sum_{s\neq j} a_{ns}a_{sn}^{(k)} \end{pmatrix} i{-}th$$

The above obtained matrix $\tilde{\mathbf{A}}$ has the following factorization.

$$\tilde{\mathbf{A}} = \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & 0 & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} & \dots & a_{1n}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} & \dots & a_{2n}^{(k)} \\ \dots & \dots & \dots & \dots \\ a_{n1}^{(k)} & a_{n2}^{(k)} & \dots & a_{nn}^{(k)} \end{pmatrix}$$

Denote the first matrix by

$$\tilde{\mathbf{A}}_1 := \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \underset{j-th}{0} & \dots & a_{nn} \end{pmatrix} i-th.$$

The matrix $\tilde{\mathbf{A}}_1$ is obtained from $\mathbf{A}$ by replacing all entries of the i-th row and the j-th column with zeroes except for 1 in the (i,j)-entry. Since elementary transformations of a matrix do not change its rank, then $\operatorname{rank} \mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right) \leq \min\left\{\operatorname{rank} \mathbf{A}^k, \operatorname{rank} \tilde{\mathbf{A}}\right\}$. By $\operatorname{rank} \tilde{\mathbf{A}}_1 \geq \operatorname{rank} \mathbf{A}^k$, the proof is completed. □

The next lemma is proved similarly.

Lemma 3.3. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$*, then* $\operatorname{rank}\left(\mathbf{A}^{k+1}\right)_{i\,.}\left(\mathbf{a}_{j\,.}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{A}^{k+1}\right)$.

We shall use the following notations. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By $\mathbf{A}_\beta^\alpha$ denote the submatrix of $\mathbf{A}$ determined by rows indexed by α and columns indexed by β. Then $\mathbf{A}_\alpha^\alpha$ denotes the principal submatrix determined by the rows and columns indexed by α. If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}_\alpha^\alpha|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} := \{\,\alpha : \alpha = (\alpha_1, \ldots, \alpha_k)\,,\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,\,m}\{i\} := \{\,\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $\quad J_{r,\,n}\{j\} := \{\,\beta : \beta \in L_{r,n}, j \in \beta\}$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

Lemma 3.4. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian with* $Ind\,\mathbf{A} = k$ *and* $\lambda \in \mathbb{R}$*, then*

$$\operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}, \tag{3.2}$$

where $c_n^{(ij)} = \operatorname{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)$ *and*

$$c_s^{(ij)} = \sum_{\beta \in J_{s,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{k}\right)\right)_{\beta}^{\beta}$$

for all $s = \overline{1, n-1}$*,* $i, j = \overline{1, n}$*.*

Proof. Denote by $\mathbf{b}_{.\,i}$ the i-th column of $\mathbf{A}^{k+1} =: (b_{ij})_{n\times n}$. Consider the Hermitian matrix $\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.i}) \in \mathbb{H}^{n\times n}$. It differs from $\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)$ in an entry b_{ii}. Taking into account Theorem 2.13, we obtain

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.\,i}) = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \tag{3.3}$$

where $d_s = \sum\limits_{\beta\in J_{s,\,n}\{i\}} |\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}|$ is the sum of all principal minors of order s that contain the i-th column for all $s = \overline{1, n-1}$ and $d_n = \det\left(\mathbf{A}^{k+1}\right)$.

Consequently, we have $\mathbf{b}_{.\,i} = \begin{pmatrix} \sum\limits_l a_{1l}^{(k)} a_{li} \\ \sum\limits_l a_{2l}^{(k)} a_{li} \\ \vdots \\ \sum\limits_l a_{nl}^{(k)} a_{li} \end{pmatrix} = \sum\limits_l \mathbf{a}_{.\,l}^{(k)} a_{li}$, where $\mathbf{a}_{.\,l}^{(k)}$ is the l-th column of $\mathbf{A}^k$ for all $l = \overline{1, n}$. Due to Theorem 2.5, Lemma 2.4 and Proposition 2.2, we obtain on the one hand

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.\,i}(\mathbf{b}_{.\,i}) = \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.\,i}(\mathbf{b}_{.\,i}) = \sum_l \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.\,l}\left(\mathbf{a}_{.\,l}^{(k)} a_{l\,i}\right) = \sum_l \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.\,l}^{(k)}\right) \cdot a_{li}. \quad (3.4)$$

On the other hand having changed the order of summation, for all $s = \overline{1, n-1}$ we have

$$d_s = \sum_{\beta\in J_{s,\,n}\{i\}} \det\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta} = \sum_{\beta\in J_{s,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta} = \sum_{\beta\in J_{s,\,n}\{i\}} \sum_l \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.\,l}^{(k)} a_{l\,i}\right)\right)_{\beta}^{\beta} = \sum_l \sum_{\beta\in J_{s,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.\,l}^{(k)}\right)\right)_{\beta}^{\beta} \cdot a_{l\,i}. \quad (3.5)$$

By substituting (3.4) and (3.5) in (3.3), and equating factors at $a_{l\,i}$ when $l = j$, we obtain (3.2). □

The following lemma can be proved similarly.

Lemma 3.5. *If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *is Hermitian with* $Ind\,\mathbf{A} = k$ *and* $\lambda \in \mathbb{R}$, *then*

$$\operatorname{rdet}_j(\lambda\mathbf{I} + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)}) = r_1^{(ij)}\lambda^{n-1} + r_2^{(ij)}\lambda^{n-2} + \ldots + r_n^{(ij)},$$

where $r_n^{(ij)} = \mathrm{rdet}_j(\mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)})$ *and* $r_s^{(ij)} = \sum\limits_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}$ *for all* $s=\overline{1,n-1}$ *and* $i,j=\overline{1,n}$.

Theorem 3.6. *If* $\mathbf{A}\in \mathrm{M}\,(n,\mathbb{H})$ *is Hermitian with* $Ind\,\mathbf{A}=k$ *and* $\mathrm{rank}\,\mathbf{A}^{k+1}=\mathrm{rank}\,\mathbf{A}^{k}=r$, *then the Drazin inverse* $\mathbf{A}^D=\left(a_{ij}^D\right)\in\mathbb{H}^{n\times n}$ *possess the following determinantal representations:*

$$a_{ij}^D=\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{k}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}\right|},\tag{3.6}$$

or

$$a_{ij}^D=\frac{\sum\limits_{\alpha\in I_{r,n}\{j\}}\mathrm{rdet}_j\left((\mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}}\left|(\mathbf{A}^{k+1})_{\alpha}^{\alpha}\right|}.\tag{3.7}$$

Proof. At first we prove (3.6). By Theorem 3.1, $\mathbf{A}^D=\lim\limits_{\lambda\to 0}\left(\lambda\mathbf{I}_n+\mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k$. The matrix $\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)\in\mathbb{H}^{n\times n}$ is a full-rank Hermitian matrix. Taking into account Theorem 2.9, it has an inverse which can be represented as a left inverse,

$$\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)^{-1}=\frac{1}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)}\begin{pmatrix}L_{11}&L_{21}&\ldots&L_{n1}\\L_{12}&L_{22}&\ldots&L_{n2}\\\ldots&\ldots&\ldots&\ldots\\L_{1n}&L_{2n}&\ldots&L_{nn}\end{pmatrix},$$

where L_{ij} is a left ij-th cofactor of a matrix $\lambda\mathbf{I}+\mathbf{A}^{k+1}$. Then, we have

$$\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k=$$
$$=\frac{1}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)}\begin{pmatrix}\sum\limits_{s=1}^{n}L_{s1}a_{s1}^{(k)}&\sum\limits_{s=1}^{n}L_{s1}a_{s2}^{(k)}&\ldots&\sum\limits_{s=1}^{n}L_{s1}a_{sn}^{(k)}\\\sum\limits_{s=1}^{n}L_{s2}a_{s1}^{(k)}&\sum\limits_{s=1}^{n}L_{s2}a_{s2}^{(k)}&\ldots&\sum\limits_{s=1}^{n}L_{s2}a_{sn}^{(k)}\\\ldots&\ldots&\ldots&\ldots\\\sum\limits_{s=1}^{n}L_{sn}a_{s1}^{(k)}&\sum\limits_{s=1}^{n}L_{sn}a_{s2}^{(k)}&\ldots&\sum\limits_{s=1}^{n}L_{sn}a_{sn}^{(k)}\end{pmatrix}.$$

By using the definition of a left cofactor, we obtain

$$\mathbf{A}^D=\lim_{\lambda\to 0}\begin{pmatrix}\frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.1}\left(\mathbf{a}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \dots & \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.1}\left(\mathbf{a}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)}\\ \dots & \dots & \dots\\ \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.n}\left(\mathbf{a}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \dots & \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.n}\left(\mathbf{a}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)}\end{pmatrix}. \tag{3.8}$$

By Theorem 2.13, we have

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{m+1}\right)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_n,$$

where $d_s=\sum\limits_{\beta\in J_{s,\,n}}\left|\left(\mathbf{A}^{k+1}\right)\,{}^{\beta}_{\beta}\right|$ is a sum of principal minors of $\mathbf{A}^{k+1}$ of order s for all $s=\overline{1,n-1}$ and $d_n=\det\mathbf{A}^{k+1}$.

Since $\operatorname{rank}\mathbf{A}^{k+1}=\operatorname{rank}\mathbf{A}^k=r$, then $d_n=d_{n-1}=\ldots=d_{r+1}=0$. It follows that $\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_r\lambda^{n-r}$.

Using (3.2), we have

$$\mathrm{cdet}_i\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)=c_1^{(ij)}\lambda^{n-1}+c_2^{(ij)}\lambda^{n-2}+\ldots+c_n^{(ij)}$$

for all $i,j=\overline{1,n}$, where $c_s^{(ij)}=\sum\limits_{\beta\in J_{s,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)\,{}^{\beta}_{\beta}$ for all $s=\overline{1,n-1}$ and $c_n^{(ij)}=\mathrm{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)$. We shall prove that $c_k^{(ij)}=0$, when $k\geq r+1$ for all $i,j=\overline{1,n}$.

Since by Lemma 3.2, $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\leq r$, then the matrix $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)$ has no more r right-linearly independent columns. Consider $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)\,{}^{\beta}_{\beta}$, when $\beta\in J_{s,n}\{i\}$. This is a principal submatrix of $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)$ of order $s\geq r+1$. Deleting both its i-th row and column, we obtain a principal submatrix of order $s-1$ of $\mathbf{A}^{k+1}$. We denote it by $\mathbf{M}$. The following cases are possible.

- Let $s=r+1$ and $\det\mathbf{M}\neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)\,{}^{\beta}_{\beta}$ keeps their right-linear independence.

Hence, they are basis in the matrix $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$, and the i-th column is the right linear combination of its basis columns. From this by Theorem 2.8, we get $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$, when $\beta\in J_{s,n}\{i\}$ and $s=r+1$.

- If $s=r+1$ and $\det\mathbf{M}=0$, then p, ($p<s$), columns are basis in $\mathbf{M}$ and in $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. So, by Theorems 2.11 and 2.8 we obtain $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$ as well.
- If $s>r+1$, then from Theorem 2.11 it follows that $\det\mathbf{M}=0$ and p, ($p<r$), columns are basis in the both matrices $\mathbf{M}$ and $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. Therefor, by Theorem 2.8, we have $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$.

Thus, in all cases, $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$, when $\beta\in J_{s,n}\{i\}$ and $r+1\leq s<n$. From here, if $r+1\leq s<n$, then

$$c_s^{(ij)}=\sum_{\beta\in J_{s,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0,$$

and $c_n^{(ij)}=\mathrm{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)=0$ for all $i,j=\overline{1,n}$.

Hence, $\mathrm{cdet}_i\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.\,j}^{(k)}\right)=c_1^{(ij)}\lambda^{n-1}+c_2^{(ij)}\lambda^{n-2}+\ldots+c_r^{(ij)}\lambda^{n-r}$ for all $i,j=\overline{1,n}$. By substituting these values in the matrix from (3.8), we obtain

$$\mathbf{A}^D=\lim_{\lambda\to0}\begin{pmatrix}\frac{c_1^{(11)}\lambda^{n-1}+\ldots+c_r^{(11)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{c_1^{(1n)}\lambda^{n-1}+\ldots+c_r^{(1n)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}}\\ \ldots & \ldots & \ldots\\ \frac{c_1^{(n1)}\lambda^{n-1}+\ldots+c_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{c_1^{(nn)}\lambda^{n-1}+\ldots+c_r^{(nn)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}}\end{pmatrix}=$$
$$\begin{pmatrix}\frac{c_r^{(11)}}{d_r} & \ldots & \frac{c_r^{(1n)}}{d_r}\\ \ldots & \ldots & \ldots\\ \frac{c_r^{(n1)}}{d_r} & \ldots & \frac{c_r^{(nn)}}{d_r}\end{pmatrix},$$

where $c_r^{(ij)} = \sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1} \right)_{.\,i} \left(\mathbf{a}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$ and $d_r = \sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1} \right) {}_{\beta}^{\beta} \right|$.

Thus, we obtained the determinantal representation of $\mathbf{A}^D$ by (3.6).

The determinantal representation of $\mathbf{A}^D$ by (3.7) can be proved similarly. □

In the following corollaries we introduce determinantal representations of the group inverse $\mathbf{A}^g$ and the projection matrices $\mathbf{A}^D\mathbf{A}$ and $\mathbf{A}\mathbf{A}^D$, respectively.

Corollary 3.1. *If* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank} \mathbf{A}^2 = \operatorname{rank} \mathbf{A} = r \le n$ *for a Hermitian matrix* $\mathbf{A} \in \mathbb{H}^{n \times n}$*, then the group inverse* $\mathbf{A}^g = \left(a_{ij}^g \right)_{n \times n}$ *possess the following determinantal representations:*

$$a_{ij}^g = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^2 \right)_{.\,i} \left(\mathbf{a}_{.j} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^2 \right) {}_{\beta}^{\beta} \right|},$$

or

$$a_{ij}^g = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{A}^2)_{j\,.} (\mathbf{a}_{i.}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| (\mathbf{A}^2) {}_{\alpha}^{\alpha} \right|}.$$

Proof. The proof follows immediately from Theorem 3.6 in view of $k = 1$. □

Corollary 3.2. *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^k = r \le n$ *for a Hermitian matrix* $\mathbf{A} \in \mathbb{H}^{n \times n}$*, then*

$$\mathbf{A}^D\mathbf{A} = \left(\frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1} \right)_{.\,i} \left(\mathbf{a}_{.j}^{(k+1)} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1} \right) {}_{\beta}^{\beta} \right|} \right)_{n \times n}, \tag{3.9}$$

and

$$\mathbf{A}\mathbf{A}^D = \left(\frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{A}^{k+1})_{j\,.} (\mathbf{a}_{i.}^{(k+1)}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| (\mathbf{A}^{k+1}) {}_{\alpha}^{\alpha} \right|} \right)_{n \times n}. \tag{3.10}$$

Proof. At first we prove (3.9). Let $\mathbf{A}^D\mathbf{A} = (v_{ij})_{n\times n}$. Using (3.6) for all $i,j=\overline{1,n}$, we have

$$v_{i\,j} = \sum_s \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_\beta^\beta}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)_\beta^\beta\right|}\cdot a_{s\,j} =$$

$$\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\sum\limits_s \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\cdot a_{s\,j}\right)\right)_\beta^\beta}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)_\beta^\beta\right|} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k+1)}\right)\right)_\beta^\beta}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)_\beta^\beta\right|}.$$

By analogy can be proved (3.10), using the determinantal representation of the Drazin inverse by (3.7). □

3.2. Determinantal Representations of the Drazin Inverse for an Arbitrary Matrix

For an arbitrary matrix $\mathbf{A}\in \mathrm{M}(n,\mathbb{H})$ with $Ind\,\mathbf{A}=k$ and $\mathrm{rank}\,\mathbf{A}^{k+1} = \mathrm{rank}\,\mathbf{A}^k = r$, we can not apply the method proposed for Hermitian matrices primarily because the lemma on the characteristic polynomial for an arbitrary quaternion matrix is not possible in general. We shall use a basic equality on the Drazin inverse and determinantal representations of the Moore-Penrose inverse by the following proposition and theorem, respectively.

Proposition 3.1. *[30] If $Ind(\mathbf{A})=k$, then*

$$\mathbf{A}^D = \mathbf{A}^k(\mathbf{A}^{2k+1})^+\mathbf{A}^k.$$

Theorem 3.7. *[10] If $\mathbf{A}\in\mathbb{H}_r^{m\times n}$, then the Moore-Penrose inverse $\mathbf{A}^+ = \left(a_{ij}^+\right)\in\mathbb{H}^{n\times m}$ possess the following determinantal representations:*

$$a_{ij}^+ = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^*\right)\right)_\beta^\beta}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^*\mathbf{A}\right)_\beta^\beta\right|}, \tag{3.11}$$

or

$$a_{ij}^+ = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}_{i.}^*)\right)_\alpha^\alpha}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^*)_\alpha^\alpha\right|}, \tag{3.12}$$

for all $i = \overline{1,n}$, $j = \overline{1,m}$.

Therefore, an entry of the Drazin inverse of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is

$$a_{ij}^{D} = \sum_{s=1}^{n} \sum_{t=1}^{n} a_{it}^{(k)} \left(a_{ts}^{(2k+1)}\right)^{+} a_{sj}^{(k)} \tag{3.13}$$

for all $i, j = \overline{1,n}$. Denote by $\hat{\mathbf{a}}_{.s}$ the s-th column of $(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k} =: \hat{\mathbf{A}} = (\hat{a}_{ij}) \in \mathbb{H}^{n\times n}$ for all $s = \overline{1,n}$. It follows from $\sum\limits_{s} \left(\mathbf{a}_{.s}^{(2k+1)}\right)^{*} a_{sj}^{(k)} = \hat{\mathbf{a}}_{.j}$ and (3.11) that

$$\sum_{s=1}^{n} \left(a_{ts}^{(2k+1)}\right)^{+} a_{sj}^{(k)} =$$

$$\sum_{s=1}^{n} \frac{\sum\limits_{\beta\in J_{r,n}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)\right)_{.t}\left(\mathbf{a}_{.s}^{(2k+1)}\right)^{*}\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} \left|\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|} \cdot a_{sj}^{(k)} =$$

$$\frac{\sum\limits_{\beta\in J_{r,n}\{t\}} \mathrm{cdet}_t \left(\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)_{.t}\left(\hat{\mathbf{a}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} \left|\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}.$$

So, the Drazin inverse $\mathbf{A}^{D}$ possess the following determinantal representation,

$$a_{ij}^{D} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta\in J_{r,n}\{t\}} \mathrm{cdet}_t \left(\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)_{.t}\left(\hat{\mathbf{a}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} \left|\left(\mathbf{A}^{2k+1}\right)^{*}\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}, \tag{3.14}$$

for all $i, j = \overline{1,n}$.

Denote by $\check{\mathbf{a}}_{t.}$ the t-th row of $\mathbf{A}^{k}(\mathbf{A}^{2k+1})^{*} =: \check{\mathbf{A}} = (\check{a}_{ij}) \in \mathbb{H}^{n\times n}$ for all $t = \overline{1,n}$. It follows from $a_{it}^{(k)} \sum\limits_{t} \left(\mathbf{a}_{t.}^{(2k+1)}\right)^{*} = \check{\mathbf{a}}_{i.}$ and (3.11) that

$$\sum_{t=1}^{n} a_{it}^{(k)} \left(a_{ts}^{(2k+1)}\right)^{+} = \sum_{t=1}^{n} a_{it}^{(k)} \cdot \frac{\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right)_{.s} \left(\mathbf{a}_{.t}^{(2k+1)} \right)^{*} \right) {\,}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right) {\,}_{\alpha}^{\alpha} \right|} = \frac{\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right)_{.s} \left(\check{\mathbf{a}}_{i\,.} \right) \right) {\,}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right) {\,}_{\alpha}^{\alpha} \right|}$$

Therefore, the Drazin inverse $\mathbf{A}^{D}$ possess the following determinantal representation,

$$a_{ij}^{D} = \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right)_{.s} \left(\check{\mathbf{a}}_{i\,.} \right) \right) {\,}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^{*} \right) {\,}_{\alpha}^{\alpha} \right|}, \qquad (3.15)$$

for all $i, j = \overline{1, n}$. Thus, we have proved the following theorem.

