ADVANCES IN MATHEMATICS RESEARCH

VOLUME 23

Albert R. Baswell
Editor

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PREFACE

In the opening chapter by Victor Martinez-Luaces, two kinds of matrices related to chemical problems are examined and an outline of their main properties about their eigenvalues is exhibited in order to demonstrate that all the ODE solutions are either stable or asymptotically stable. In chapter two by Ivan Kyrchei, the Cramer rules for the weighted Moore-Penrose solutions of left and right systems of quaternion linear equations are obtained. Next, in chapter three, Tadeusz Antczak showcases numerous sets of saddle point criteria for a new class of nonconvex nonsmooth discrete minimax fractional programing problems. Marcia de F. B. Binelo, Airam T. Z. R. Sausen, Paulo S. Sausen, and Manuel O. Binelo provide a summary of electric mathematical models used for the prediction of batteries charge and discharge behavior in chapter four. In chapter five, general methodology for the precise modeling and performance assessment of launch vehicles dedicated to microsatellites is proposed by M. Pontani, M. Palloney, and P. Teofilattoz. In chapter six, Nodari Vakhania exemplifies ties and relationships among some optimization problems such as scheduling and transportation issues. In chapter seven, a geometry without using points in established by N. L. Bushwick, bringing the book to a close.

In Chapter 1, two kinds of matrices related to chemical problems are analyzed. Firstly, the focus will be put on first order chemical kinetics mechanisms (FOCKM), which are modeled through ODE linear systems, where their associated matrices (FOCKM-matrices) have a particular structure. A summary of the main properties of their eigenvalues will be discussed in this Chapter. Taking into account these results it is possible to prove that all the ODE solutions are stable or asymptotically stable.

The second class of problems to consider are mixing problems (MP), also analyzed in previous works. These problems led to linear ODE systems, for which the associated matrices (MP-matrices) have different structures depending on whether or not there is recirculation of fluids. It can be observed that all the matrix eigenvalues have a non-positive real part and if the mixing problem involves three or less components, then all the eigenvalues have a negative real part and so, the corresponding ODE solutions are asymptotically stable.

Both types of matrices (FOCKM and MP matrices) have similarities and differences and the latter are important enough to obtain different qualitative behaviors of the ODE solutions as analyzed in Chapter 1.

The theory of noncommutative column-row determinants (previously introduced by the author) is extended to determinantal representations of the weighted Moore-Penrose inverse over the quaternion skew field in Chapter 2. To begin with, the authors introduce the weighted singular value decomposition (WSVD) of a quaternion matrix.

Similarly as the singular value decomposition can be used for expressing the Moore-Penrose inverse, Chapter 2 gives the representation of the weighted Moore-Penrose inverse by WSVD. Using this representation, limit and determinantal representations of the weighted Moore-Penrose inverse of a quaternion matrix are derived within the framework of the theory of column-row determinants. By using the obtained analogs of the adjoint matrix, the authors get the Cramer rules for the weighted Moore-Penrose solutions of left and right systems of quaternion linear equations, and for solutions of two-sided restricted quaternion matrix equation in all cases with respect to weighted matrices.

Numerical examples to illustrate the main results are given.

In Chapter 3, the authors present several sets of saddle point criteria for a new class of nonconvex nonsmooth discrete minimax fractional programming problems in which the involving functions are $\left(\Psi, \Phi, \rho\right)$-univex and/or $\left(\Psi, \Phi, \rho\right)$-pseudounivex. The results extend and generalize the corresponding results established earlier in the literature for such nonsmooth optimization problems.

Battery behavior modeling, under different use conditions, can be relatively complex due to the nonlinear nature of the charge and discharge processes. Understanding these dynamics by leveraging mathematical models, favors the development of more efficient batteries and also provides tools for software developers to better manage device resources. A review of electrical mathematical models used in the prediction of battery charge and discharge behavior is presented in Chapter 4. The class of electrical models has been used in various battery modeling applications, including mobile devices and electrical vehicles. The scientific investigation of such models is motivated by their capacity to provide important electrical information such as current, voltage, state of charge, and also some nonlinear aspects of the problem while keeping a relatively low complexity. Six subclasses of electrical models (Simple models, Thévenin-based models, Impedance-based models, Runtime-based models, Combined models and Generic models) along with a discussion of the main characteristics of each. This will demonstrate the evolution of electrical models through successive modification and combination, resulting in varying levels of accuracy and complexity.