Theorem 3.8. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^{k} = r$*, then the Drazin inverse* $\mathbf{A}^{D}$ *possess the determinantal representations (3.14) and (3.15).*

Using obtained determinantal representations (3.14) and (3.15), we have the following corollaries. Their proofs are similarly to the proofs of Corollaries **??** and **??**, respectively.

Corollary 3.3. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank} \mathbf{A}^{2} = \operatorname{rank} \mathbf{A}^{2} = r$*, then the group inverse* $\mathbf{A}^{g} = \left(a_{ij}^{g}\right)_{n \times n}$ *possess the following determinantal representations*

$$a_{ij}^{g} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\mathbf{A}^{3}\right)^{*} \left(\mathbf{A}^{3}\right)_{.\,t} \left(\hat{\mathbf{a}}_{.\,j} \right) \right) {\,}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{3}\right)^{*} \left(\mathbf{A}^{3}\right) {\,}_{\beta}^{\beta} \right|},$$

$$a_{ij}^{g} = \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{3} \left(\mathbf{A}^{3}\right)^{*} \right)_{.s} \left(\check{\mathbf{a}}_{i\,.} \right) \right) {\,}_{\alpha}^{\alpha} \right) a_{sj}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left(\mathbf{A}^{3} \left(\mathbf{A}^{3}\right)^{*} \right) {\,}_{\alpha}^{\alpha} \right|},$$

for all $i, j = \overline{1, n}$.

Corollary 3.4. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^{k} = r$*, then*

$$\mathbf{A}^D \mathbf{A} = \left(\frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1} \right)^* \right)_{.s} (\check{\mathbf{a}}_{i\,.}) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k+1)}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1} \right)^* \right) {}_{\alpha}^{\alpha} \right|} \right),$$

and

$$\mathbf{A}\mathbf{A}^D = \left(\frac{\sum\limits_{t=1}^{n} a_{it}^{(k+1)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\mathbf{A}^{2k+1} \right)^* \left(\mathbf{A}^{2k+1} \right)_{.\,t} (\hat{\mathbf{a}}_{.\,j}) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1} \right)^* \left(\mathbf{A}^{2k+1} \right) {}_{\beta}^{\beta} \right|} \right),$$

where $\mathbf{A}^k (\mathbf{A}^{2k+1})^* = \check{\mathbf{A}} = (\check{a}_{ij})$ *and* $(\mathbf{A}^{2k+1})^* \mathbf{A}^k = \hat{\mathbf{A}} = (\hat{a}_{ij})$.

4. Cramer's Rule of the Drazin Inverse Solutions of Some Matrix Equations

One of the main applications of the determinantal representation of an inverse matrix by the classical adjoint matrix is the Cramer rule. In this section we shall show that the obtained determinantal representations give the exact analogues of Cramer's rule for the Drazin inverse solutions of some matrix equations.

For an arbitrary matrix over the quaternion skew field, $\mathbf{A} \in \mathbb{H}^{m \times n}$, we denote by

- $\mathcal{R}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \ \mathbf{y} = \mathbf{A}\mathbf{x}, \ \mathbf{x} \in \mathbb{H}^n\}$, the column right space of $\mathbf{A}$,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \ \mathbf{A}\mathbf{x} = 0\}$, the right null space of $\mathbf{A}$,
- $\mathcal{R}_l(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \ \mathbf{y} = \mathbf{x}\mathbf{A}, \ \mathbf{x} \in \mathbb{H}^m\}$, the column left space of $\mathbf{A}$,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \ \mathbf{x}\mathbf{A} = 0\}$, the left null space of $\mathbf{A}$.

Consider a matrix equation

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{4.1}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times m}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{A} = k_1$ and $Ind\,\mathbf{B} = k_2$.

It's well known (see, e.g., [12]) that the equation (4.1) with restrictions

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r(\mathbf{A}^{k_1}),\ \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r(\mathbf{B}^{k_2}),$$
$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{A}^{k_1}),\ \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l(\mathbf{B}^{k_2}),$$

has a unique solution $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D$.

4.1. The Case of Hermitian Matrices

Denote $\mathbf{A}^{k_1}\mathbf{D}\mathbf{B}^{k_2} =: \widetilde{\mathbf{D}} = (\widetilde{d}_{ij}) \in \mathbb{H}^{n\times m}$.

Theorem 4.1. *If* $\mathbf{A}$, $\mathbf{B}$ *are Hermitian,* $\operatorname{rank}\mathbf{A}^{k_1+1} = \operatorname{rank}\mathbf{A}^{k_1} = r_1 \le n$ *for* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, and* $\operatorname{rank}\mathbf{B}^{k_2+1} = \operatorname{rank}\mathbf{B}^{k_2} = r_2 \le m$ *for* $\mathbf{B} \in \mathbb{H}^{m\times m}$*, then, for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ *of (4.1), we have*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{.\,j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)^{\alpha}_{\alpha}\right|}, \tag{4.2}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j\,.}\left(\mathbf{d}^{\,\mathbf{A}}_{i\,.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)^{\alpha}_{\alpha}\right|}, \tag{4.3}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j.}\left(\tilde{\mathbf{d}}_{l.}\right)\right)^{\alpha}_{\alpha}\right) \in \mathbb{H}^{n\times 1},\quad l = \overline{1,n}, \tag{4.4}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.i}\left(\tilde{\mathbf{d}}_{.t}\right)\right)^{\beta}_{\beta}\right) \in \mathbb{H}^{1\times m},\quad t = \overline{1,m}, \tag{4.5}$$

are the column vector and the row vector, respectively. $\tilde{\mathbf{d}}_{i.}$ *and* $\tilde{\mathbf{d}}_{.j}$ *are the* i*-th row and the* j*-th column of* $\widetilde{\mathbf{D}}$ *for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*.*

Proof. An entry of the Drazin inverse solution $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ is

$$x_{ij} = \sum_{s=1}^{m}\left(\sum_{t=1}^{n} a_{it}^D d_{ts}\right) b_{sj}^D \tag{4.6}$$

for all $i = \overline{1,n}$, $j = \overline{1,m}$, where by Theorem 3.6

$$a_{it}^D = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}, \tag{4.7}$$

$$b_{sj}^D = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j\,.}\left(\mathbf{b}_{s.}^{(k_2)}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|}. \tag{4.8}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the s-th column of $\mathbf{A}^{k_1}\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times m}$ for all $s = \overline{1,m}$. It follows from $\sum\limits_{t}\mathbf{a}_{.\,t}^{(k_1)}d_{ts} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{t=1}^{n} a_{it}^D d_{ts} = \sum_{t=1}^{n}\frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}\cdot d_{ts} =$$

$$\frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}}\sum\limits_{t=1}^{n} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta}\cdot d_{ts}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are respectively the unit row-vector and the unit column-vector whose components are 0, except the s-th components, which are 1. Substituting (4.7) and (4.8) in (4.6), we obtain

$$x_{ij} = \sum_{s=1}^{m}\frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}\frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j\,.}\left(\mathbf{b}_{s.}^{(k_2)}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{t=1}^{n}\mathbf{e}_{.\,t}\hat{d}_{ts},\; \mathbf{b}_{s.}^{(k_2)} = \sum_{l=1}^{m} b_{sl}^{(k_2)}\mathbf{e}_{l.},\; \sum_{s=1}^{m}\hat{d}_{ts}b_{sl}^{(k_2)} = \widetilde{d}_{tl},$$

then we have

$$x_{ij} = \frac{\sum\limits_{s=1}^{m}\sum\limits_{l=1}^{m}\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)\,{}^{\beta}_{\beta}\hat{d_{ts}}b_{sl}^{(k_2)}\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,{}^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|(\mathbf{B}^{k_2+1})\,{}^{\alpha}_{\alpha}\right|} =$$

$$\frac{\sum\limits_{t=1}^{n}\sum\limits_{l=1}^{m}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,l})\right)\,{}^{\beta}_{\beta}\,\widetilde{d}_{tl}\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{t.})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,{}^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|(\mathbf{B}^{k_2+1})\,{}^{\alpha}_{\alpha}\right|}. \tag{4.9}$$

Denote by

$$d_{il}^{\mathbf{A}} := \sum_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\tilde{\mathbf{d}}_{.\,l}\right)\right)^{\beta}_{\beta} = \sum_{t=1}^{n}\sum_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)^{\beta}_{\beta}\widetilde{d}_{tl}$$

the l-th component of a row-vector $\mathbf{d}^{\mathbf{A}}_{i\,.} = (d^{\mathbf{A}}_{i1}, ..., d^{\mathbf{A}}_{im})$ for all $l = \overline{1, m}$. Substituting it in (4.9), we have

$$x_{ij} = \frac{\sum\limits_{l=1}^{m} d_{it}^{\mathbf{A}}\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)\,{}^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|(\mathbf{B}^{k_2+1})^{\alpha}_{\alpha}\right|}.$$

Since $\sum\limits_{l=1}^{m} d_{il}^{\mathbf{A}}\mathbf{e}_{l.} = \mathbf{d}^{\mathbf{A}}_{i\,.}$, then it follows (4.3).

If we denote by

$$d_{tj}^{\mathbf{B}} := \sum_{l=1}^{m}\widetilde{d}_{tl}\sum_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)^{\alpha}_{\alpha} = \sum_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\widetilde{\mathbf{d}}_{t.})\right)^{\alpha}_{\alpha},$$

the t-th component of a column-vector $\mathbf{d}^{\mathbf{B}}_{.\,j} = (d^{\mathbf{B}}_{1j}, ..., d^{\mathbf{B}}_{nj})^T$ for all $t = \overline{1, n}$ and substitute it in (4.9), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)^{\beta}_{\beta} d_{tj}^{\mathbf{B}}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|(\mathbf{A}^{k_1+1})^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|(\mathbf{B}^{k_2+1})^{\alpha}_{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{n} \mathbf{e}_{.t} d_{tj}^{\mathbf{B}} = \mathbf{d}_{.j}^{\mathbf{B}}$, then it follows (4.2). □

Consider a matrix equation

$$\mathbf{A}\mathbf{X} = \mathbf{D}, \tag{4.10}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, $\mathbf{A}$ is Hermitian, and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{A} = k$. We denote $\mathbf{A}^k\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times m}$. Putting $\mathbf{B} = \mathbf{I}$ in (4.1) we evidently obtain the following corollary.

Corollary 4.1. *If* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^k = r \leq n$ *for Hermitian* $\mathbf{A} \in \mathbb{H}^{n\times n}$, *then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D} = (x_{ij})$ *of (4.10), we have*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.\,i} \left(\hat{\mathbf{d}}_{.j}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right) {}_{\beta}^{\beta}\right|}, \tag{4.11}$$

where $\hat{\mathbf{d}}_{.j}$ *is the* j*-th column of* $\hat{\mathbf{D}}$ *for* $j = \overline{1, m}$.

Consider a matrix equation

$$\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{4.12}$$

where $\mathbf{B} \in \mathbb{H}^{m\times m}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, $\mathbf{B}$ is Hermitian and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{B} = k$ and denote $\mathbf{D}\mathbf{B}^k =: \check{\mathbf{D}} = (\check{d}_{ij}) \in \mathbb{H}^{n\times m}$. Putting $\mathbf{A} = \mathbf{I}$ in (4.1) we evidently obtain the following corollary.

Corollary 4.2. *If* $\operatorname{rank} \mathbf{B}^{k+1} = \operatorname{rank} \mathbf{B}^k = r \leq n$ *for Hermitian* $\mathbf{B} \in \mathbb{H}^{n\times n}$, *then for the Drazin inverse solution* $\mathbf{X} = \mathbf{D}\mathbf{B}^D =: (x_{ij})$ *of (4.12), we have for* $i = \overline{1, n}$, $j = \overline{1, m}$

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j \left(\left(\mathbf{B}^{k+1}\right)_{j\,.} \left(\check{\mathbf{d}}_{i\,.}\right)\right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{B}^{k+1}\right) {}_{\alpha}^{\alpha}\right|}. \tag{4.13}$$

where $\check{\mathbf{d}}_{i.}$ *is the* i*-th row of* $\check{\mathbf{D}}$ *for* $i = \overline{1, n}$.

4.2. The Case of Arbitrary Matrices

Using the determinantal representations (3.14) for arbitrary $\mathbf{A} \in \mathbb{H}^{n\times n}$ and (3.15) for arbitrary $\mathbf{B} \in \mathbb{H}^{m\times m}$, we obtain the following theorem and corollaries by analogy to Theorem 4.1, Corollaries 4.1 and 4.2, respectively.

Denote $(\mathbf{A}^{2k_1+1})^*\mathbf{A}^{k_1}\mathbf{D}\mathbf{B}^{k_2}(\mathbf{B}^{2k_2+1})^* =: \widetilde{\mathbf{D}} = (\widetilde{d}_{ij}) \in \mathbb{H}^{n\times m}$.

Theorem 4.2. *If* $\operatorname{rank}\mathbf{A}^{k_1+1} = \operatorname{rank}\mathbf{A}^{k_1} = r_1 \leq n$ *for* $\forall\mathbf{A} \in \mathbb{H}^{n\times n}$*, and* $\operatorname{rank}\mathbf{B}^{k_2+1} = \operatorname{rank}\mathbf{B}^{k_2} = r_2 \leq m$ *for* $\forall\mathbf{B} \in \mathbb{H}^{m\times m}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ *of (4.1), we have*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k_1)} \sum\limits_{\beta\in J_{r_1,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{.\,t}\left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,\,m}}\left|\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{\alpha}^{\alpha}\right|} \tag{4.14}$$

or

$$x_{ij} = \frac{\sum\limits_{s=1}^{m}\left(\sum\limits_{\alpha\in I_{r_2,\,m}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{.\,s}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k_2)}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,\,m}}\left|\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{\alpha}^{\alpha}\right|}, \tag{4.15}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{s=1}^{m}\left(\sum_{\alpha\in I_{r_2,\,m}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{.\,s}\left(\widetilde{\mathbf{d}}_{q.}\right)\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k_2)}\right) \in \mathbb{H}^{n\times 1}, \tag{4.16}$$

and

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{t=1}^{n} a_{it}^{(k_1)} \sum_{\beta\in J_{r_1,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{.\,t}\left(\widetilde{\mathbf{d}}_{.p}\right)\right)_{\beta}^{\beta}\right) \in \mathbb{H}^{1\times m}, \tag{4.17}$$

and $\tilde{\mathbf{d}}_{p.}$*,* $\tilde{\mathbf{d}}_{.q}$ *are the* p*-th row and the* q*-th column of* $\widetilde{\mathbf{D}}$*, respectively, for all* $q = \overline{1,n}$*,* $p = \overline{1,m}$*.*

Corollary 4.3. *If* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, then, for the*

Drazin inverse solution $\mathbf{X} = \mathbf{A}^D\mathbf{D} = (x_{ij})$ *of (4.10), we have*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,t} \left(\hat{\mathbf{d}}_{\,.\,j}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|}, \tag{4.18}$$

where $\hat{\mathbf{d}}_{.j}$ *is the* j*-th column of* $\hat{\mathbf{D}} = (\mathbf{A}^{2k_1+1})^*\mathbf{A}^{k_1}\mathbf{D}$ *for all* $j = \overline{1, m}$, $i = \overline{1, n}$.

Corollary 4.4. *If* $\operatorname{rank} \mathbf{B}^{k+1} = \operatorname{rank} \mathbf{B}^k = r \le m$ *for* $\mathbf{B} \in \mathbb{H}^{m\times m}$*, then, for the Drazin inverse solution* $\mathbf{X} = \mathbf{D}\mathbf{B}^D =: (x_{ij})$ *of (4.12), we have*

$$x_{ij} = \frac{\sum\limits_{s=1}^{m} \left(\sum\limits_{\alpha \in I_{r,\,m}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{B}^{2k+1} \left(\mathbf{B}^{2k+1}\right)^* \right)_{.s} \left(\check{\mathbf{d}}_{\,i\,.}\right) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \left(\mathbf{B}^{2k+1} \left(\mathbf{B}^{2k+1}\right)^* \right) {}_{\alpha}^{\alpha} \right|}, \tag{4.19}$$

where $\check{\mathbf{d}}_{i.}$ *is the* i*-th row of* $\check{\mathbf{D}} = \mathbf{D}\mathbf{B}^{k_2}(\mathbf{B}^{2k_2+1})^*$ *for all* $i = \overline{1, n}$, $j = \overline{1, m}$.

4.3. Examples

In this section, we give examples to illustrate our results.

1. Let us consider the matrix equation

$$\mathbf{AXB} = \mathbf{D}, \tag{4.20}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & i \\ k & 1 \\ 1 & j \end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{pmatrix}$, $\det \mathbf{A} = \det \mathbf{A}^2 = 0$, and $\det \begin{pmatrix} 1 & k \\ -k & 2 \end{pmatrix} = 1$, $\det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} = 2$, then, by Theorem 2.11, $Ind\, \mathbf{A} = 1$ and $r_1 = \operatorname{rank} \mathbf{A} = 2$. Similarly, since $\mathbf{B}^2 = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$, then $Ind\, \mathbf{B} = 1$ and $r_2 = \operatorname{rank} \mathbf{B} = 1$.

Because $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, we shall find the Drazin inverse solution $\mathbf{X}^d = (x_{ij}^d)$ of (4.20) by the equations (4.2)-(4.4). We have $\sum\limits_{\alpha\in I_{1,\,2}} \left|\left(\mathbf{B}^2\right)\,{}_{\alpha}^{\alpha}\right| = 2+2=4$,

$$\sum_{\beta\in J_{2,\,3}} \left|\left(\mathbf{A}^2\right)\,{}_{\beta}^{\beta}\right| = \det\begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} + \det\begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix} + \det\begin{pmatrix} 6 & 4j \\ -4j & 3 \end{pmatrix} = 4.$$

Since

$$\widetilde{\mathbf{D}} = \mathbf{ADB} = \begin{pmatrix} 1-i & 1+i \\ -i+j & 1-k \\ 1+i & -1+i \end{pmatrix},$$

then by (4.4)

$$\mathbf{d}_{.\,j}^{\mathbf{B}} = \left(\sum_{\alpha\in I_{1,2}\{j\}} \operatorname{rdet}_j \left(\left(\mathbf{B}^2\right)_{1.} \left(\tilde{\mathbf{d}}_{l.}\right)\right)_{\alpha}^{\alpha} \right) \in \mathbb{H}^{n\times 1}, \quad l=1,2,3; \;\; j=1,2.$$

Thus, we have

$$\mathbf{d}_{.1}^{\mathbf{B}} = \begin{pmatrix} 1-i \\ -i+j \\ 1+i \end{pmatrix}, \quad \mathbf{d}_{.2}^{\mathbf{B}} = \begin{pmatrix} 1+i \\ 1-k \\ -1+i \end{pmatrix}.$$

So

$$\left(\mathbf{A}^2\right)_{.\,1}\left(\mathbf{d}_{.1}^{\mathbf{B}}\right) = \begin{pmatrix} 1-i & 4k & -3i \\ -i+j & 6 & 4j \\ 1+i & -4j & 3 \end{pmatrix},$$

and finally we obtain

$$x_{11}^d = \frac{\sum\limits_{\beta\in J_{2,\,3}\{1\}} \operatorname{cdet}\left(\left(\mathbf{A}^2\right)_{.\,1}\left(\mathbf{d}_{.1}^{\mathbf{B}}\right)\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{2,3}} \left|(\mathbf{A}^2)_{\beta}^{\beta}\right| \sum\limits_{\alpha\in I_{1,2}} |(\mathbf{B}^2)_{\alpha}^{\alpha}|} =$$

$$\frac{1}{16}\left(\operatorname{cdet}_1\begin{pmatrix} 1-i & 4k \\ -i+j & 6 \end{pmatrix} + \operatorname{cdet}_1\begin{pmatrix} 1-i & -3i \\ 1+i & 3 \end{pmatrix}\right) = \frac{3-i+2j}{8}.$$

Similarly,

$$x_{12}^d = \frac{1}{16}\left(\mathrm{cdet}_1\begin{pmatrix}1+i & 4k\\ 1-k & 6\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}1+i & -3i\\ 1+i & 3\end{pmatrix}\right) = \frac{1+3i-2k}{8},$$

$$x_{21}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i\\ -4k & -i+j\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}-i+j & 4j\\ 1+i & 3\end{pmatrix}\right) = \frac{-3i-j+4k}{8},$$

$$x_{22}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+i\\ -4k & 1-k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}1-k & 4j\\ -1+i & 3\end{pmatrix}\right) = \frac{3+4j+k}{8},$$

$$x_{31}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i\\ 3i & 1+i\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & -i+j\\ -4j & 1+i\end{pmatrix}\right) = \frac{1+3i+2k}{8},$$

$$x_{32}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+i\\ 3i & -1+i\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 1-k\\ -4j & -1+i\end{pmatrix}\right) = \frac{-3+i+2j}{8}.$$

So,

$$\mathbf{X}^d = \frac{1}{8}\begin{pmatrix} 3-i+2j & 1+3i-2k\\ -3i-j+4k & 3+4j+k\\ 1+3i+2k & -3+i+2j \end{pmatrix}$$

is the Drazin inverse solution of (4.20).

2. Let us consider the matrix equation

$$\mathbf{AX} = \mathbf{D}, \tag{4.21}$$

where

$$\mathbf{A} = \begin{pmatrix} i & j & k\\ 1 & -k & j\\ 1 & 0 & i \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} 1 & i\\ k & 1\\ 1 & j \end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix} -1+j+k & -i+k & -1\\ i+j-k & -1+j & i\\ 2i & j & -1+k \end{pmatrix}$, $\mathbf{A}^*\mathbf{A} = \begin{pmatrix} 3 & -2k & i+2j\\ 2k & 2 & -2i\\ -i+2j & 2i & 3 \end{pmatrix}$,

$$(\mathbf{A}^2)^*\mathbf{A}^2 = \begin{pmatrix} 10 & 2+2j-6k & 2+2i+4j+2k\\ 2-2j+6k & 5 & -3i+j+2k\\ 2-2i-4j-2k & 3-j-2k & 4 \end{pmatrix},$$

$$\det \mathbf{A}^*\mathbf{A} = \det(\mathbf{A}^2)^*\mathbf{A}^2 = 0, \det\begin{pmatrix} 3 & -2k \\ 2k & 2 \end{pmatrix} = 2,$$

$$\det\begin{pmatrix} 10 & 2+2j-6k \\ 2-2j+6k & 5 \end{pmatrix} = 6,$$

then, by Theorem 2.11, $Ind\,\mathbf{A} = 1$ and $r = \operatorname{rank}\mathbf{A} = 2$. We shall find the Drazin inverse solution $\mathbf{X}^d = (x_{ij}^d)$ of (4.21) by (4.18). Since

$$(\mathbf{A}^3)^*\mathbf{A}^3 = \begin{pmatrix} 23 & 2+3i+5j-17k & 8+4i+15j+2k \\ 2-3i-5j+17k & 15 & 3-13i+2j+5k \\ 8-4i-15j-2k & 3+13i-2j-5k & 15 \end{pmatrix},$$

then

$$\sum_{\beta\in J_{2,\,3}} \left|\left((\mathbf{A}^3)^*\mathbf{A}^3\right){}_{\beta}^{\beta}\right| = \det\begin{pmatrix} 23 & 2+3i+5j-17k \\ 2-3i-5j+17k & 15 \end{pmatrix}$$

$$+\det\begin{pmatrix} 15 & 3-13i+2j+5k \\ 3+13i-2j-5k & 15 \end{pmatrix}$$

$$+\det\begin{pmatrix} 23 & 8+4i+15j+2k \\ 8-4i-15j-2k & 15 \end{pmatrix} = 72.$$

Further,

$$\hat{\mathbf{D}} = (\mathbf{A}^3)^*\mathbf{A}\mathbf{D} = \begin{pmatrix} -11-9i-6j+2k & 9-6i-j \\ -5+5i-4j-10k & -1-2i-7j+6k \\ -10-4i+7j-3k & 3-4i-7j-4k \end{pmatrix},$$

and

$$\left((\mathbf{A}^3)^*\mathbf{A}^3\right)_{.\,1}\left(\hat{\mathbf{d}}_{.\,1}\right) = \begin{pmatrix} -11-9i-6j+2k & 2+3i+5j-17k & 8+4i+15j+2k \\ -5+5i-4j-10k & 15 & 3-13i+2j+5k \\ -10-4i+7j-3k & 3+13i-2j-5k & 15 \end{pmatrix}.$$

Therefore, finally we obtain

$$x_{11}^d = \frac{\sum\limits_{t=1}^{3} a_{1t} \sum\limits_{\beta \in J_{2,3}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{A}^3\right)^* \mathbf{A}^3\right)_{.\,t} \left(\hat{\mathbf{d}}_{\,.\,1}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left| \left(\left(\mathbf{A}^3\right)^* \mathbf{A}^3\right)_{\beta}^{\beta} \right|} =$$

$$\frac{i}{76}\left(\operatorname{cdet}_1 \begin{pmatrix} 1 & 2+3i+5j-17k \\ k & 5 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 1 & 8+4i+15j+2k \\ 1 & 15 \end{pmatrix}\right) +$$
$$\frac{j}{76}\left(\operatorname{cdet}_2 \begin{pmatrix} 23 & 1 \\ 2-3i-5j+17k & k \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & 3-13i+2j+5k \\ 1 & 15 \end{pmatrix}\right) +$$
$$\frac{k}{76}\left(\operatorname{cdet}_2 \begin{pmatrix} 23 & 1 \\ 8-4i-15j-2k & 1 \end{pmatrix} + \operatorname{cdet}_2 \begin{pmatrix} 15 & k \\ 3+13i-2j-5k & 1 \end{pmatrix}\right) =$$
$$\frac{1}{76}(7-17i+5j-3k)$$

Similarly,

$$x_{12}^d = \frac{i}{76}(13+29i-13j+13k) + \frac{j}{76}(37+3i+14j+18k) - \frac{k}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(-33-11i+3j-23k),$$
$$x_{21}^d = \frac{1}{76}(-5-9i-12j-4k) + \frac{k}{76}(-5+18i-5j+8k) + \frac{j}{76}(25+6i+28j-k) =$$
$$\frac{1}{76}(-49-13i+29j-15k),$$
$$x_{22}^d = \frac{1}{76}(13+29i-13j+13k) + \frac{k}{76}(37+3i+14j+18k) - \frac{j}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(-47+5i-17j+71k),$$
$$x_{31}^d = \frac{1}{76}(-5-9i-12j-4k) + \frac{0}{76}(-5+18i-5j+8k) + \frac{i}{76}(25+6i+28j-k) =$$
$$\frac{1}{76}(-11+16i-11j+24k),$$
$$x_{32}^d = \frac{1}{76}(13+29i-13j+13k) + \frac{0}{76}(37+3i+14j+18k) - \frac{i}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(34+22i-3j+55k).$$

Thus,we have the Drazin inverse solution of (4.21),

$$\mathbf{X}^d = \frac{1}{76}\begin{pmatrix} 7-17i+5j-3k & -33-11i+3j-23k \\ -49-13i+29j-15k & -47+5i-17j+71k \\ -11+16i-11j+24k & 34+22i-3j+55k \end{pmatrix}.$$

5. Applications of the Determinantal Representations of the Drazin Inverse to Some Differential Matrix Equations

In [40], applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients have been done. In [41], we recently have obtained determinantal representations of solutions of some singular differential complex-valued matrix equations. In this chapter we extend studies conducted in [41] from the complex field to the quaternion skew field.

5.1. Background for Quaternion-valued Differential Equations (QDE)

Consider a quaternion-valued function of real variable, $\mathbf{f} : \mathbb{R} \to \mathbb{H}$, ($t \in \mathbb{R}$ is a real variable), such that $\mathbf{f}(t) = f_0(t) + f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$. The first derivative of a quaternionic function $\mathbf{f}(t)$ with respect to the real variable t denote by,

$$\mathbf{f}'(t) := \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} = \frac{\mathrm{d}f_0(t)}{\mathrm{d}t} + \frac{\mathrm{d}f_1(t)}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}f_2(t)}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}f_3(t)}{\mathrm{d}t}\mathbf{k}.$$

It is easy to prove the following proposition on properties of the derivative of quaternionic functions.

Proposition 5.1. *If* $\mathbf{q} : \mathbb{R} \to \mathbb{H}$ *and* $\mathbf{r} : \mathbb{R} \to \mathbb{H}$ *are differentiable, then* $(\mathbf{q} \pm \mathbf{r})(t)$, $\mathbf{qr}(t)$ *and, for any integer* $n \geq 1$, $\mathbf{q}^n$ *are differentiable, and*

$$(\mathbf{q} \pm \mathbf{r})'(t) = \mathbf{q}'(t) \pm \mathbf{r}'(t), \tag{5.1}$$

$$(\mathbf{qr})'(t) = \mathbf{q}'(t)\mathbf{r}(t) + \mathbf{q}(t)\mathbf{r}'(t), \tag{5.2}$$

$$[\mathbf{q}^n(t)]' = \sum_{j=0}^{n-1} \mathbf{q}^j(t)\mathbf{q}'(t)\mathbf{q}^{n-j}(t). \tag{5.3}$$

If $f_i(t)$ for all $l = \overline{0,3}$ is integrable on $[a, b] \subset \mathbb{R}$, the $\mathbf{f}(t)$ is integrable and

$$\int_a^b \mathbf{f}(t)\mathrm{d}t = \int_a^b f_0(t)\mathrm{d}t + \int_a^b f_1(t)\mathrm{d}t\,\mathbf{i} + \int_a^b f_2(t)\mathrm{d}t\,\mathbf{j} + \int_a^b f_3(t)\mathrm{d}t\,\mathbf{k}.$$

Consider a matrix valued function $\mathbf{A}(t) = (\mathbf{a}_{ij}(t)) \in \mathbb{H}^{n\times n} \otimes \mathbb{R}$, where $\mathbf{a}_{ij}(t)$ are quaternion-valued functions with the real variable t for all $i, j = \overline{1, n}$. Then

$$\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{a}_{ij}(t)}{\mathrm{d}t}\right)_{n\times n}, \qquad \int_a^b \mathbf{A}(t)\mathrm{d}t = \left(\int_a^b \mathbf{a}_{ij}(t)\mathrm{d}t\right)_{n\times n}.$$

We need the exponential of $q \in \mathbb{H}$ that can be defined by putting,

$$\exp q = \sum_{n=0}^{\propto} \frac{q^n}{n!}. \tag{5.4}$$

From the definition of a quaternionic exponential (5.4), we evidently have the following properties.

Proposition 5.2. *If $q, r \in \mathbb{H}$ are such that $qr = rq$, then $\exp(q + r) = (\exp q)(\exp r)$.*

Proposition 5.3. *If $\mathbf{q} : \mathbb{R} \to \mathbb{H}$ is differentiable and $\mathbf{q}'(t)\mathbf{q}(t) = \mathbf{q}(t)\mathbf{q}'(t)$, then*

$$[\exp \mathbf{q}(t)]' = [\exp \mathbf{q}(t)]\, \mathbf{q}'(t).$$

In [42], the linear quaternion differential equations,

$$\mathbf{q}'(t) = \mathbf{a}(t)\mathbf{q}(t), \tag{5.5}$$

and

$$\mathbf{q}'(t) = \mathbf{q}(t)\mathbf{a}(t), \tag{5.6}$$

with the initial condition $q(t_0) = q_0$ have been considered and the following proposition has been derived.

Proposition 5.4. *Let $\mathbf{q}(t) = \mathbf{\Phi}_l(t)q_0$ and $\mathbf{q}(t) = q_0\mathbf{\Phi}_r(t)$ be solutions of (5.5) and (5.6), respectively. If*

$$\mathbf{a}(t)\int_{t_0}^t \mathbf{a}(\tau)\mathrm{d}\tau = \int_{t_0}^t \mathbf{a}(\tau)\mathrm{d}\tau\, \mathbf{a}(t), \tag{5.7}$$

then

$$\mathbf{\Phi}_l(t) = \mathbf{\Phi}_r(t) = \exp\left(\int_{t_0}^t \mathbf{a}(\tau)\mathrm{d}t\right).$$

If $\mathbf{a}$ is constant, then, evidently, $\int_{t_0}^{t} \mathbf{a}(\tau)\mathrm{d}t = \mathbf{a}\,(t - t_0)$, and $\mathbf{\Phi}_l(t) = \mathbf{\Phi}_r(t) = \exp(\mathbf{a}\,(t - t_0))$.

The similar result has been obtained in [43] as well. In [43], the following nonhomogeneous differential equation corresponding to (5.5) has been considered,

$$\mathbf{q}'(t) = \mathbf{a}(t)\mathbf{q}(t) + \mathbf{f}(t), \tag{5.8}$$

where $\mathbf{f} : [0, T] \to \mathbb{H}$ and $\mathbf{a} : [0, T] \to \mathbb{H}$. It has been shown, if condition (5.7) is satisfied, then the solutions of (5.8) are given by

$$\mathbf{q}(t) = \exp\left(\int_0^t \mathbf{a}(\tau)\mathrm{d}\tau\right)\left(\mathbf{q}(0) + \int_0^t \exp\left(\int_0^s (-\mathbf{a}(\tau))\,\mathrm{d}\tau\right)\mathbf{f}(s)\mathrm{d}s\right), \quad (t \in [0, T]). \tag{5.9}$$

In the special case when $\mathbf{a}$ is constant and $\mathbf{q}(0) = 1$, then the solutions of (5.8) are given by

$$\mathbf{q}(t) = \exp\left(\mathbf{a}t\right)\left(\int_0^t \exp\left(-\mathbf{a}s\right)\mathbf{f}(s)\mathrm{d}s\right), \quad (t \in [0, T]). \tag{5.10}$$

5.2. Determinantal Representations of Solutions of Some Singular Differential Quaternion-Matrix Equations

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{A}\mathbf{X} = \mathbf{B}, \tag{5.11}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{H}^{n\times n}$ is unknown. By (5.10) the general solution of (5.11) is found to be

$$\mathbf{X}(t) = \exp\left(-\mathbf{A}t\right)\left(\int \exp\left(\mathbf{A}t\right)dt\right)\mathbf{B}.$$

If $\mathbf{A}$ is invertible, then

$$\int \exp\left(\mathbf{A}t\right)dt = \mathbf{A}^{-1}\exp\left(\mathbf{A}t\right) + \mathbf{G}, \tag{5.12}$$

where $\mathbf{G}$ is an arbitrary $n \times n$ quaternionic matrix.

Since $\mathbf{A}^{-1}\exp\left(\mathbf{A}\right) = \exp\left(\mathbf{A}\right)\mathbf{A}^{-1}$, then the general solution of (5.11) is $\mathbf{X}(t) = \{\mathbf{A}^{-1} + \exp\left(-\mathbf{A}t\right)\mathbf{G}\}\mathbf{B}$. If $\mathbf{A}$ is noninvertible, then due to [30] the following theorem can be expended to quaternion matrices.

Theorem 5.1. *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index k, then*

$$\int \exp(\mathbf{A}t)\,\mathrm{d}t = \mathbf{A}^D \exp(\mathbf{A}t) + (\mathbf{I}-\mathbf{A}\mathbf{A}^D)t\left[\mathbf{I}+\frac{\mathbf{A}}{2}t+\frac{\mathbf{A}^2}{3!}t^2+\ldots+\frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right]+\mathbf{G}. \tag{5.13}$$

Proof. Differentiate the right-hand side of (5.13), and use the series expansion for $\exp(\mathbf{A}t)$. ☐

Using (5.13) and the series expansion for $\exp(-\mathbf{A}t)$, we get an explicit form for a general solution of (5.11),

$$\mathbf{X}(t) = \left\{\mathbf{A}^D + (\mathbf{I}-\mathbf{A}\mathbf{A}^D)t\left(\mathbf{I}-\frac{\mathbf{A}}{2}t+\frac{\mathbf{A}^2}{3!}t^2-\ldots(-1)^{k-1}\frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right)+\mathbf{G}\right\}\mathbf{B}.$$

If we put $\mathbf{G}=\mathbf{0}$, then the following partial solution of (5.11) is obtained,

$$\mathbf{X}(t) = \mathbf{A}^D\mathbf{B}+(\mathbf{B}-\mathbf{A}^D\mathbf{A}\mathbf{B})t-\frac{1}{2}(\mathbf{A}\mathbf{B}-\mathbf{A}^D\mathbf{A}^2\mathbf{B})t^2+\ldots\frac{(-1)^{k-1}}{k!}(\mathbf{A}^{k-1}\mathbf{B}-\mathbf{A}^D\mathbf{A}^k\mathbf{B})t^k. \tag{5.14}$$

Theorem 5.2. *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index k and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$*, then the partial solution (5.14),* $\mathbf{X}(t) = (x_{ij})$*, possess the following determinantal representation,*

1. when $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is Hermitian,*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|} + \left(b_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+1)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t-$$

$$-\frac{1}{2}\left(\widehat{b}^{(1)}_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+2)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^2+\ldots$$

$$\frac{(-1)^k}{k!}\left(\widehat{b}^{(k-1)}_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(2k)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^k \tag{5.15}$$

where $\mathbf{A}^l\mathbf{B} =: \widehat{\mathbf{B}}^{(l)} = (\widehat{b}^{(l)}_{ij}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{k, 2k}$*;*

2. when $\mathbf{A}$ *is arbitrary,*

$$x_{ij} = \frac{\sum\limits_{s=1}^{n} a_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{s\}} \operatorname{cdet}_s \left(\left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,s} \left(\widehat{\mathbf{d}}_{.j}^{(0)}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|}$$

$$+ \left(b_{ij} - \frac{\sum\limits_{s=1}^{n} a_{is}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{s\}} \operatorname{cdet}_s \left(\left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,s} \left(\widehat{\mathbf{d}}_{.j}^{(1)}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|} \right) t$$

$$-\frac{1}{2} \left(\widehat{b}_{ij}^{(1)} - \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{s\}} \operatorname{cdet}_s \left(\left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,s} \left(\widehat{\mathbf{d}}_{.j}^{(2)}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|} \right) t^2 + \ldots$$

$$\frac{(-1)^k}{k!} \left(\widehat{b}_{ij}^{(k-1)} - \frac{\sum\limits_{s=1}^{n} a_{is}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{s\}} \operatorname{cdet}_s \left(\left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,s} \left(\widehat{\mathbf{d}}_{.j}^{(k)}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|} \right) t^k \tag{5.16}$$

where $(\mathbf{A}^{2k+1})^* \mathbf{A}^{k+l} \mathbf{B} = \hat{\mathbf{A}} \mathbf{A}^l \mathbf{B} =: \widehat{\mathbf{D}}^{(l)} = (\widehat{d}_{ij}^{(l)}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{1,k}$ *and for all* $i, j = \overline{1,n}$*.*

Proof. 1. Using the determinantal representation of $\mathbf{A}^D$ by (3.6), we obtain the following determinantal representation of the matrix $\mathbf{A}^D \mathbf{A}^m \mathbf{B} := (y_{ij})$,

$$y_{ij} = \sum_{s=1}^{n} a_{is}^{D} \sum_{t=1}^{n} a_{st}^{(m)} b_{tj} = \sum_{\beta\in J_{r,n}\{i\}} \frac{\sum\limits_{s=1}^{n} \operatorname{cdet}_i \left(\mathbf{A}_{.\,i}^{k+1} \left(\mathbf{a}_{.s}{}^{(k)} \right) \right)_{\beta}^{\beta} \cdot \sum\limits_{t=1}^{n} a_{st}^{(m)} b_{tj}}{\sum\limits_{\beta\in J_{r,n}} \left| (\mathbf{A}^{k+1})_{\beta}^{\beta} \right|} =$$

$$\sum_{\beta\in J_{r,n}\{i\}} \frac{\sum\limits_{t=1}^{n} \operatorname{cdet}_i \left(\mathbf{A}_{.\,i}^{k+1} \left(\mathbf{a}_{.t}{}^{(k+m)} \right) \right)_{\beta}^{\beta} \cdot b_{tj}}{\sum\limits_{\beta\in J_{r,n}} \left| (\mathbf{A}^{k+1})_{\beta}^{\beta} \right|} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\mathbf{A}_{.\,i}^{k+1} \left(\widehat{\mathbf{b}}_{.j}^{(k+m)} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| (\mathbf{A}^{k+1}) {}_{\beta}^{\beta} \right|}$$

for all $i, j = \overline{1,n}$ and $m = \overline{1,k}$. From this and the determinantal representation of the Drazin inverse solution (4.11), it follows (5.15).