Multistage launch vehicles of reduced size, such as "Super Strypi" or "Sword", are currently investigated for the purpose of providing launch opportunities for microsatellites. Chapter 5 proposes a general methodology for the accurate modeling and performance evaluation of launch vehicles dedicated to microsatellites. For illustrative purposes, the approach at hand is applied to the Scout rocket, a micro-launcher used in the past. Aerodynamics and propulsion are modeled with high fidelity through interpolation of available data. Unlike the original Scout, the terminal optimal ascent path is determined for the upper stage, using a firework algorithm in conjunction with the Euler-Lagrange equations and the Pontryagin minimum principle. Firework algorithms represent a recently-introduced heuristic technique, not requiring any starting guess and inspired by the firework explosions in the night sky. The numerically results prove that this methodology is easy-to-implement, robust, precise and computationally effective, although it uses an accurate aerodynamic and propulsive model.

Scheduling and transportation problems are important real-life problems having a wide range of applications in production process, computer systems and routing optimization when the goods are to be distributed to the customers using scarce available resources. In Chapter 6, the authors illustrate ties and relationships among some of these optimization problems. They consider scheduling problem with release and due dates, batch scheduling and vehicle routing problems. As the authors will show here, although these problems seem to have a little in common, a closer look at their parametric and structural properties can give us more insight into the “hidden” ties among these problems that may lead to efficient solution methods.

The construction presented in Chapter 7, like systems of Aristotelian continua presented elsewhere, is designed to establish a geometry without using points. However, it goes further in that the foundation consists of elements that are completely abstract, rather than line segments whose universe is a line. Furthermore, the result could be modified to represent elements and spaces of multiple dimensions.

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Chapter 2

DETERMINANTAL REPRESENTATIONS OF THE QUATERNION WEIGHTED MOORE-PENROSE INVERSE AND ITS APPLICATIONS

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Abstract

The theory of noncommutative column-row determinants (previously introduced by the author) is extended to determinantal representations of the weighted Moore-Penrose inverse over the quaternion skew field in the chapter. To begin with, we introduce the weighted singular value decomposition (WSVD) of a quaternion matrix. Similarly as the singular value decomposition can be used for expressing the Moore-Penrose inverse, we give the representation of the weighted Moore-Penrose inverse by WSVD. Using this representation, limit and determinantal representations of the weighted Moore-Penrose inverse of a quaternion matrix are derived within the framework of the theory of column-row determinants. By using the obtained analogs of the adjoint matrix, we get the Cramer rules for the weighted Moore-Penrose solutions of left and right systems

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of quaternion linear equations, and for solutions of two-sided restricted quaternion matrix equation in all cases with respect to weighted matrices.

Numerical examples to illustrate the main results are given.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion skew field

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \,|\, i^2 = j^2 = k^2 = -1,\, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, and by $\mathbb{H}_r^{m\times n}$ the set of all $m \times n$ matrices over $\mathbb{H}$ with a rank r. Let $\mathrm{M}\left(n, \mathbb{H}\right)$ be the ring of $n \times n$ quaternion matrices and $\mathbf{I}$ be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, we denote by $\mathbf{A}^*$, $\mathrm{rank}\,\mathbf{A}$ the conjugate transpose (Hermitian adjoint) matrix and the rank of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices can be extended to quaternion matrices as follows.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$, denoted by $\mathbf{A}^\dagger$, is the unique matrix $\mathbf{X} \in \mathbb{H}^{n\times m}$ satisfying the following equations [1],

$$\mathbf{AXA} = \mathbf{A}; \tag{1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{2}$$

$$(\mathbf{AX})^* = \mathbf{AX}; \tag{3}$$

$$(\mathbf{XA})^* = \mathbf{XA}. \tag{4}$$

Let Hermitian positive definite matrices $\mathbf{M}$ and $\mathbf{N}$ of order m and n, respectively, be given. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, **the weighted Moore-Penrose inverse** of $\mathbf{A}$ is the unique solution $\mathbf{X} = \mathbf{A}^\dagger_{M,N}$ of the matrix equations (1) and (2) and the following equations in $\mathbf{X}$ [2]:

$$(3M)\ (\mathbf{MAX})^* = \mathbf{MAX};\ (4N)\ (\mathbf{NXA})^* = \mathbf{NXA}.$$

In particular, when $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{N} = \mathbf{I}_n$, the matrix $\mathbf{X}$ satisfying the equations (1), (2), (3M), (4N) is the Moore-Penrose inverse $\mathbf{A}^\dagger$.

It is known various representations of the weighted Moore-Penrose. In particular, limit representations have been considered in [3, 4]. Determinantal representations of the complex (real) weighted Moore-Penrose have been derived by full-rank factorization in [5], by limit representation in [6] using the method first introduced in [7], and by minors in [8]. A basic method for finding the Moore-Penrose inverse is based on the singular value decomposition (SVD). It is available for quaternion matrices, (see, e.g. [9, 10]). The weighted Moore-Penrose inverse $\mathbf{A}^{\dagger}_{M,N} \in \mathbb{C}^{m\times n}$ can be explicitly expressed by the weighted singular value decomposition (WSVD) which at first has been obtained in [11] by Cholesky factorization. In [12], WSVD of real matrices with singular weights has been derived using weighted orthogonal matrices and weighted pseudoorthogonal matrices.