2. The proof of (5.16) is similar to the proof of (5.15) by using the determinantal representation of $\mathbf{A}^D$ by (3.14). □

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{X}\mathbf{A} = \mathbf{B} \tag{5.17}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{H}^{n\times n}$ is unknown. The general solution of (5.17) is found to be

$$\mathbf{X}(t) = \mathbf{B}\exp(-\mathbf{A}t)\left(\int \exp(\mathbf{A}t)\, dt\right).$$

If $\mathbf{A}$ is invertible, then by (5.12) the general solution of (5.17) is $\mathbf{X}(t) = \mathbf{B}\{\mathbf{A}^{-1} + \exp(-\mathbf{A}t)\,\mathbf{G}\}$. If $\mathbf{A}$ is noninvertible, then an explicit form for a general solution of (5.17) is

$$\mathbf{X}(t) =$$
$$\mathbf{B}\left\{\mathbf{A}^D + (\mathbf{I} - \mathbf{A}\mathbf{A}^D)t\left(\mathbf{I} - \frac{\mathbf{A}}{2}t + \frac{\mathbf{A}^2}{3!}t^2 + ...(-1)^{k-1}\frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right) + \mathbf{G}\right\}.$$

If we put $\mathbf{G} = \mathbf{0}$, then we obtain the following partial solution of (5.17),

$$\mathbf{X}(t) = \mathbf{B}\mathbf{A}^D + (\mathbf{B} - \mathbf{B}\mathbf{A}\mathbf{A}^D)t - \frac{1}{2}(\mathbf{B}\mathbf{A} - \mathbf{B}\mathbf{A}^2\mathbf{A}^D)t^2 + ...\frac{(-1)^{k-1}}{k!}(\mathbf{B}\mathbf{A}^{k-1} - \mathbf{B}\mathbf{A}^k\mathbf{A}^D)t^k. \tag{5.18}$$

Theorem 5.3. *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index* k *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$*, then the partial solution (5.18),* $\mathbf{X}(t) = (x_{ij})$*, possess the following determinantal representation,*

1. when $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is Hermitian,*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k)}_{.i}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|} + \left(b_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k+1)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t$$
$$- \frac{1}{2}\left(\check{b}^{(1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k+2)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t^2 + ...$$
$$\frac{(-1)^k}{k!}\left(\check{b}^{(k-1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(2k)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t^k,$$

where $\mathbf{BA}^l =: \check{\mathbf{B}}^{(l)} = (\check{b}_{ij}^{(l)}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{k, 2k}$*;*

2. when $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is arbitrary,*

$$x_{ij} = \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^* \right)_{.s} (\check{\mathbf{d}}^{(0)}) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} |(\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*) {}_{\alpha}^{\alpha}|}$$

$$+ \left(b_{ij} - \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^* \right)_{.s} (\check{\mathbf{d}}^{(1)}) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} |(\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*) {}_{\alpha}^{\alpha}|} \right) t$$

$$- \frac{1}{2} \left(\check{b}_{ij}^{(1)} - \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^* \right)_{.s} (\check{\mathbf{d}}^{(2)}) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} |(\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*) {}_{\alpha}^{\alpha}|} \right) t^2 + \ldots$$

$$\frac{(-1)^k}{k!} \left(\check{b}_{ij}^{(k-1)} - \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1}\right)^* \right)_{.s} (\check{\mathbf{d}}^{(k)}) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} |(\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*) {}_{\alpha}^{\alpha}|} \right) t^k,$$

where $\mathbf{BA}^{k+l}(\mathbf{A}^{2k+1})^* = \mathbf{BA}^l\check{\mathbf{A}} =: \check{\mathbf{D}}^{(l)} = (\check{d}_{ij}^{(l)}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{1, k}$ *and for all* $i, j = \overline{1, n}$*.*

Proof. The proof is similar to the proof of Theorem 5.2 by using the determinantal representation of the Drazin inverse (3.6) and (3.14), respectively. □

5.3. An Example

Let us consider the matrix equation

$$\mathbf{X}' + \mathbf{AX} = \mathbf{B}, \tag{5.19}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} i & j & k \\ 1 & -k & j \\ 1 & 0 & i \end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{pmatrix}$, $\det \mathbf{A} = \det \mathbf{A}^2 = 0$, and $\det \begin{pmatrix} 1 & k \\ -k & 2 \end{pmatrix} = 1$, $\det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} = 2$, then, by Theorem 2.11, $Ind\, \mathbf{A} = 1$ and $r_1 = \operatorname{rank} \mathbf{A} = 2$. Since $\mathbf{A}$ is Hermitian and $Ind\, \mathbf{A} = 1$, then we shall find the solutions $(x_{ij}) \in \mathbb{H}^{3\times 3}$ by (5.15),

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{2,3}\{i\}} \operatorname{cdet}_i \left(\mathbf{A}^2_{.\,i}\left(\widehat{\mathbf{b}}^{(1)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left|(\mathbf{A}^2)^{\beta}_{\beta}\right|} + \left(b_{ij} - \frac{\sum\limits_{\beta \in J_{2,3}\{i\}} \operatorname{cdet}_i \left(\mathbf{A}^2_{.\,i}\left(\widehat{\mathbf{b}}^{(2)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left|(\mathbf{A}^2)^{\beta}_{\beta}\right|}\right) t$$

for all $i, j = 1, 2, 3$. We have, $\sum\limits_{\beta \in J_{2,3}} \left|(\mathbf{A}^2)^{\beta}_{\beta}\right| = 4$,

$$\widehat{\mathbf{B}}^{(1)} = \mathbf{AB} = \begin{pmatrix} k & 1+j & 1-i+k \\ 2 & i-2k & 1+2j-k \\ -j & i+k & 1+i-j \end{pmatrix},$$

$$\widehat{\mathbf{B}}^{(2)} = \mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4k & 4+3j & 3-4i+3k \\ 6 & 4i-6k & 4+6j-4k \\ -4j & 4i+3k & 4+3i-3j \end{pmatrix}.$$

Therefore,

$$x_{11} = \frac{1}{4}\left(\operatorname{cdet}_1 \begin{pmatrix} k & 4k \\ 2 & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & -3i \\ -j & 3 \end{pmatrix}\right) + \left(i - \frac{1}{4}\left[\operatorname{cdet}_1 \begin{pmatrix} 4k & 4k \\ 6 & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 4k & -3i \\ -4j & 3 \end{pmatrix}\right]\right) t = \frac{1}{4}(-2k) + \left(i - \frac{1}{4}[0]\right) t = -0.5k + (i)t;$$

$$x_{12} = \frac{1}{4}\left(\operatorname{cdet}_1 \begin{pmatrix} 1+j & 4k \\ i-2k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 1+j & -3i \\ i+k & 3 \end{pmatrix}\right) + \left(j - \frac{1}{4}\left[\operatorname{cdet}_1 \begin{pmatrix} 4+3j & 4k \\ 4i-6k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 4+3j & -3i \\ 4i+3k & 3 \end{pmatrix}\right]\right) t = \frac{1}{4}(-2+2j) + \left(j - \frac{1}{4}[2j]\right) t = -0.5 + 0.5j + (0.5j)\, t;$$

$$x_{13} = \frac{1}{4}\left(\operatorname{cdet}_1 \begin{pmatrix} 1-i+k & 4k \\ 1+2j-k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 1-i+k & -3i \\ 1+ij & 3 \end{pmatrix}\right) + \left(k - \frac{1}{4}\left[\operatorname{cdet}_1 \begin{pmatrix} 3-4i+3j & 4k \\ 4+6j-4k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 3-4i+4k & -3i \\ 4+3i-3j & 3 \end{pmatrix}\right]\right) t = \frac{1}{4}(2+2i+2k) + \left(k - \frac{1}{4}[2+11k]\right) t = 0.5 + 0.5i + 0.5k + (-0.5 - 4.5k)\, t;$$

$$x_{21}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & k\\ -4k & 2\end{pmatrix}+\mathrm{cdet}_1\begin{pmatrix}2 & 4j\\ -j & 3\end{pmatrix}\right)+\left(1-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4k\\ -4k & 6\end{pmatrix}+\right.\right.$$
$$\left.\left.\mathrm{cdet}_1\begin{pmatrix}6 & 4j\\ -4j & 3\end{pmatrix}\right]\right)t=\frac{1}{4}(4)+\left(1-\frac{1}{4}\left[-4k\right]\right)t=1+(1+k)t;$$

$$x_{22}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+j\\ -4k & i-2k\end{pmatrix}+\mathrm{cdet}_1\begin{pmatrix}i-2k & 4j\\ i+k & 3\end{pmatrix}\right)+$$
$$\left(-k-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4+3j\\ -4k & 4i-6k\end{pmatrix}+\mathrm{cdet}_1\begin{pmatrix}4i-6k & 4j\\ 4i+3k & 3\end{pmatrix}\right]\right)t=$$
$$\frac{1}{4}(-2i-4k)+\left(-k-\frac{1}{4}\left[-4k\right]\right)t=-0.5i-k;$$

$$x_{23}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i+k\\ -4k & 1+2j-k\end{pmatrix}+\mathrm{cdet}_1\begin{pmatrix}1+2j-k & 4j\\ 1+i-j & 3\end{pmatrix}\right)+$$
$$\left(j-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 3-4i+3k\\ -4k & 4+6j-4k\end{pmatrix}+\mathrm{cdet}_1\begin{pmatrix}4+6j-4k & 4j\\ 4+3i-3j & 3\end{pmatrix}\right]\right)t=$$
$$\frac{1}{4}(-2+4j+2k)+\left(j-\frac{1}{4}\left[4j\right]\right)t=-0.5+j+0.5k;$$

$$x_{31}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & k\\ 3i & -j\end{pmatrix}+\mathrm{cdet}_2\begin{pmatrix}6 & 2\\ -4j & -j\end{pmatrix}\right)+\left(1-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4k\\ 3i & -4j\end{pmatrix}+\right.\right.$$
$$\left.\left.\mathrm{cdet}_2\begin{pmatrix}6 & 6\\ -4j & -4j\end{pmatrix}\right]\right)t=\frac{1}{4}(2j)+\left(i-\frac{1}{4}\left[0\right]\right)t=0.5j+t;$$

$$x_{32}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+j\\ 3i & i+k\end{pmatrix}+\mathrm{cdet}_2\begin{pmatrix}6 & i-2k\\ -4j & i+k\end{pmatrix}\right)+$$
$$\left(0-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4+3j\\ 3i & 4i+3k\end{pmatrix}+\mathrm{cdet}_2\begin{pmatrix}6 & 4i-6k\\ -4j & 4i+3k\end{pmatrix}\right]\right)t=$$
$$\frac{1}{4}(-2i+2k)+\left(-\frac{1}{4}\left[2k\right]\right)t=-0.5i+0.5k+(-0.5k)t;$$

$$x_{33}=\frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i+k\\ 3i & 1+i-j\end{pmatrix}+\mathrm{cdet}_2\begin{pmatrix}6 & 1+2j-k\\ -4j & 1+i-j\end{pmatrix}\right)+$$
$$\left(i-\frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 3-4i+3jk\\ 3i & 4+3i-3j\end{pmatrix}+\mathrm{cdet}_2\begin{pmatrix}6 & 4+6j-4k\\ -4j & 4+3i-3j\end{pmatrix}\right]\right)t=$$
$$\frac{1}{4}(-2+2i-2j)+\left(i-\frac{1}{4}\left[2i-2j\right]\right)t=-0.5+0.5i-0.5j+(0.5i-0.5j)t.$$

6. Determinantal Representations of the W-Weighted Drazin Inverse for an Arbitrary Matrix

The properties of the complex W-weighted Drazin inverse can be found in [1,44, 45, 46, 47, 48]. These properties can be generalized to $\mathbb{H}$. In particular, if $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ and $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$, then

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}\left((\mathbf{WA})^{D}\right)^{2} = \left((\mathbf{AW})^{D}\right)^{2}\mathbf{A}, \tag{6.1}$$

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = (\mathbf{WA})^{D}, \mathbf{WA}_{d,\mathbf{W}} = (\mathbf{AW})^{D}. \tag{6.2}$$

Determinantal representations W-weighted Drazin inverse of complex matrices have been received by a full-rank factorization in [37] and by a limit representation in [49].

Through the theory of column-row determinants, a determinantal representation W-weighted Drazin inverse over the quaternion skew-field for the first time has been obtained in [14] by the following theorem.

Theorem 6.1. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and and* $\operatorname{rank}(\mathbf{AW})^{k} = s$. *Suppose that* $\mathbf{B} \in \mathbb{H}^{n\times(n-s)}_{n-s}$ *and* $\mathbf{C}^{*} \in \mathbb{H}^{m\times(m-s)}_{m-s}$ *are of full-ranks and*

$$\mathcal{R}_r(\mathbf{B}) = \mathcal{N}_r\left((\mathbf{WA})^{k}\right),\ \ \mathcal{N}_r(\mathbf{C}) = \mathcal{R}_r\left((\mathbf{AW})^{k}\right),$$
$$\mathcal{R}_l(\mathbf{C}) = \mathcal{N}_l\left((\mathbf{AW})^{k}\right),\ \ \mathcal{N}_l(\mathbf{B}) = \mathcal{R}_l\left((\mathbf{WA})^{k}\right).$$

Denote

$$\mathbf{M} = \begin{bmatrix} \mathbf{WAW} & \mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}.$$

Then the W-weighted Drazin inverse $\mathbf{A}_{d,\mathbf{W}} = (a)_{ij} \in \mathbb{H}^{n\times m}$ *has the following determinantal representations:*

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} L_{ki} m^{*}_{kj}}{\det \mathbf{M}^{*}\mathbf{M}}, i = \overline{1,m}, j = \overline{1,n}, \tag{6.3}$$

or

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} m^{*}_{ik} R_{jk}}{\det \mathbf{M}\mathbf{M}^{*}}, i = \overline{1,m}, j = \overline{1,n}, \tag{6.4}$$

where L_{ij} *are the left* (ij)*-th cofactor of* $\mathbf{M}^{*}\mathbf{M}$ *and* R_{ij} *are the right* (ij)*-th cofactor of* $\mathbf{M}\mathbf{M}^{*}$*, respectively, for all* $i, j = \overline{1, m+n-s}$.

As can be seen, the auxiliary matrices $\mathbf{B}$ and $\mathbf{C}$ have been used in the determinantal representations (6.3) and (6.4). In this chapter we escape it. We shall derive determinantal representations of the W-weighted Drazin inverse of an arbitrary matrix $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ by using the determinantal representations of the Drazin inverse, of the Moore-Penrose inverse, and the limit representation of the W-weighted Drazin inverse in some particular case.

6.1. Determinantal Representations of the W-Weighted Drazin Inverse by using Determinantal Representations of the Drazin Inverse

Let $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{W} \in \mathbb{H}^{n\times m}$. Denote $\mathbf{WA} =: \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ and $\mathbf{AW} =: \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$. Due to Theorem 3.6 for an arbitrary element of the Drazin inverse $\mathbf{U}^D$, we have the following determinantal representations,

$$u_{ij}^{D,1} = \frac{\sum\limits_{t=1}^{n} u_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{U}^{2k+1}\right)^* \left(\mathbf{U}^{2k+1}\right)_{.\,t} (\hat{\mathbf{u}}_{.\,j})\right) {}^{\beta}_{\beta}\right)}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{U}^{2k+1}\right)^* \left(\mathbf{U}^{2k+1}\right) {}^{\beta}_{\beta}\right|} \tag{6.5}$$

or

$$u_{ij}^{D,2} = \frac{\sum\limits_{s=1}^{n} \left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \mathrm{rdet}_s \left(\left(\mathbf{U}^{2k+1} \left(\mathbf{U}^{2k+1}\right)^*\right)_{.\,s} (\check{\mathbf{u}}_{i\,.})\right) {}^{\alpha}_{\alpha}\right) u_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{U}^{2k+1} \left(\mathbf{U}^{2k+1}\right)^*\right) {}^{\alpha}_{\alpha}\right|} \tag{6.6}$$

where $\hat{\mathbf{u}}_{.j}$ is the j-th column of $(\mathbf{U}^{2k+1})^*\mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n\times n}$, and $\check{\mathbf{u}}_{i.}$ is the i-th row of $\mathbf{U}^k(\mathbf{U}^{2k+1})^* =: \check{\mathbf{U}} = (\check{u}_{ij}) \in \mathbb{H}^{n\times n}$ for all $i, j = \overline{1, n}$, and $r = \mathrm{rank}\, \mathbf{U}^{k+1} = \mathrm{rank}\, \mathbf{U}^k$.

Then, by (6.1), we can obtain the following determinantal representations of $\mathbf{A}_{d,\mathbf{W}} = (a_{ij}^{d,\mathbf{W}}) \in \mathbb{H}^{m\times n}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^{n} a_{iq} (u_{qj}^{D})^{(2)} \tag{6.7}$$

where

$$(u_{qj}^{D})^{(2)} = \sum_{p=1}^{n} u_{qp}^{D,l} u_{pj}^{D,f} \tag{6.8}$$

for all $l, f = \overline{1,2}$. $u_{ij}^{D,1}$ and $u_{ij}^{D,2}$ are represented by (6.5) and (6.6), respectively.

Similarly, using $\mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$, we have the following determinantal representations of $\mathbf{A}_{d,\mathbf{W}}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^{m} (v_{iq}^{D})^{(2)} a_{qj}. \tag{6.9}$$

The first factor is one of the four possible equations

$$(v_{iq}^{D})^{(2)} = \sum_{p=1}^{m} v_{ip}^{D,l} v_{pq}^{D,f} \tag{6.10}$$

for all $l, f = \overline{1,2}$. An element of the Drazin inverse $\mathbf{V}^D$ can be represented by

$$v_{ij}^{D,1} = \frac{\sum\limits_{t=1}^{m} v_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,m}\{t\}} \operatorname{cdet}_t\left(\left(\mathbf{V}^{2k+1}\right)^{*}\left(\mathbf{V}^{2k+1}\right)_{.\,t}(\hat{\mathbf{v}}_{.\,j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\right)^{*}\left(\mathbf{V}^{2k+1}\right)_{\beta}^{\beta}\right|} \tag{6.11}$$

or

$$v_{ij}^{D,2} = \frac{\sum\limits_{s=1}^{m} \left(\sum\limits_{\alpha\in I_{r,\,m}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^{*}\right)_{.\,s}(\check{\mathbf{v}}_{i\,.})\right)_{\alpha}^{\alpha}\right) v_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^{*}\right)_{\alpha}^{\alpha}\right|}, \tag{6.12}$$

where $\hat{\mathbf{v}}_{.s}$ is the s-th column of $(\mathbf{V}^{2k+1})^{*}\mathbf{V}^{k} =: \hat{\mathbf{V}} = (\hat{v}_{ij}) \in \mathbb{H}^{m\times m}$, and $\check{\mathbf{v}}_{t.}$ is the t-th row of $\mathbf{V}^{k}(\mathbf{V}^{2k+1})^{*} =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m\times m}$ for all $s, t = \overline{1,m}$, and $r = \operatorname{rank}\mathbf{V}^{k+1} = \operatorname{rank}\mathbf{V}^{k}$.

6.2. Determinantal Representations of the W-Weighted Drazin Inverse by using Determinantal Representations of the Moore-Penrose Inverse

Consider the general algebraic structures (GAS) of the matrices $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}^{n\times m}$, $\mathbf{A}^{+} \in \mathbb{H}^{n\times m}$, $\mathbf{W}^{+} \in \mathbb{H}^{m\times n}$ and $\mathbf{A}_{d,\mathbf{W}} \in \mathbb{H}^{m\times n}$ with $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ (e.g., [44,45,46,47]).

Let exist $\mathbf{L} \in \mathbb{H}^{m\times m}$ and $\mathbf{Q} \in \mathbb{H}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{L}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}\mathbf{Q}^{-1}, \quad \mathbf{W} = \mathbf{Q}\begin{bmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{bmatrix}\mathbf{L}^{-1}.$$

Then

$$\mathbf{A}^{+} = \mathbf{Q}\begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{L}^{-1}, \quad \mathbf{W}^{+} = \mathbf{L}\begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{L}\begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

where $\mathbf{L}, \mathbf{Q}, \mathbf{A}_{11}, \mathbf{W}_{11}$ are invertible matrices, and $\mathbf{A}_{22}\mathbf{W}_{22}, \mathbf{W}_{22}\mathbf{A}_{22}$ are nilpotent matrices. Due to [47], the following theorem can be expanded to $\mathbb{H}$.

Theorem 6.2. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}^{n\times m}$ *such that* $\mathbf{A}_{22}\mathbf{W}_{22}$ *and* $\mathbf{W}_{22}\mathbf{A}_{22}$ *be nilpotent matrices of index* k *in GAS form. Then the weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *can be written as matrix expression involving the Moore-Penrose inverse,*

$$\mathbf{A}_{d,\mathbf{W}} = \left\{(\mathbf{AW})^{k}\left[(\mathbf{AW})^{2k+1}\right]^{+}(\mathbf{AW})^{k}\right\}\mathbf{W}^{+}, \tag{6.13}$$

where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$.

Similarly, the following theorem can be obtained.

Theorem 6.3. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}^{n\times m}$ *such that* $\mathbf{A}_{22}\mathbf{W}_{22}$ *and* $\mathbf{W}_{22}\mathbf{A}_{22}$ *be nilpotent matrices of index* k *in GAS form. Then the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *can be written as the following matrix expression,*

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\}, \tag{6.14}$$

where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$.

Proof. Since $\mathbf{W}_{22}\mathbf{A}_{22}$ is a nilpotent matrix of index k, then due to GAS of $\mathbf{A}$, $\mathbf{W}$ and their generalized inverses, we have the following Jordan canonical forms,

$$\mathbf{WA} = \mathbf{Q}\begin{bmatrix} \mathbf{W}_{11}\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22}\mathbf{A}_{22} \end{bmatrix}\mathbf{Q}^{-1}, \quad (\mathbf{WA})^{k} = \mathbf{Q}\begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

$$\left[(\mathbf{WA})^{2k+1}\right]^{+} = \mathbf{Q}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}.$$

Simple computing of $\mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\}$ proves the theorem,

$$\mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\} =$$

$$\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} =$$

$$\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{k}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} =$$

$$\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} =$$

$$\mathbf{L}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} = \mathbf{A}_{d,\mathbf{W}}.$$

□

Using (6.13), an entry $a_{ij}^{d,\mathbf{W}}$ of the W-weighted Drazin inverse $\mathbf{A}_{d,\mathbf{W}}$ can be obtained as follows

$$a_{ij}^{d,\mathbf{W}} = \sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{l=1}^{m} v_{is}^{(k)}\left(v_{st}^{(2k+1)}\right)^{+} v_{tl}^{(k)} w_{lj}^{+} \tag{6.15}$$

for all $i = \overline{1,m}$, $j = \overline{1,n}$.

Denote by $\check{\mathbf{w}}_{t.}$ the t-th row of $\mathbf{V}^{k}\mathbf{W}^{*} =: \check{\mathbf{W}} = (\check{w}_{ij}) \in \mathbb{H}^{m\times n}$ for all $t = \overline{1,m}$. It follows from $\sum_{l} v_{tl}^{(k)}\mathbf{w}_{l.}^{*} = \check{\mathbf{w}}_{t.}$ and (3.12) that

$$\sum_{l=1}^{m} v_{tl}^{(k)} w_{lj}^{+} = \sum_{l=1}^{m} v_{tl}^{(k)} \cdot \frac{\sum\limits_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{WW}^{*})_{j.}(\mathbf{w}_{l.}^{*})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_1,n}} \left|(\mathbf{WW}^{*})_{\alpha}^{\alpha}\right|} =$$

$$\frac{\sum\limits_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{WW}^{*})_{j.}(\check{\mathbf{w}}_{t.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_1,n}} \left|(\mathbf{WW}^{*})_{\alpha}^{\alpha}\right|}, \tag{6.16}$$

where $r_1 = \operatorname{rank}\mathbf{W}$. Similarly, denote by $\check{\mathbf{v}}_{i.}$ the t-th row of $\mathbf{V}^k(\mathbf{V}^{2k+1})^* =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m\times m}$ for all $t = \overline{1,m}$. It follows from $\sum\limits_s v_{is}^{(k)}\left(\mathbf{v}_{s.}^{(2k+1)}\right)^* = \check{\mathbf{v}}_{i.}$ and (3.12) that

$$\sum_{s=1}^{m} v_{is}^{(k)}\left(v_{st}^{(2k+1)}\right)^{+} = \sum_{s=1}^{m} v_{is}^{(k)} \cdot \frac{\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\mathbf{v}_{s.}^{(2k+1)}\right)^*\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|} = \frac{\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\check{\mathbf{v}}_{i.}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}, \quad (6.17)$$

where $r = \operatorname{rank}\mathbf{W}^{k+1} = \operatorname{rank}\mathbf{W}^k$. Using (6.16) and (6.17) in (6.15), we obtain the following determinantal representation of $\mathbf{A}_{d,\mathbf{W}}$,

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{m}\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\check{\mathbf{v}}_{i.}\right)\right)_{\alpha}^{\alpha} \sum\limits_{\alpha\in I_{r_1,\,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{W}\mathbf{W}^*\right)_{j.}\left(\check{\mathbf{w}}_{t.}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right| \sum\limits_{\alpha\in I_{r_1,\,n}} \left|\left(\mathbf{W}\mathbf{W}^*\right)_{\alpha}^{\alpha}\right|} \quad (6.18)$$

Thus, we have proved the following theorem.

Theorem 6.4. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $r = \operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^k$. *Then the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *possesses the determinantal representation (6.18), where* $\mathbf{V} = \mathbf{AW}$, $\check{\mathbf{V}} = \mathbf{V}^k(\mathbf{V}^{2k+1})^*$, *and* $\check{\mathbf{W}} = \mathbf{V}^k\mathbf{W}^*$.

Similarly we have the following theorem.

Theorem 6.5. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $r = \operatorname{rank}(\mathbf{WA})^{k+1} = \operatorname{rank}(\mathbf{WA})^k$. *Then the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *possesses the following determinantal representation,*

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\hat{\mathbf{w}}_{.t}\right)\right)_{\beta}^{\beta} \sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{u}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right| \sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|} \quad (6.19)$$

where $\mathbf{U} = \mathbf{WA}$, $\hat{\mathbf{U}} = (\mathbf{U}^{2k+1})^*\mathbf{U}^k$, *and* $\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^k$.

Proof. Using (6.14), an entry $a_{ij}^{d,\mathbf{W}}$ of the W-weighted Drazin inverse $\mathbf{A}_{d,\mathbf{W}}$ can be obtained as follows

$$a_{ij}^{d,\mathbf{W}} = \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{l=1}^{n} w_{is}^{+} u_{st}^{(k)} \left(u_{tl}^{(2k+1)}\right)^{+} u_{lj}^{(k)} \tag{6.20}$$

for all $i = \overline{1, m}$, $j = \overline{1, n}$. Denote by $\hat{\mathbf{w}}_{.t}$ the t-th column of $\mathbf{W}^*\mathbf{U}^k =: \hat{\mathbf{W}} = (\hat{w}_{ij}) \in \mathbb{H}^{m\times n}$ for all $t = \overline{1, n}$. It follows from $\sum\limits_{t} \mathbf{w}_{.s}^{*} u_{st}^{(k)} = \hat{\mathbf{w}}_{.t}$ and (3.11) that

$$\sum_{s=1}^{n} w_{is}^{+} u_{st}^{(k)} = \sum_{s=1}^{n} \frac{\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i \left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\mathbf{w}_{.s}^{*}\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right|} \cdot u_{st}^{(k)} = \frac{\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\hat{\mathbf{w}}_{.t}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right|}, \tag{6.21}$$

where $r_1 = \operatorname{rank}\mathbf{W}$. Similarly, denote by $\hat{\mathbf{u}}_{.j}$ the j-th column of $(\mathbf{U}^{2k+1})^*\mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n\times n}$ for all $j = \overline{1, n}$. It follows from $\sum\limits_{l} \left(\mathbf{u}_{.l}^{(2k+1)}\right)^{*} u_{lj}^{(k)} = \hat{\mathbf{u}}_{.j}$ and (3.11) that

$$\sum_{l=1}^{n} \left(u_{tl}^{(2k+1)}\right)^{+} u_{lj}^{(k)} = \sum_{l=1}^{n} \frac{\sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{.t}\left(\mathbf{u}_{.l}^{(2k+1)}\right)^{*}\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|} \cdot u_{lj}^{(k)} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{u}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|}, \tag{6.22}$$

where $r = \operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^{k}$. Using the equations (6.22) and (6.21) in (6.20), we obtain (6.19). □

6.3. Determinantal Representations of the W-Weighted Drazin Inverse in Some Special Case

In this subsection we consider the determinantal representation of the W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ in a special case, when $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ and $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ are Hermitian. Then, for the determinantal representations of their Drazin inverse we can use (3.6) and (3.7).

For Hermitian matrix, we apply the method, which consists of the theorem on the limit representation of the Drazin inverse, lemmas on rank of matrices and on characteristic polynomial. By analogy to the complex case [39] we have the following limit representations of the W-weighted Drazin inverse,

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1}(\mathbf{AW})^k\mathbf{A} \tag{6.23}$$

and

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\mathbf{A}(\mathbf{WA})^k\left(\lambda \mathbf{I}_n + (\mathbf{WA})^{k+2}\right)^{-1} \tag{6.24}$$

where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Denote by $\mathbf{v}_{.j}^{(k)}$ and $\mathbf{v}_{i.}^{(k)}$ the j-th column and the i-th row of $\mathbf{V}^k$, respectively. Denote by $\bar{\mathbf{V}}^k := (\mathbf{AW})^k\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\bar{\mathbf{W}} = \mathbf{WAW} \in \mathbb{H}^{n\times m}$.

Lemma 6.6. *If* $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ *with* $Ind\mathbf{V} = k$*, then*

$$\operatorname{rank}\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{V}^{k+2}\right). \tag{6.25}$$

Proof. We have $\mathbf{V}^{k+2} = \bar{\mathbf{V}}^k\bar{\mathbf{W}}$. Let $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) \in \mathbb{H}^{m\times m}$, $(s \neq i)$, be a matrix with $-\bar{w}_{j\,s}$ in the (i,s)-entry, 1 in all diagonal entries, and 0 in others. The matrix $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s})$, $(s \neq i)$, is a matrix of an elementary transformation. It follows that

$$\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.\,j}^{(k)}\right)\cdot\prod_{s\neq i}\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) = \begin{pmatrix} \sum\limits_{s\neq j}\bar{v}_{1s}^{(k)}\bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum\limits_{s\neq j}\bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{s\neq j}\bar{v}_{ms}^{(k)}\bar{w}_{s1} & \dots & \underset{i-th}{\bar{v}_{mj}^{(k)}} & \dots & \sum\limits_{s\neq j}\bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix}.$$

We have the next factorization of the obtained matrix.

$$\begin{pmatrix} \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{s1} & \ldots & \bar{v}_{1j}^{(k)} & \ldots & \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{s1} & \ldots & \bar{v}_{mj}^{(k)} & \ldots & \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix} =$$
$$\qquad\qquad\qquad\qquad i-th$$

$$= \begin{pmatrix} \bar{v}_{11}^{(k)} & \bar{v}_{12}^{(k)} & \ldots & \bar{v}_{1n}^{(k)} \\ \bar{v}_{21}^{(k)} & \bar{v}_{22}^{(k)} & \ldots & \bar{v}_{2n}^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ \bar{v}_{m1}^{(k)} & \bar{v}_{m2}^{(k)} & \ldots & \bar{v}_{mn}^{(k)} \end{pmatrix} \begin{pmatrix} \bar{w}_{11} & \ldots & 0 & \ldots & \bar{w}_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{w}_{n1} & \ldots & 0 & \ldots & \bar{w}_{nm} \end{pmatrix} j-th.$$
$$\qquad\qquad\qquad\qquad\qquad\qquad i-th$$

Denote $\tilde{\mathbf{W}} := \begin{pmatrix} \bar{w}_{11} & \ldots & 0 & \ldots & \bar{w}_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{w}_{n1} & \ldots & 0 & \ldots & \bar{w}_{nm} \end{pmatrix} j-th.$ The matrix $\tilde{\mathbf{W}}$ is
$i-th$

obtained from $\bar{\mathbf{W}} = \mathbf{WAW}$ by replacing all entries of the j-th row and the ith column with zeroes except for 1 in the (i, j)-entry. Since elementary transformations of a matrix do not change a rank, then $\operatorname{rank} \mathbf{V}_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) \leq \min\left\{\operatorname{rank} \bar{\mathbf{V}}^k, \operatorname{rank} \tilde{\mathbf{W}}\right\}$. It is obvious that

$$\operatorname{rank} \bar{\mathbf{V}}^k = \operatorname{rank} (\mathbf{AW})^k\mathbf{A} \geq \operatorname{rank} (\mathbf{AW})^{k+2},$$
$$\operatorname{rank} \tilde{\mathbf{W}} \geq \operatorname{rank} \mathbf{WAW} \geq \operatorname{rank} (\mathbf{AW})^{k+2}.$$

From this the inequality (3.1) follows immediately. □

The next lemma is proved similarly.

Lemma 6.7. *If* $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ *with* $Ind\,\mathbf{U} = k$, *then*

$$\operatorname{rank} \left(\mathbf{U}^{k+2}\right)_{i\,.}\left(\bar{\mathbf{u}}_{j\,.}^{(k)}\right) \leq \operatorname{rank} \left(\mathbf{U}^{k+2}\right),$$

where $\bar{\mathbf{U}}^k := \mathbf{A}(\mathbf{WA})^k \in \mathbb{H}^{m\times n}$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

Lemma 6.8. *If* $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ *is Hermitian with* $Ind\mathbf{V} = k$ *and* $\lambda \in \mathbb{R}$*, then*

$$\mathrm{cdet}_i \left(\lambda \mathbf{I}_m + \mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}, \quad (6.26)$$

where $c_n^{(ij)} = \mathrm{cdet}_i\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)$ *and*

$$c_s^{(ij)} = \sum_{\beta\in J_{s,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$$

for all $s = \overline{1, n-1}$*,* $i, j = \overline{1, n}$*.*

Proof. Consider the Hermitian matrix $\left(t\mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i}(\mathbf{v}_{.\,i}^{(k+2)}) \in \mathbb{H}^{n\times n}$. Taking into account Theorem 2.13, we obtain

$$\det\left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i}\left(\mathbf{v}_{.\,i}^{(k+2)}\right) = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \qquad (6.27)$$

where $d_s = \sum\limits_{\beta\in J_{s,\,n}\{i\}} |\left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta}|$ is the sum of all principal minors of order s that contain the i-th column for all $s = \overline{1, n-1}$ and $d_n = \det\left(\mathbf{V}^{k+2}\right)$. Consequently, we have $\mathbf{v}_{.\,i}^{(k+2)} = \begin{pmatrix} \sum\limits_l \bar{v}_{1l}^{(k)}\bar{w}_{li} \\ \sum\limits_l \bar{v}_{2l}^{(k)}\bar{w}_{li} \\ \vdots \\ \sum\limits_l \bar{v}_{nl}^{(k)}\bar{w}_{li} \end{pmatrix} = \sum\limits_l \bar{\mathbf{v}}_{.\,l}^{(k)}\bar{w}_{li}$, where $\bar{\mathbf{v}}_{.\,l}^{(k)}$ is the l-th column of $\bar{\mathbf{V}}^k = (\mathbf{AW})^k\mathbf{A}$ and $\mathbf{WAW} = \bar{\mathbf{W}} = (\bar{w}_{li})$ for all $l = \overline{1, n}$. By Theorem 2.5, we obtain on the one hand

$$\det\left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i}\left(\mathbf{v}_{.\,i}^{(k+2)}\right) = \mathrm{cdet}_i\left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i}\left(\mathbf{v}_{.\,i}^{(k+2)}\right) =$$
$$= \sum_l \mathrm{cdet}_i\left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,l}\left(\bar{\mathbf{v}}_{.\,l}^{(k)}\bar{w}_{li}\right) = \sum_l \mathrm{cdet}_i\left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.\,l}^{(k)}\right)\cdot \bar{w}_{li} \qquad (6.28)$$

On the other hand, having changed the order of summation, for all $s = \overline{1, n-1}$ we have

$$d_s = \sum_{\beta \in J_{s,\,n}\{i\}} \det \left(\mathbf{V}^{k+2}\right) {}_{\beta}^{\beta} = \sum_{\beta \in J_{s,\,n}\{i\}} \operatorname{cdet}_i \left(\mathbf{V}^{k+2}\right) {}_{\beta}^{\beta} =$$
$$\sum_{\beta \in J_{s,\,n}\{i\}} \sum_{l} \operatorname{cdet}_i \left(\left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.\,l}^{(k)} \bar{w}_{l\,i}\right)\right) {}_{\beta}^{\beta} =$$
$$\sum_{l} \sum_{\beta \in J_{s,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.\,l}^{(k)}\right)\right) {}_{\beta}^{\beta} \cdot \bar{w}_{l\,i}. \quad (6.29)$$

By substituting (6.28) and (6.29) in (6.27), and equating factors at $\bar{w}_{l\,i}$ when $l = j$, we obtain (6.26). □

By analogy can be proved the following lemma.

Lemma 6.9. *If* $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$ *is Hermitian with* $Ind\mathbf{U} = k$ *and* $\lambda \in \mathbb{R}$, *then*

$$\operatorname{rdet}_j(\lambda \mathbf{I} + \mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)}) = r_1^{(ij)} \lambda^{n-1} + r_2^{(ij)} \lambda^{n-2} + \ldots + r_n^{(ij)},$$

where $r_s^{(ij)} = \sum_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)})\right) {}_{\alpha}^{\alpha}$ *and* $r_n^{(ij)} = \operatorname{rdet}_j(\mathbf{U}^{k+2})_{j\,.}(\bar{\mathbf{u}}_{i.}^{(k)})$ *for all* $s = \overline{1, n-1}$ *and* $i, j = \overline{1, n}$.

Theorem 6.10. *If* $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W} \in \mathbb{H}^{n \times m}$, *and* $\mathbf{AW} \in \mathbb{H}^{m \times m}$ *is Hermitian with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $\operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^k = r$, *then the W-weighted Drazin inverse* $\mathbf{A}_{d,W} = \left(a_{ij}^{d,W}\right) \in \mathbb{H}^{m \times n}$ *with respect to* $\mathbf{W}$ *possess the following determinantal representations:*

$$a_{ij}^{d,W} = \frac{\sum\limits_{\beta \in J_{r,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW})_{.\,i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left|(\mathbf{AW})^{k+2} {}_{\beta}^{\beta}\right|}, \quad (6.30)$$

where $\bar{\mathbf{v}}_{.j}^{(k)}$ *is the* j*-th column of* $\bar{\mathbf{V}}^k = (\mathbf{AW})^k \mathbf{A}$ *for all* $j = \overline{1, m}$.

Proof. The matrix $\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1} \in \mathbb{H}^{m \times m}$ is a full-rank Hermitian matrix. Taking into account Theorem 2.9 it has an inverse, which we represent as a left inverse matrix

$$\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)^{-1}=\frac{1}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)}\begin{pmatrix} L_{11} & L_{21} & \ldots & L_{m1}\\ L_{12} & L_{22} & \ldots & L_{m2}\\ \ldots & \ldots & \ldots & \ldots\\ L_{1m} & L_{2m} & \ldots & L_{mm}\end{pmatrix},$$

where L_{ij} is a left ij-th cofactor of a matrix $\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}$. Then we have

$$\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)^{-1}(\mathbf{AW})^k\mathbf{A}=$$
$$=\frac{1}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)}\begin{pmatrix} \sum\limits_{s=1}^{m}L_{s1}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m}L_{s1}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m}L_{s1}\bar{v}_{sn}^{(k)}\\ \sum\limits_{s=1}^{m}L_{s2}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m}L_{s2}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m}L_{s2}\bar{v}_{sn}^{(k)}\\ \ldots & \ldots & \ldots & \ldots\\ \sum\limits_{s=1}^{m}L_{sm}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m}L_{sm}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m}L_{sm}\bar{v}_{sn}^{(k)}\end{pmatrix}.$$

By (6.23) and using the definition of a left cofactor, we obtain

$$\mathbf{A}_{d,W}=\lim_{\alpha\to 0}\begin{pmatrix} \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)_{.1}\left(\bar{\mathbf{v}}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)} & \ldots & \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)_{.1}\left(\bar{\mathbf{v}}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)}\\ \ldots & \ldots & \ldots\\ \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)_{.n}\left(\bar{\mathbf{v}}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)} & \ldots & \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)_{.m}\left(\bar{\mathbf{v}}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)}\end{pmatrix}. \tag{6.31}$$

By Theorem 2.13, we have

$$\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)=\lambda^m+d_1\lambda^{m-1}+d_2\lambda^{m-2}+\ldots+d_m,$$

where $d_s=\sum\limits_{\beta\in J_{s,\,m}}\left|\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)\,{}_{\beta}^{\beta}\right|$ is a sum of principal minors of $(\mathbf{AW})^{k+2}$ of order s for all $s=\overline{1,m-1}$ and $d_m=\det(\mathbf{AW})^{k+2}$.

Since $\mathrm{rank}(\mathbf{AW})^{k+2}=\mathrm{rank}(\mathbf{AW})^{k+1}=\mathrm{rank}(\mathbf{AW})^{k}=r$, then $d_m=d_{m-1}=\ldots=d_{r+1}=0$. It follows that $\det\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)=\lambda^m+d_1\lambda^{m-1}+d_2\lambda^{m-2}+\ldots+d_r\lambda^{m-r}$.

Using (6.26) we have

$$\mathrm{cdet}_i\left(\lambda\mathbf{I}_m+(\mathbf{AW})^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)=c_1^{(ij)}\lambda^{m-1}+c_2^{(ij)}\lambda^{m-2}+\ldots+c_m^{(ij)}$$

for $i=\overline{1,m}$ and $j=\overline{1,n}$, where $c_s^{(ij)}=\sum\limits_{\beta\in J_{s,\,m}\{i\}}\mathrm{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)\,{}_{\beta}^{\beta}$ for all $s=\overline{1,m-1}$ and $c_m^{(ij)}=\mathrm{cdet}_i(\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)$.

We shall prove that $c_k^{(ij)} = 0$, when $k \geq r+1$ for $i = \overline{1, m}$ and $j = \overline{1, n}$.

Since by Lemma 3.2 $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) \leq r$, then the matrix $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right)$ has no more r right-linearly independent columns.

Consider $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$, when $\beta \in J_{s,m}\{i\}$. It is a principal submatrix of $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right)$ of order $s \geq r+1$. Deleting both its i-th row and column, we obtain a principal submatrix of order $s-1$ of $(\mathbf{AW})^{k+2}$. We denote it by $\mathbf{M}$. The following cases are possible.

- Let $s = r+1$ and $\det \mathbf{M} \neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$ keeps their right-linear independence. Hence, they are basis in a matrix $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$, and the i-th column is the right linear combination of its basis columns. From this by Theorem 2.8, we get $\operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta} = 0$, when $\beta \in J_{s,n}\{i\}$ and $s = r+1$.
- If $s = r+1$ and $\det \mathbf{M} = 0$, than p, $(p < s)$, columns are basis in $\mathbf{M}$ and in $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$. Therefore, by Theorem 2.8, $\operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta} = 0$ as well.
- If $s > r+1$, then $\det \mathbf{M} = 0$ and p, $(p < r)$, columns are basis in the both matrices $\mathbf{M}$ and $\left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta}$. Therefore, by Theorem 2.8, we also have $\operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta} = 0$.

Thus, in all cases we have $\operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta} = 0$, when $\beta \in J_{s,m}\{i\}$ and $r+1 \leq s < m$. From here if $r+1 \leq s < m$, then

$$c_s^{(ij)} = \sum_{\beta \in J_{s,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) {}_{\beta}^{\beta} = 0,$$

and $c_m^{(ij)} = \operatorname{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) \right) = 0$ for all $i, j = \overline{1, n}$.

Hence, $\operatorname{cdet}_i \left(\lambda \mathbf{I} + (\mathbf{AW})^{k+2} \right)_{.i} \left(\bar{\mathbf{v}}_{.j}^{(k)} \right) = c_1^{(ij)} \lambda^{m-1} + c_2^{(ij)} \lambda^{m-2} + \ldots + c_r^{(ij)} \lambda^{m-r}$ for $i = \overline{1, m}$ and $j = \overline{1, n}$. By substituting these values in the matrix from (6.31), we obtain

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\begin{pmatrix} \frac{c_1^{(11)}\lambda^{m-1}+...+c_r^{(11)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} & \ldots & \frac{c_1^{(1n)}\lambda^{m-1}+...+c_r^{(1n)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} \\ \ldots & \ldots & \ldots \\ \frac{c_1^{(m1)}\lambda^{m-1}+...+c_r^{(m1)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} & \ldots & \frac{c_1^{(mn)}\lambda^{m-1}+...+c_r^{(mn)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+...+d_r\lambda^{m-r}} \end{pmatrix} = \begin{pmatrix} \frac{c_r^{(11)}}{d_r} & \ldots & \frac{c_r^{(1n)}}{d_r} \\ \ldots & \ldots & \ldots \\ \frac{c_r^{(m1)}}{d_r} & \ldots & \frac{c_r^{(mn)}}{d_r} \end{pmatrix}.$$

Here $c_r^{(ij)} = \sum\limits_{\beta\in J_{r,\,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{AW})^{k+2}_{\,.\,i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}$ and $d_r = \sum\limits_{\beta\in J_{r,\,m}} \left|(\mathbf{AW})^{k+2}\,{}^{\beta}_{\beta}\right|$. Thus, we have the determinantal representation of $\mathbf{A}_{d,W}$ by (6.30). □

The following theorem can be proved similarly.

Theorem 6.11. *If* $\mathbf{A}\in\mathbb{H}^{m\times n}$, $\mathbf{W}\in\mathbb{H}^{n\times m}$, *and* $\mathbf{WA}\in\mathbb{H}^{n\times n}$ *is Hermitian with* $k=\max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $\operatorname{rank}(\mathbf{WA})^{k+1}=\operatorname{rank}(\mathbf{WA})^{k}=r$, *then the W-weighted Drazin inverse* $\mathbf{A}_{d,W}=\left(a_{ij}^{d,W}\right)\in\mathbb{H}^{m\times n}$ *with respect to* $\mathbf{W}$ *possess the following determinantal representations:*

$$a_{ij}^{d,W} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{WA})^{k+2}_{j\,.}(\bar{\mathbf{u}}^{(k)}_{i.})\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}}\left|(\mathbf{WA})^{k+2}\,{}^{\alpha}_{\alpha}\right|}. \tag{6.32}$$

where $\bar{\mathbf{u}}^{(k)}_{i.}$ *is the* i*-th row of* $\bar{\mathbf{U}}^k=\mathbf{A}(\mathbf{WA})^k$ *for all* $i=\overline{1,n}$.

6.4. An Example

Let us consider the matrices

$$\mathbf{A}=\begin{pmatrix} 0 & i & 0 \\ k & 1 & i \\ 1 & 0 & 0 \\ 1 & -k & -j \end{pmatrix},\ \mathbf{W}=\begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix}.$$

Then

$$\mathbf{V} := \mathbf{AW} = \begin{pmatrix} -k & -j & 0 & i \\ -1-j & i+k & j & 1+j \\ k & 0 & i & 0 \\ -i+k & 1-j & i & i-k \end{pmatrix}, \ \mathbf{U} := \mathbf{WA} = \begin{pmatrix} i & j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\operatorname{rank}\mathbf{W} = 3$, $\operatorname{rank}\mathbf{V} = 3$, $\operatorname{rank}\mathbf{V}^3 = \operatorname{rank}\mathbf{V}^2 = 2$, $\operatorname{rank}\mathbf{U}^2 = \operatorname{rank}\mathbf{U} = 2$. Therefore, $Ind\,\mathbf{V} = 2$, $Ind\,\mathbf{U} = 1$, and $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\} = 2$.

It's evident that obtaining the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ by using the matrix $\mathbf{U}$ by (6.19) is more convenient. We have

$$\mathbf{U}^2 = \begin{pmatrix} -1 & i+k & 0 \\ 0 & -1 & \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{U}^5 = \begin{pmatrix} i & 2+3j & 0 \\ 0 & k & \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathbf{U}^5)^* = \begin{pmatrix} -i & 0 & 0 \\ 2-3j & -k & \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\mathbf{U}^5\right)^* \mathbf{U}^5 = \begin{pmatrix} 1 & -2i-3k & 0 \\ 2i+3k & 14 & \\ 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{U}} = (\mathbf{U}^5)^* \mathbf{U}^2 = \begin{pmatrix} i & 1+j & 0 \\ -2+3j & -i+6k & \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{W}^* = \begin{pmatrix} -k & j & 0 \\ 0 & -k & 1 \\ -i & 0 & 0 \\ 0 & 1 & k \end{pmatrix}, \mathbf{W}^*\mathbf{W} = \begin{pmatrix} 2 & i & -j & j \\ -i & 2 & 0 & -2k \\ j & 0 & 1 & 0 \\ -j & 2k & 0 & 2 \end{pmatrix},$$

$$\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^2 = \begin{pmatrix} -k & 1-2j & 0 \\ 0 & i+k & 0 \\ i & 1+j & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

By (6.19),

$$a_{11}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{3} \sum\limits_{\beta \in I_{3,4}\{1\}} \operatorname{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1} (\hat{\mathbf{w}}_{.t}) \right) {}_{\beta}^{\beta} \sum\limits_{\beta \in J_{2,3}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{U}^5 \right)^* \mathbf{U}^5 \right)_{.t} (\hat{\mathbf{u}}_{.1}) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{3,4}} \left| (\mathbf{W}^*\mathbf{W}) {}_{\beta}^{\beta} \right| \sum\limits_{\beta \in J_{2,3}} \left| \left((\mathbf{U}^5)^* \mathbf{U}^5 \right) {}_{\beta}^{\beta} \right|},$$

where

$$\sum_{\beta \in I_{3,4}\{1\}} \operatorname{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1} (\hat{\mathbf{w}}_{.1}) \right) {}_{\beta}^{\beta} =$$

$$\operatorname{cdet}_1 \begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0,$$

$$\sum_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}(\hat{\mathbf{w}}_{.2})\right)_{\beta}^{\beta} = -2j, \sum_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}(\hat{\mathbf{w}}_{.3})\right)_{\beta}^{\beta} = 0,$$

$$\sum_{\beta\in J_{3,\,4}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right| = 2,$$

and

$$\sum_{\beta\in J_{2,\,3}\{1\}} \operatorname{cdet}_1\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.1}(\hat{\mathbf{u}}_{.1})\right)_{\beta}^{\beta} =$$

$$\operatorname{cdet}_1\begin{pmatrix} i & -2i-3k \\ -2+3j & 14 \end{pmatrix} + \operatorname{cdet}_1\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = i,$$

$$\sum_{\beta\in J_{2,\,3}\{2\}} \operatorname{cdet}_2\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.2}(\hat{\mathbf{u}}_{.1})\right)_{\beta}^{\beta} = 0,$$

$$\sum_{\beta\in J_{2,\,3}\{3\}} \operatorname{cdet}_3\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.3}(\hat{\mathbf{u}}_{.1})\right)_{\beta}^{\beta} = 0, \sum_{\beta\in J_{2,\,3}} \left|\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{\beta}^{\beta}\right| = 1.$$

Therefore,

$$a_{11}^{d,\mathbf{W}} = \frac{(0\cdot i)+(-2j\cdot 0)+(0\cdot 0)}{2\cdot 1} = 0.$$

Continuing in the same way, we finally get,

$$\mathbf{A}_{d,W} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.33}$$

By (3.11), we obtain

$$\left(\mathbf{U}^5\right)^+ = \begin{pmatrix} -i & -3+2j & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix},\ (\mathbf{AW})^D = \mathbf{U}^D = \mathbf{U}^2\left(\mathbf{U}^5\right)^+\mathbf{U}^2 = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify (6.33) by (6.2). Indeed,

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix} = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix} = (\mathbf{AW})^D.$$

We also obtain the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ by (6.7), then we have

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}\left((\mathbf{WA})^D)\right)^2 = \begin{pmatrix} 0 & -i & 0 \\ -k & 6+5i & 0 \\ -1 & 5i+5k & 0 \\ -1 & 5i+6k & 0 \end{pmatrix}, \tag{6.34}$$

The W-weighted Drazin inverse in (6.34) different from (6.33). It can be explained that the Jordan normal form of $\mathbf{WA}$ is unique only up to the order of the Jordan blocks. We get their complete equality, if $\mathbf{A}_{d,\mathbf{W}}$ from (6.34) be left-multiply by the nonsingular matrix $\mathbf{P}$ which is the product of multiplication of the following elementary matrices,

$$\mathbf{P} = \mathbf{P}_{2,4}(-k)\cdot\mathbf{P}_{4,3}(-1)\cdot\mathbf{P}_{3,4}(-6)\cdot\mathbf{P}_{4,1}(-j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 7 & -6 \\ -j & 0 & -1 & 1 \end{pmatrix}.$$

7. Cramer's Rule for the W-weighted Drazin Inverse Solution

7.1. Background of the Problem

In [46], Wei has established Cramer's rule for solving of a general restricted equation

$$\mathbf{WAWx} = \mathbf{b},\ \ \mathbf{x} \in \mathcal{R}\left[(\mathbf{AW})^{k_1}\right], \tag{7.1}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W} \in \mathbb{C}^{n\times m}$ with $Ind\,(\mathbf{AW}) = k_1$, $Ind\,(\mathbf{WA}) = k_2$ and $\text{rank}\,(\mathbf{AW})^{k_1} = r_1$, $\text{rank}\,(\mathbf{WA})^{k_2} = r_2$. He proofed if $\mathbf{b} \in \mathcal{R}\left[(\mathbf{W})^{k_2}\mathbf{A}\right]$ and $r_1 = r_2$, then (7.1) has a unique solution, $\mathbf{x} = \mathbf{A}_{d,W}\mathbf{b}$, which can be presented by the following Cramer rule,

$$x_j = \det\begin{pmatrix} \mathbf{WAW}(j \longrightarrow \mathbf{b}) & \mathbf{U}_1 \\ \mathbf{V}_1(j \longrightarrow 0) & 0 \end{pmatrix} \Big/ \det\begin{pmatrix} \mathbf{WAW} & \mathbf{U}_1 \\ \mathbf{V}_1 & 0 \end{pmatrix}, \tag{7.2}$$

where $\mathbf{U}_1 \in \mathbb{C}^{n\times n-r_2}_{n-r_2}$, $\mathbf{V}_1^* \in \mathbb{C}^{m\times m-r_1}_{m-r_1}$ are matrices whose columns form bases for $\mathcal{N}((\mathbf{WA})^{k_2})$ and $\mathcal{N}((\mathbf{AW})^{k_1})$, respectively.

Recently, within the framework of the theory of column-row determinants Song [14] has considered a characterization of the W-weighted Drazin inverse over the quaternion skew and presented Cramer's rule of the restricted matrix equation,

$$\mathbf{W}_1\mathbf{A}\mathbf{W}_1\mathbf{X}\mathbf{W}_2\mathbf{B}\mathbf{W}_2 = \mathbf{D}, \tag{7.3}$$

$$\begin{array}{ll} \mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r\left((\mathbf{A}\mathbf{W}_1)^{k_1}\right) & \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r\left((\mathbf{W}_2\mathbf{B})^{k_2}\right), \\ \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l\left((\mathbf{B}\mathbf{W}_2)^{k_2}\right), & \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l\left((\mathbf{W}_1\mathbf{A})^{k_1}\right), \end{array} \tag{7.4}$$

where $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{H}^{n\times m}$, $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W}_2 \in \mathbb{H}^{q\times p}$, and $\mathbf{D} \in \mathbb{H}^{n\times p}$ with $k_1 = \max\{Ind(\mathbf{A}\mathbf{W}_1), Ind\,(\mathbf{W}_1\mathbf{A})\}$, $k_2 = \max\{Ind(\mathbf{B}\mathbf{W}_2), Ind\,(\mathbf{W}_2\mathbf{B})\}$, and $\operatorname{rank}(\mathbf{A}\mathbf{W}_1)^{k_1} = s_1$, $\operatorname{rank}(\mathbf{B}\mathbf{W}_2)^{k_2} = s_2$. He proved that if

$$\mathcal{R}_r(\mathbf{D}) \in \mathcal{R}_r\left((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2}\right),\ \mathcal{R}_l(\mathbf{D}) \in \mathcal{R}_l\left((\mathbf{A}\mathbf{W}_1)^{k_1}, (\mathbf{B}\mathbf{W}_2)^{k_2}\right)$$

and there exist auxiliary matrices of full column rank, $\mathbf{L}_1 \in \mathbb{H}^{n\times n-s_1}_{n-s_1}$, $\mathbf{M}_1^* \in \mathbb{H}^{m\times m-s_1}_{m-s_1}$, $\mathbf{L}_2 \in \mathbb{H}^{q\times q-s_2}_{q-s_2}$, $\mathbf{M}_2^* \in \mathbb{H}^{p\times p-s_2}_{p-s_2}$ with additional terms of their ranges and null spaces, then the restricted matrix equation (7.3) has a unique solution,

$$\mathbf{X} = \mathbf{A}_{d,\mathbf{W}_1}\mathbf{D}\mathbf{B}_{d,\mathbf{W}_2}.$$

Using auxiliary matrices, $\mathbf{L}_1$, $\mathbf{M}_1$, $\mathbf{L}_2$, $\mathbf{M}_2$, Song presented its Cramer's rule by analogy to (7.2). In this chapter we avoid such approach and obtain explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of matrix equations by using only given matrices.

7.2. Cramer's Rules for the W-weighted Drazin Inverse Solutions of Some Matrix Equations

Consider the matrix equation (7.3) with the constraints (7.4). Denote $\mathbf{A}\mathbf{D}\mathbf{B} =: \tilde{\mathbf{D}} = \left(\tilde{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, and $\bar{\mathbf{V}}\mathbf{D}\bar{\mathbf{U}} =: \bar{\mathbf{D}} = \left(\bar{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, where $\bar{\mathbf{V}} := (\mathbf{A}\mathbf{W}_1)^{k_1}\mathbf{A}$, $\bar{\mathbf{U}} := \mathbf{B}(\mathbf{W}_2\mathbf{B})^{k_2}$.

Theorem 7.1. *Suppose* $\mathbf{D} \in \mathbb{H}^{n\times p}$, $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{H}^{n\times m}_{r_1}$ *with* $k_1 = \max\{Ind(\mathbf{A}\mathbf{W}_1), Ind\,(\mathbf{W}_1\mathbf{A})\}$, *and* $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W}_2 \in \mathbb{H}^{q\times p}_{r_2}$ *with* $k_2 = \max\{Ind(\mathbf{B}\mathbf{W}_2), Ind\,(\mathbf{W}_2\mathbf{B})\}$, *where* $\operatorname{rank}(\mathbf{A}\mathbf{W}_1)^{k_1} = s_1$, $\operatorname{rank}(\mathbf{B}\mathbf{W}_2)^{k_2} = s_2$. *If* $\mathcal{R}_r(\mathbf{D}) \in \mathcal{R}_r\left((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2}\right)$, $\mathcal{R}_l(\mathbf{D}) \in \mathcal{R}_l\left((\mathbf{A}\mathbf{W}_1)^{k_1}, (\mathbf{B}\mathbf{W}_2)^{k_2}\right)$, *then the restricted matrix equation (7.3) has a unique solution,*

$$\mathbf{X} = \mathbf{A}_{d,\mathbf{W}_1}\mathbf{D}\mathbf{B}_{d,\mathbf{W}_2}, \tag{7.5}$$

which possess the following determinantal representations for all $i = \overline{1,m}$, $j = \overline{1,q}$.

i)

$$x_{ij} = \sum_{l=1}^{m} \sum_{f=1}^{q} (v_{il}^{D})^{(2)} \tilde{d}_{lf} (u_{fj}^{D})^{(2)}, \tag{7.6}$$

where $(v_{il}^{D}) = \mathbf{V}^{D}$ *is the Drazin inverse of* $\mathbf{V} = \mathbf{AW}_1$ *and* $(v_{il}^{D})^{(2)}$ *can be obtained by (6.10), and* $(u_{fj}^{D}) = \mathbf{U}^{D}$ *is the Drazin inverse of* $\mathbf{U} = \mathbf{W}_2\mathbf{B}$ *and* $(u_{qj}^{D})^{(2)}$ *can be obtained by (6.8).*

ii) If $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ *and* $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ *are Hermitian, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{s_1,\, m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\mathbf{d}^{\mathbf{B}}_{.\,j} \right) \right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{s_1,\, m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}, \tag{7.7}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\mathbf{d}^{\mathbf{A}}_{i\,.}) \right)^{\alpha}_{\alpha}}{\sum\limits_{\beta \in J_{s_1,\, m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}, \tag{7.8}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha \in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\bar{\mathbf{d}}_{t.}) \right)^{\alpha}_{\alpha} \right) \in \mathbb{H}^{n\times 1}, \quad t = \overline{1,n} \tag{7.9}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta \in J_{s_1,\, m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{d}}_{.\,l} \right) \right)^{\beta}_{\beta} \right) \in \mathbb{H}^{1\times q}, \quad l = \overline{1,q} \tag{7.10}$$

are the column vector and the row vector, respectively. $\bar{\mathbf{d}}_{i.}$ *and* $\bar{\mathbf{d}}_{.j}$ *are the i-th row and the j-th column of* $\bar{\mathbf{D}}$ *for all* $i = \overline{1,n}$, $j = \overline{1,p}$.

Proof. The existence and uniqueness of the solution (7.5) can be proved similar as in ([14], Theorem 5.2).

To derive Cramer's rule (7.6) we use (6.1). Then, we obtain

$$\mathbf{X} = \left((\mathbf{AW}_1)^{D}) \right)^2 \mathbf{ADB} \left((\mathbf{W}_2\mathbf{B})^{D}) \right)^2 . \tag{7.11}$$

Denote $\mathbf{ADB} =: \tilde{\mathbf{D}} = \left(\tilde{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, $\mathbf{V} := \mathbf{AW}_1$, and $\mathbf{U} := \mathbf{W}_2\mathbf{B}$. The equation (7.11) can be written component-wise as follows

$$x_{ij} = \sum_{s=1}^{p}\sum_{t=1}^{n}(a_{it}^{d,W1})d_{ts}(b_{sj}^{d,W_2}) = \sum_{s=1}^{p}\sum_{t=1}^{n}\left(\sum_{l=1}^{m}(v_{il}^{D})^{(2)}a_{lt}\right)d_{ts}\left(\sum_{f=1}^{q}b_{sf}(u_{fj}^{D})^{(2)}\right)$$

By changing the order of summation, from here it follows (7.6).

ii) If $\mathbf{A} \in \mathbb{H}_{r_1}^{m\times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{p\times q}$ and $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ and $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ are Hermitian, then by Theorems 6.10 and 6.11 the W-weighted Drazin inverses $\mathbf{A}_{d,W_1} = \left(a_{ij}^{d,W_1}\right) \in \mathbb{H}^{m\times n}$ and $\mathbf{B}_{d,W_2} = \left(b_{ij}^{d,W_2}\right) \in \mathbb{H}^{q\times p}$ posses the following determinantal representations respectively,

$$a_{ij}^{d,W_1} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{AW_1}\right)^{k_1+2}_{.\,i}\left(\bar{\mathbf{v}}_{.j}\right)\right)\,{}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\;m}}\left|\left(\mathbf{AW_1}\right)^{k_1+2}\,{}^{\beta}_{\beta}\right|}, \tag{7.12}$$

where $\bar{\mathbf{v}}_{.j}$ is the j-th column of $\bar{\mathbf{V}} = (\mathbf{AW_1})^{k_1}\mathbf{A}$ for all $j = \overline{1,m}$, and

$$b_{ij}^{d,W_2} = \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{W_2B}\right)^{k_2+2}_{j\,.}(\bar{\mathbf{u}}_{i.})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{s_2,\,q}}\left|\left(\mathbf{W_2B}\right)^{k_2+2}\,{}^{\alpha}_{\alpha}\right|}, \tag{7.13}$$

where $\bar{\mathbf{u}}_{i.}$ is the i-th row of $\bar{\mathbf{U}} = \mathbf{B}(\mathbf{W_2B})^{k_2}$ for all $i = \overline{1,p}$. By component-wise writing (7.5) we obtain,

$$x_{ij} = \sum_{s=1}^{p}\left(\sum_{t=1}^{n}a_{it}^{d,W1}d_{ts}\right)\cdot b_{sj}^{d,W_2} \tag{7.14}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the s-th column of $\bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW_1})^{k_1}\mathbf{AD} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{m\times p}$

for all $s=\overline{1,p}$. It follows from $\sum\limits_{t}\bar{\mathbf{v}}_{.\,t}d_{ts}=\hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{t=1}^{n} a_{it}^{d,W1} d_{ts} = \sum_{t=1}^{n} \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{v}}_{.t}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right|} \cdot d_{ts} =$$

$$\frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \sum\limits_{t=1}^{n} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{v}}_{.t}\right) \right) {}_{\beta}^{\beta} \cdot d_{ts}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right|} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\hat{\mathbf{d}}_{.\,s}\right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right|} \tag{7.15}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are respectively the unit row-vector and the unit column-vector whose components are 0, except the s-th components, which are 1. Substituting (7.15) and (7.13) in (7.14), we obtain

$$x_{ij} = \sum_{s=1}^{p} \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\hat{\mathbf{d}}_{.\,s}\right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right|} \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\bar{\mathbf{u}}_{s.}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}_{\alpha}^{\alpha} \right|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{t=1}^{n} \mathbf{e}_{.\,t}\hat{d_{ts}},\ \bar{\mathbf{u}}_{s.} = \sum_{l=1}^{q} \bar{u}_{sl}\mathbf{e}_{l.},\ \sum_{s=1}^{p} \hat{d_{ts}}\bar{u}_{sl} = \bar{d}_{tl}, \tag{7.16}$$

then we have

$$x_{ij} =$$

$$\frac{\sum\limits_{s=1}^{p}\sum\limits_{t=1}^{n}\sum\limits_{l=1}^{q}\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\mathbf{e}_{.\,t}\right) \right) {}_{\beta}^{\beta} \hat{d_{ts}}\bar{u}_{sl} \sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\mathbf{e}_{l.}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}_{\alpha}^{\alpha} \right|} =$$

$$\frac{\sum\limits_{t=1}^{n}\sum\limits_{l=1}^{q}\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left((\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\mathbf{e}_{.\,t}\right) \right) {}_{\beta}^{\beta} \bar{d}_{tl} \sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\mathbf{e}_{l.}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}_{\alpha}^{\alpha} \right|}. \tag{7.17}$$

Denote by

$$d_{il}^{\mathbf{A}} := \sum_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{AW}_1)_{.\,i}^{k_1+2} \left(\bar{\mathbf{d}}_{.\,l} \right) \right) {}_{\beta}^{\beta} = \sum_{t=1}^{n} \sum_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{AW}_1)_{.\,i}^{k_1+2} \left(\mathbf{e}_{.\,t} \right) \right) {}_{\beta}^{\beta} \bar{d}_{tl}$$

the l-th component of a row-vector $\mathbf{d}_{i\,.}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, ..., d_{iq}^{\mathbf{A}})$ for all $l = \overline{1, q}$. Substituting it in (7.17), we have

$$x_{ij} = \frac{\sum\limits_{l=1}^{q} d_{il}^{\mathbf{A}} \sum\limits_{\alpha \in I_{s_2, q}\{j\}} \mathrm{rdet}_j \left((\mathbf{W}_2\mathbf{B})_{j\,.}^{k_2+2} (\mathbf{e}_{l.}) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{s_1,\, m}} \left| (\mathbf{AW}_1)^{k_1+2} {}_{\beta}^{\beta} \right| \sum\limits_{\alpha \in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}_{\alpha}^{\alpha} \right|}.$$

Since $\sum\limits_{l=1}^{q} d_{il}^{\mathbf{A}} \mathbf{e}_{l.} = \mathbf{d}_{i\,.}^{\mathbf{A}}$, then it follows (7.8).

If we denote by

$$d_{tj}^{\mathbf{B}} := \sum_{l=1}^{q} \bar{d}_{tl} \sum_{\alpha \in I_{s_2, q}\{j\}} \mathrm{rdet}_j \left((\mathbf{W}_2\mathbf{B})_{j\,.}^{k_2+2} (\mathbf{e}_{l.}) \right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{s_2, q}\{j\}} \mathrm{rdet}_j \left((\mathbf{W}_2\mathbf{B})_{j\,.}^{k_2+2} (\bar{\mathbf{d}}_{t.}) \right)_{\alpha}^{\alpha}$$

the t-th component of a column-vector $\mathbf{d}_{.\,j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, ..., d_{nj}^{\mathbf{B}})^T$ for all $t = \overline{1, n}$ and substituting it in (7.17), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{AW}_1)_{.\,i}^{k_1+2} \left(\mathbf{e}_{.\,t} \right) \right) {}_{\beta}^{\beta} \, d_{tj}^{\mathbf{B}}}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) {}_{\beta}^{\beta} \right| \sum\limits_{\alpha \in I_{r_2, p}} \left| (\mathbf{BB}^*)_{\alpha}^{\alpha} \right|}.$$

Since $\sum\limits_{t=1}^{n} \mathbf{e}_{.t} d_{tj}^{\mathbf{B}} = \mathbf{d}_{.\,j}^{\mathbf{B}}$, then it follows (7.7). □

Remark 7.2. *To establish the Cramer rule of (7.3) we shall not use the determinantal representations (6.30) and (6.30) for (7.5) because corresponding determinantal representations of it's solution will be too cumbersome. But they are suitable in the following corollaries.*

Remark 7.3. *In the complex case, i.e.* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{C}_{r_1}^{n\times m}$, $\mathbf{W}_2 \in \mathbb{C}_{r_2}^{q\times p}$, *and* $\mathbf{D} \in \mathbb{C}^{n\times p}$, *we can substitute usual determinants for all corresponding row and column determinants in (7.6), (7.7) and (7.7).*

Because in the case ii), the conditions $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ *and* $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ *be Hermitian are not necessary, then we have,*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \left| (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|},$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\mathbf{d}^{\mathbf{A}}_{i\,.}) {}^{\alpha}_{\alpha} \right|}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|},$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha\in I_{s_2,q}\{j\}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\bar{\mathbf{d}}_{t.})^{\alpha}_{\alpha} \right| \right) \in \mathbb{C}^{n\times 1}, \quad t = \overline{1,n}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta\in J_{s_1,\,m}\{i\}} \left| (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{d}}_{.\,l}\right)^{\beta}_{\beta} \right| \right) \in \mathbb{C}^{1\times q}, \quad l = \overline{1,q}$$

are the column vector and the row vector, respectively. $\bar{\mathbf{d}}_{i.}$ *and* $\bar{\mathbf{d}}_{.j}$ *are the i-th row and the j-th column of* $\bar{\mathbf{D}}$ *for all* $i = \overline{1,n}$, $j = \overline{1,p}$. *These determinantal representations are most applicable for the complex case.*

Corollary 7.1. *Suppose the following restricted matrix equation is given,*

$$\mathbf{WAWX} = \mathbf{D}, \tag{7.18}$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r\left((\mathbf{AW})^k\right),\ \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l\left((\mathbf{WA})^k\right), \tag{7.19}$$

where $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$, *and* $\mathbf{D} \in \mathbb{H}^{n\times p}$. *If* $\mathcal{R}_r(\mathbf{D}) \subset \mathcal{R}_r\left((\mathbf{AW})^k\right)$ *and* $\mathcal{N}_l(\mathbf{D}) \supset \mathcal{N}_l\left((\mathbf{WA})^k\right)$, *then the restricted matrix equation (7.18-7.19) has a unique solution,*

$$\mathbf{X} = \mathbf{A}_{d,W}\mathbf{D}, \tag{7.20}$$

which possess the following determinantal representations for all $i = \overline{1, m}$, $j = \overline{1, p}$,

i)

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\beta \in J_{r_1,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{W}^*\mathbf{W})_{.t} (\hat{\mathbf{w}}_{.t}) \right)_{\beta}^{\beta} \sum\limits_{\beta \in J_{r,\, n}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{U}^{2k+1}\right)^* \mathbf{U}^{2k+1} \right)_{.t} (\hat{\mathbf{d}}_{.j}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_1,\, m}} \left| (\mathbf{W}^*\mathbf{W})_{\beta}^{\beta} \right| \sum\limits_{\beta \in J_{r,\, n}} \left| ((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \tag{7.21}$$

where $\mathbf{U} = \mathbf{WA}$, $\hat{\mathbf{d}}_{.j}$ *is the* j*-th column of* $\hat{\mathbf{D}} = \hat{\mathbf{U}}\mathbf{D} = (\mathbf{U}^{2k+1})^*\mathbf{U}^k\mathbf{D}$, $\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^k$, *and* $r = \mathrm{rank}(\mathbf{WA})^{k+1} = \mathrm{rank}(\mathbf{WA})^k$.

ii)

$$x_{ij} = \sum_{q=1}^{m} (v_{iq}^{D})^{(2)} r_{qj}, \tag{7.22}$$

where $(v_{iq}^{D})^{(2)}$ *can be obtained by (6.10) and* $\mathbf{AD} = \mathbf{R} = (r_{qj}) \in \mathbb{H}^{m \times p}$.

iii) If $\mathbf{AW} \in \mathbb{H}^{m \times m}$ *is Hermitian, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{AW})^{k+2}_{.i} (\mathbf{f}_{.j}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\, m}} \left| (\mathbf{AW})^{k+2}{}_{\beta}^{\beta} \right|}, \tag{7.23}$$

where $\mathbf{f}_{.j}$ *is the* j*-th column of* $\mathbf{F} = \bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW})^k\mathbf{AD}$.

Proof. To derive a Cramer's rule (7.21), we use the determinantal representation (6.19) for $\mathbf{A}_{d,W}$. Then

$$x_{ij} = \sum_{s=1}^{p} a_{is}^{d,\mathbf{W}} d_{sj} =$$

$$\sum_{s=1}^{p} \left[\frac{\sum\limits_{t=1}^{n} \sum\limits_{\beta \in J_{r_1,\, m}\{i\}} \mathrm{cdet}_i (\mathbf{W}^*\mathbf{W})_{.t} (\hat{\mathbf{w}}_{.t})_{\beta}^{\beta} \sum\limits_{\beta \in J_{r,\, n}\{t\}} \mathrm{cdet}_t \left(\left(\mathbf{U}^{2k+1}\right)^* \mathbf{U}^{2k+1} \right)_{.t} (\hat{\mathbf{u}}_{.s})_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_1,\, m}} \left| (\mathbf{W}^*\mathbf{W})_{\beta}^{\beta} \right| \sum\limits_{\beta \in J_{r,\, n}} \left| ((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \right] d_{sj} \tag{7.24}$$

Denote $\hat{\mathbf{D}} = \hat{\mathbf{U}}\mathbf{D} = (\mathbf{U}^{2k+1})^*\mathbf{U}^k\mathbf{D}$, where $\hat{\mathbf{D}} = \left(\hat{d}_{sj}\right) \in \mathbb{H}^{n \times p}$. Since

$$\sum_{s=1}^{p} \hat{\mathbf{u}}_{.s} d_{sj} = \hat{\mathbf{d}}_{.j},$$

where $\hat{\mathbf{d}}_{.j}$ is the j-th column of $\hat{\mathbf{D}}$, then (7.21) follows from (7.24).

Cramer's rules (7.22) and (7.23) immediately follow from Theorem 7.1 by putting $\mathbf{W}_1 = \mathbf{W}$, $\mathbf{W}_2\mathbf{B} = \mathbf{I}$. □

Remark 7.4. *In the complex case, i.e.* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W} \in \mathbb{C}^{n\times m}_{r_1}$, *and* $\mathbf{D} \in \mathbb{C}^{n\times p}$, *we substitute usual determinants for all corresponding row and column determinants in (7.21), (7.22), and (7.23).*

Note that in the case iii), the condition $\mathbf{AW} \in \mathbb{C}^{m\times m}$ *be Hermitian is not necessary, then in the complex case (7.23) will have the form*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,m}\{i\}} \left| \left((\mathbf{AW})^{k+2}_{.\,i} (\mathbf{f}_{.j}) \right) {}^{\beta}_{\beta} \right|}{\sum\limits_{\beta\in J_{r,\,m}} \left| (\mathbf{AW})^{k+2} {}^{\beta}_{\beta} \right|},$$

where $\mathbf{f}_{.j}$ *is the* j*-th column of* $\mathbf{F} = \bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW})^k\mathbf{AD}$.

Corollary 7.2. *Suppose the following restricted matrix equation is given,*

$$\mathbf{XWBW} = \mathbf{D}, \tag{7.25}$$

$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l\left((\mathbf{BW})^k\right),\ \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r\left((\mathbf{BA})^k\right), \tag{7.26}$$

where $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W} \in \mathbb{H}^{q\times p}_{r_1}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WB})\}$, *and* $\mathbf{D} \in \mathbb{H}^{n\times p}$. *If* $\mathcal{R}_l(\mathbf{D}) \subset \mathcal{R}_l\left((\mathbf{BW})^k\right)$ *and* $\mathcal{N}_r(\mathbf{D}) \supset \mathcal{N}_r\left((\mathbf{WB})^k\right)$, *then the restricted matrix equation (7.25-7.26) has a unique solution,*

$$\mathbf{X} = \mathbf{DB}_{d,W}, \tag{7.27}$$

which possess the following determinantal representations for $i = \overline{1,n}$, $j = \overline{1,q}$,

i)

$$x_{ij} = \frac{\sum\limits_{l=1}^{p} \sum\limits_{\alpha\in I_{r,\,p}\{l\}} \mathrm{rdet}_l \left(\left(\mathbf{V}^{2k+1} \left(\mathbf{V}^{2k+1} \right)^* \right)_{l.} (\check{\mathbf{d}}_{i\,.}) \right) {}^{\alpha}_{\alpha} \sum\limits_{\alpha\in I_{r_1,q}\{j\}} \mathrm{rdet}_j \left((\mathbf{WW}^*)_{j\,.} (\check{\mathbf{w}}_{l.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left| \left(\mathbf{V}^{2k+1} \left(\mathbf{V}^{2k+1} \right)^* \right) {}^{\alpha}_{\alpha} \right| \sum\limits_{\alpha\in I_{r_1,\,n}} \left| (\mathbf{WW}^*) {}^{\alpha}_{\alpha} \right|} \tag{7.28}$$

where $\mathbf{V}=\mathbf{BW}$, $\check{\mathbf{d}}_{i.}$ *is the* i*-th row of* $\check{\mathbf{D}}=\mathbf{D}\check{\mathbf{V}}=\mathbf{D}\mathbf{V}^{k}(\mathbf{V}^{2k+1})^{*}$, $\check{\mathbf{w}}_{l.}$ *is the* l*-th row of* $\check{\mathbf{W}}=\mathbf{V}^{k}\mathbf{W}^{*}$, *and* $r=\operatorname{rank}(\mathbf{BW})^{k+1}=\operatorname{rank}(\mathbf{BW})^{k}$.

ii)

$$x_{ij}=\sum_{t=1}^{q} l_{it}(u_{tj}^{D})^{(2)}, \tag{7.29}$$

where $(u_{tj}^{D})^{(2)}$ *can be obtained by (6.8) and* $\mathbf{DB}=\mathbf{L}=(l_{it})\in\mathbb{H}^{n\times q}$.

iii) If $\mathbf{WB}\in\mathbb{H}^{q\times q}$ *is Hermitian, then*

$$x_{ij}=\frac{\sum\limits_{\alpha\in I_{r,q}\{j\}} \operatorname{rdet}_{j}\left((\mathbf{WB})_{j\,.}^{k+2}(\mathbf{g}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,q}} \left|(\mathbf{WB})^{k+2}{}_{\alpha}^{\alpha}\right|}. \tag{7.30}$$

where $\mathbf{g}_{i.}$ *is the* i*-th row of* $\mathbf{G}=\mathbf{DB}(\mathbf{WB})^{k}$ *for all* $i=\overline{1,n}$.

Proof. The proof is similar to the proof of Corollary 7.1 in the point i), and follows from Theorem 7.1 by putting $\mathbf{W}_2=\mathbf{W}$, $\mathbf{AW}_1=\mathbf{I}$. □

Remark 7.5. *In the complex case, i.e.* $\mathbf{B}\in\mathbb{C}^{p\times q}$, $\mathbf{W}\in\mathbb{C}_{r_1}^{q\times p}$, *and* $\mathbf{D}\in\mathbb{C}^{n\times p}$, *we substitute usual determinants for all corresponding row and column determinants in (7.28), (7.29), and (7.30). Herein the condition* $\mathbf{WB}\in\mathbb{C}^{n\times n}$ *be Hermitian is not necessary, then in the complex case (7.30) can be represented as follows,*

$$x_{ij}=\frac{\sum\limits_{\alpha\in I_{r,q}\{j\}} \left|\left((\mathbf{WB})_{j\,.}^{k+2}(\mathbf{g}_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,q}} \left|(\mathbf{WB})^{k+2}{}_{\alpha}^{\alpha}\right|}.$$

where $\mathbf{g}_{i.}$ *is the* i*-th row of* $\mathbf{G}=\mathbf{DB}(\mathbf{WB})^{k}$ *for all* $i=\overline{1,n}$.

7.3. Examples

1. Let us consider the matrix equation

$$\mathbf{WAWX}=\mathbf{D} \tag{7.31}$$

with the restricted conditions (7.19), where $\mathbf{W}$ and $\mathbf{A}$ are the same as in Example 64., and

$$\mathbf{D} = \begin{pmatrix} k & i \\ i & -j \\ 1 & -i \end{pmatrix}.$$

Therefore, the matrices $\mathbf{V} = \mathbf{AW}$, $\mathbf{U} = \mathbf{WA}$, $\left(\mathbf{U}^5\right)^* \mathbf{U}^5$, $\mathbf{W}^*$, $\mathbf{W}^*\mathbf{W}$, $\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^2$ are the same that in Example 64. as well, and

$$\hat{\mathbf{D}} = (\mathbf{U}^5)^*\mathbf{U}^2\mathbf{D} = \begin{pmatrix} i-j-k & -j \\ 1+3i+6j-2k & 4i-2k \\ 0 & 0 \end{pmatrix}.$$

So, by (7.21)

$$x_{11} = \frac{\sum\limits_{t=1}^{3} \sum\limits_{\beta\in I_{3,\,4}\{1\}} \mathrm{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1}\left(\hat{\mathbf{w}}_{.t}\right)\right) {}_{\beta}^{\beta} \sum\limits_{\beta\in J_{2,\,3}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.t} \left(\hat{\mathbf{d}}_{.1}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{3,\,4}} \left|(\mathbf{W}^*\mathbf{W})\,{}_{\beta}^{\beta}\right| \sum\limits_{\beta\in J_{2,\,3}} \left|(\left(\mathbf{U}^5\right)^* \mathbf{U}^5)\,{}_{\beta}^{\beta}\right|},$$

where

$$\sum_{\beta\in I_{3,4}\{1\}} \mathrm{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1}\left(\hat{\mathbf{w}}_{.1}\right)\right) {}_{\beta}^{\beta} =$$

$$\mathrm{cdet}_1 \begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} + \mathrm{cdet}_1 \begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \mathrm{cdet}_1 \begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0,$$

$$\sum_{\beta\in I_{3,4}\{1\}} \mathrm{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1}\left(\hat{\mathbf{w}}_{.2}\right)\right) {}_{\beta}^{\beta} = -2j,$$

$$\sum_{\beta\in I_{3,4}\{1\}} \mathrm{cdet}_1 \left((\mathbf{W}^*\mathbf{W})_{.1}\left(\hat{\mathbf{w}}_{.3}\right)\right) {}_{\beta}^{\beta} = 0, \sum_{\beta\in J_{3,4}} \left|(\mathbf{W}^*\mathbf{W})\,{}_{\beta}^{\beta}\right| = 2,$$

and

$$\sum_{\beta\in J_{2,3}\{1\}} \mathrm{cdet}_1 \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.1} \left(\hat{\mathbf{d}}_{.1}\right)\right) {}_{\beta}^{\beta} =$$

$$\mathrm{cdet}_1 \begin{pmatrix} i-j-k & -2i-3k \\ 1+3i+6j-2k & 14 \end{pmatrix} + \mathrm{cdet}_1 \begin{pmatrix} i-j-k & 0 \\ 0 & 0 \end{pmatrix} = -2i-j-k,$$

$$\sum_{\beta\in J_{2,3}\{2\}} \mathrm{cdet}_2 \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.2}(\hat{\mathbf{d}}_{.1})\right)_{\beta}^{\beta} = j,$$

$$\sum_{\beta\in J_{2,3}\{3\}} \mathrm{cdet}_3 \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.3}(\hat{\mathbf{d}}_{.1})\right)_{\beta}^{\beta} = 0, \sum_{\beta\in J_{2,3}} \left|\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{\beta}^{\beta}\right| = 1.$$

Therefore,

$$\begin{aligned} x_{11} =& \frac{0\cdot(-2i-j-k)+(-2j)\cdot j+0\cdot 0}{2\cdot 1} = 1,\\ x_{12} =& \frac{0\cdot(-2+2j)+(-2j)\cdot i+0\cdot 0}{2\cdot 1} = k,\\ x_{21} =& \frac{2j\cdot(-2i-j-k)+(10i-4k)\cdot j+0\cdot 0}{2\cdot 1} = 1+i+7k, \end{aligned}$$

$$\begin{aligned} x_{22} =& \frac{2j\cdot(-2+2j)+(10i-4k)\cdot i+0\cdot 0}{2\cdot 1} = -7-4j,\\ x_{31} =& \frac{10i\cdot(-2i-j-k)+j\cdot j+0\cdot 0}{2\cdot 1} = 9.5+5j-5k,\\ x_{32} =& \frac{10i\cdot(-2+2j)+j\cdot i+0\cdot 0}{2\cdot 1} = -10i+9.5k, \end{aligned}$$

We finally get,

$$\mathbf{X} = \begin{pmatrix} 1 & k \\ 1+i+7k & -7-4j \\ 9.5+5j-5k & -10i+9.5k \end{pmatrix}.$$

2. Let now we consider the matrix equation

$$\mathbf{W}_1\mathbf{A}\mathbf{W}_1\mathbf{X}\mathbf{W}_2\mathbf{B}\mathbf{W}_2 = \mathbf{D}, \tag{7.32}$$

with the constraints (7.4), where

$$\mathbf{A} = \begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix}, \mathbf{W}_1 = \begin{pmatrix} k & -j & 0 \\ 0 & k & 1 \\ i & 0 & 0 \\ 0 & 1 & -k \end{pmatrix}, \mathbf{W}_2 = \begin{pmatrix} k & -i \\ j & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} k & j & 0 \\ j & 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} i & -1 \\ k & 0 \\ 0 & j \\ -1 & 0 \end{pmatrix}.$$

Since the following matrices are Hermitian

$$\mathbf{V} = \mathbf{AW}_1 = \begin{pmatrix} -2 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{U} = \mathbf{W}_2\mathbf{B} = \begin{pmatrix} 0 & -i & -i \\ i & -1 & 0 \\ i & 0 & -1 \end{pmatrix},$$

then we can find the W-weighted Drazin inverse solution of (7.32) by its determinantal representation (7.7). We have

$$k_1 = \max\{Ind(\mathbf{AW}_1), Ind\,(\mathbf{W}_1\mathbf{A})\} = 1,$$
$$k_2 = \max\{Ind(\mathbf{BW}_2), Ind\,(\mathbf{W}_2\mathbf{B})\} = 1,$$

and $s_1 = \operatorname{rank}(\mathbf{AW}_1) = 2$, $s_2 = \operatorname{rank}(\mathbf{W}_2\mathbf{B}) = 2$. Since

$$(\mathbf{AW}_1)^3 = \begin{pmatrix} -13 & 8i & 0 \\ -8i & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\mathbf{W}_2\mathbf{B})^3 = \begin{pmatrix} 0 & -3i & -3i \\ 3i & -3 & 0 \\ 3i & 0 & 3 \end{pmatrix},$$

then

$$\sum_{\beta\in J_{2,3}} \left|(\mathbf{AW}_1)^3\,{}^{\beta}_{\beta}\right| = 1, \sum_{\alpha\in I_{2,3}} \left|(\mathbf{W}_2\mathbf{B})^3\,{}^{\alpha}_{\alpha}\right| = -27.$$

Therefore,

$$\bar{\mathbf{D}} = \mathbf{AW}_1\mathbf{ADBW}_2\mathbf{B} = \begin{pmatrix} 2i+j & -7+k & -5+2k \\ -1+k & -5i-j & -4i-2j \\ 0 & 0 & 0 \end{pmatrix}.$$

By (7.9), we can get

$$\mathbf{d}^{\mathbf{B}}_{.1} = \begin{pmatrix} 36i-9j \\ -27-9k \\ 0 \end{pmatrix}, \quad \mathbf{d}^{\mathbf{B}}_{.2} = \begin{pmatrix} -27 \\ -18i \\ 0 \end{pmatrix}, \quad \mathbf{d}^{\mathbf{B}}_{.3} = \begin{pmatrix} 9-9k \\ 9i+3j \\ 0 \end{pmatrix}.$$

Since

$$(\mathbf{AW}_1)^3_{.1}\left(\mathbf{d}^{\mathbf{B}}_{.1}\right) = \begin{pmatrix} 36i-9j & 8i & 0 \\ -27-9k & -5 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

then finally we obtain

$$x_{11} = \frac{\sum\limits_{\beta \in J_{2,3}\{1\}} \operatorname{cdet}_1 \left(\left(\mathbf{AW}_1 \right)^3_{.1} \left(\mathbf{d}^{\mathbf{B}}_{.1} \right) \right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left| \left(\mathbf{AW}_1 \right)^3 {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{2,3}} \left| \left(\mathbf{W}_2 \mathbf{B} \right)^3 {}^{\alpha}_{\alpha} \right|} = \frac{36i - 27j}{-27} = \frac{-4i + 3j}{3},$$

Similarly,

$$x_{12} = \frac{\operatorname{cdet}_1 \begin{pmatrix} -27 & 8i \\ -18i & -5 \end{pmatrix}}{-27} = \frac{1}{3}, \; x_{13} = \frac{\operatorname{cdet}_1 \begin{pmatrix} 9 - 9k & 8i \\ 9i - 3j & -5 \end{pmatrix}}{-27} = \frac{-9 - 7k}{9},$$

$$x_{21} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & 36i - 9j \\ -8i & -27 - 9k \end{pmatrix}}{-27} = \frac{-7 - 5k}{3}, \; x_{22} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & -27 \\ -8i & -18i \end{pmatrix}}{-27} = \frac{-2i}{3},$$

$$x_{23} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & -9 - 9k \\ -8i & 9i + 3j \end{pmatrix}}{-27} = \frac{15i - 11j}{9}, \; x_{31} = x_{32} = x_{33} = 0.$$

So, the W-weighted Drazin inverse solution of (7.32) are

$$\mathbf{X} = \frac{1}{9} \begin{pmatrix} -12i + 9j & 3 & -9 - 7k \\ -21 - 15k & -6i & 15i - 11j \\ 0 & 0 & 0 \end{pmatrix}.$$

Conclusion

In this chapter, we have obtained determinantal representations of the Drazin and W-weighted Drazin inverses over the quaternion skew field. We have derived determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field by the theory of column-row determinants recently introduced by the author. Using obtained determinantal representations of the Drazin inverse we have get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ in both cases when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian and arbitrary. We also have obtain determinantal representations of solutions of the differential quaternion-matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where $\mathbf{A}$ is noninvertible.

Also, we have obtained new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. We have gave determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$ have been obtained.

References

[1] R. E. Cline, T. N. E. Greville, A Drazin inverse for rectangular matrices. *Linear Algebra Appl.* 1980, 29, 53–62.

[2] I. Kyrchei, Cramer's rule for quaternion systems of linear equations. *Fundam Prikl Mat.* 2007, 13 (4), 67–94. (Russian)

[3] I. Kyrchei, The theory of the column and row determinants in a quaternion linear algebra. In: Albert R. Baswell (Ed.), *Advances in Mathematics Research* 15, Nova Sci. Publ., New York, 2012, pp. 301–359.

[4] J. Dieudonnè, Les determinants sur un corps non-commutatif. *Bull Soc Math France.* 1943, 71, 27–45.

[5] E. Study, Zur Theorie der linearen Gleichungen. *Acta Math.* 1920, 42, 1–61.

[6] E. H. Moore, On the determinant of an hermitian matrix of quaternionic elements. *Bull Amer Math Soc.* 1922, 28, 161–162.

[7] F. J. Dyson, Quaternion determinants. *Helv Phys Acta* 1972, 45, 289–302.

[8] L. Chen, Definition of determinant and Cramer solution over the quaternion field. *Acta Math Sinica (N.S.)* 1991, 7, 171–180.

[9] I. M. Gelfand, V. S. Retakh, Determinants of matrices over noncommutative rings. *Functional Anal Appl.* 1991, 25, 91–102.

[10] I. Kyrchei, Determinantal representations of the Moore-Penrose inverse over the quaternion skew field and corresponding Cramer's rules. *Linear Multilinear A.* 2011, 59, 413–431.

[11] I. Kyrchei, Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations. *Linear Algebra Appl.* 2013, 438, 136–152.

[12] G. Song, Q. Wang, H. Chang, Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field. *Comput Math. Appl.* 2011, 61, 1576–1589.

[13] G. J. Song, Q. W. Wang, Condensed Cramer rule for some restricted quaternion linear equations. *Appl Math Comput.* 2011, 218, 3110–3121.

[14] G. Song, Characterization of the W-weighted Drazin inverse over the quaternion skew field with applications. *Electron J Linear Al.* 2013, 26, 1–14.

[15] I. Kyrchei, Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations. *Appl Math Comput.* 2014, 238, 193–207.

[16] I. Kyrchei, Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. *Appl Math Comput.* 2015, 264, 453–465.

[17] I. Kyrchei, Explicit determinantal representation formulas of W-weighted Drazin inverse solutions of some matrix equations over the quaternion skew field. *Math Probl Eng.* 2016, Article ID 8673809, 13 pages.

[18] L. Huang, W. So, On left eigenvalues of a quaternionic matrix. *Linear Algebra Appl.* 2001, 323, 105–116.

[19] W. So, Quaternionic left eigenvalue problem. *Southeast Asian Bulletin of Mathematics* 2005, 29, 555–565.

[20] R. M. W. Wood, Quaternionic eigenvalues. *Bull Lond Math Soc.* 1985, 17, 137–138.

[21] J. L. Brenner, Matrices of quaternions. *Pac J Math.* 1951, 1, 329–335.

[22] E. Macías-Virgós, M.J. Pereira-Sáez, A topological approach to left eigenvalues of quaternionic matrices. *Linear Multilinear A.* 2014, 62, 139–158.

[23] A. Baker, Right eigenvalues for quaternionic matrices: a topological approach. *Linear Algebra Appl.* 1999, 286, 303–309.

[24] T. Dray, C. A. Manogue, The octonionic eigenvalue problem. *Adv Appl Clifford Al.* 1998, 8, 341–364.

[25] F. Zhang, Quaternions and matrices of quaternions. *Linear Algebra Appl.* 1997, 251, 21–57.

[26] D. R. Farenick, B. A. F. Pidkowich, The spectral theorem in quaternions. *Linear Algebra Appl.* 2003, 371, 75–102.

[27] F. O. Farid, Q. W. Wang, F. Zhang, On the eigenvalues of quaternion matrices. *Linear Multilinear A.* 2011, 59, 451–473.

[28] P. Lancaster, M. Tismenitsky, *Theory of matrices*; Acad. Press.: New York, 1969.

[29] M. P. Drazin, Pseudoinverse in associative rings and semigroups. *Am Math Monthly* 1958, 65, 506–514.

[30] S. L. Campbell, C. D. Meyer, *Generalized inverse of linear transformations*, Corrected reprint of the 1979 original; Dover Publications, Inc.: New York, 1991.

[31] A. Ben-Israel, T. N. E. Greville, *Generalized inverses: Theory and Applications, second ed.*; Springer: New York, 2003.

[32] L. Zhang, A characterization of the Drazin inverse. *Linear Algebra Appl.* 2001, 335, 183–188.

[33] R. E. Hartwig, G. Wang, Y. M. Wei, Some additive results on Drazin inverse. *Appl Math Comput.* 2001, 322, 207–217.

[34] I. A. Khan, Q. W. Wang, The Drazin inverses in an arbitrary semiring *Linear Multilinear A.* 2011, 59, 1019–1029.

[35] P. S. Stanimirović, D. S. Djordjević, Full-rank and determinantal representation of the Drazin inverse. *Linear Algebra Appl.* 2000, 311, 131–151.

[36] I. Kyrchei, Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules. *Linear Multilinear A.* 2008, 56, 453–469.

[37] X. Liu, G. Zhu, G. Zhou, Y. Yu, An analog of the adjugate matrix for the Outer Inverse $\mathbf{A}^{(2)}_{T,S}$. *Math Probl Eng.* 2012, Article ID 591256, 14 pages.

[38] I. Kyrchei, Cramer's rule for generalized inverse solutions. In: Ivan I.Kyrchei (Ed.): *Advances in Linear Algebra Research*; New York: Nova Sci. Publ., 2015, pp. 79–132.

[39] C. D. Meyer Jr., Limits and the index of a square matrix. *SIAM J Appl Math.* 1974, 26, 506–515.

[40] S. L. Campbell, C. D. Meyer Jr, N. J. Rose, Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. *SIAM J Appl Math.* 1976, 31, 411–425.

[41] I. Kyrchei, Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations. *Appl Math Comput.* 2013, 219, 7632–7644.

[42] S. Gupta, Linear quaternion equations with application to spacecraft attitude propagation. *IEEE Aerospace conference proceedings* 1998, 1, 69–76.

[43] J. Campos, J. Mawhin, Periodic solutions of quaternionic-values ordinary differential equations. *Annali di Matematica* 2006, 185, S109–S127.

[44] Y. Wei, Integral representation of the W-weighted Drazin inverse. *Appl Math Comput.* 2003, 144, 3–10.

[45] Y. Wei, C.-W. Woo, T. Lei, A note on the perturbation of the W- weighted Drazin inverse. *Appl Math Comput.* 2004, 149, 423–430.

[46] Y. Wei, A characterization for the W-weighted Drazin inverse and a Cramer rule for the W-weighted Drazin inverse solution. *Appl Math Comput.* 2002, 125, 303–310.

[47] Z. Al-Zhour, A. Kiliçman, M. H. Abu Hassa, New representations for weighted Drazin inverse of matrices. *Int Journal of Math Analysis* 2007, 1, 697–708.

[48] D. Mosić, D. S. Djordjević, Additive results for the Wg-Drazin inverse. *Linear Algebra Appl.* 2010, 432, 2847–2860.

[49] X. Liu, Y. Yu, H. Wang, Determinantal representation of weighted generalized inverses, *Appl Math Comput.* 2009, 208, 556–563.

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