But why determinantal representations of generalized inverses are so important? When we return to the usual inverse, its determinantal representation is the matrix with cofactors in entries that gives direct method of its finding and makes it applicable in Cramer's rule for systems of linear equations. The same be wanted for generalized inverses. But there is not so unambiguous even for complex or real matrices. Therefore, there are various determinantal representations of generalized inverses because of looking of their explicit more applicable expressions (see, e.g. [13]).

The understanding of the problem for determinantal representing of generalized inverses as well as solutions and generalized inverse solutions of quaternion matrix equations, only now begins to be decided due to the theory of column-row determinants introduced in [10, 14]. Within the framework of the theory of noncommutative column-row determinants and using SVD of quaternion matrices, the limit and determinantal representations of the Moore-Penrose inverse over the quaternion skew field have been obtained in [15].

Song at al. [16, 17] have studied the weighted Moore-Penrose inverse over the quaternion skew field and obtained its determinantal representation within the framework of the theory of column-row determinants as well. But WSVD of quaternion matrices has not been considered and for obtaining a determinantal representation there was used auxiliary matrices which different from $\mathbf{A}$, and weights $\mathbf{M}$ and $\mathbf{N}$.

Weighted singular value decomposition (WSVD) and a representation of the weighted Moore-Penrose inverse of a quaternion matrix by WSVD recently have been derived by the author in [18]. The main goals of the chapter are introducing WSVD of quaternion matrices and representation of the weighted Moore-Penrose inverse over the quaternion skew field by WSVD, and then with their help, obtaining its limit and determinantal representations. As applications obtained determinantal representations, we give analogs of Cramer's rule for left and right systems of quaternion linear equations, the quaternion restricted matrix equation $\mathbf{AXB} = \mathbf{D}$, and consequently, $\mathbf{AX} = \mathbf{D}$ and $\mathbf{XB} = \mathbf{D}$.

It need to note that currently the theory of column-row determinants of quaternion matrices is active developing. Within the framework of column-row determinants, determinantal representations of various kind of generalized inverses, (generalized inverses) solutions of quaternion matrix equations recently have been derived as by the author (see, e.g. [19–23]) so by other researchers (see, e.g. [24–27]).

In this chapter we shall adopt the following notation.

Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By $\mathbf{A}^{\alpha}_{\beta}$ denote the submatrix of $\mathbf{A}$ determined by the rows indexed by α, and the columns indexed by β. Then, $\mathbf{A}{\,}^{\alpha}_{\alpha}$ denotes a principal submatrix determined by the rows and columns indexed by α. If $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}^{\alpha}_{\alpha}|$ denote the corresponding principal minor of $\det \mathbf{A}$, since $\mathbf{A}{\,}^{\alpha}_{\alpha}$ is Hermitian as well. For $1 \leq k \leq n$, denote by $L_{k,n} := \{\, \alpha : \alpha = (\alpha_1, \ldots, \alpha_k)\,,\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$ the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let

$$I_{r,\,m}\{i\} := \{\, \alpha : \alpha \in L_{r,m}, i \in \alpha\}, \quad J_{r,\,n}\{j\} := \{\, \beta : \beta \in L_{r,n}, j \in \beta\}.$$

The chapter is organized as follows. We start with some basic concepts and results from the theory of row-column determinants in Section 2. Some provisions of quaternion matrices and quaternion vector spaces are considered in Section 3. Weighted singular value decomposition and a representation of the weighted Moore-Penrose inverse of quaternion matrices by WSVD are considered in Subsection 4.1, and its limit representations in Subsection 4.2. In Section 5, we give the determinantal representations of the weighted Moore-Penrose inverse in both different cases. In Subsection 5.1, the matrices $\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{MA}$ and $\mathbf{AN}^{-1}\mathbf{A}^{*}\mathbf{M}$ are Hermitian, and they are non-Hermitian in Subsection 5.2. In Section 6, we obtain explicit representation formulas of the weighted Moore-Penrose solutions (analogs of Cramer's rule) of the left and right systems of linear equations over the quaternion skew field. In Section 7, we give analogs of Cramer's rule for the quaternion restricted matrix equation $\mathbf{AXB} = \mathbf{D}$, and consequently, $\mathbf{AX} = \mathbf{D}$ and $\mathbf{XB} = \mathbf{D}$. We consider all possible cases. In Section 8, we give numerical examples to illustrate the main results.

2. Preliminaries

For a quadratic matrix $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ can be define n row determinants and n column determinants as follows.

Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 2.1. *[14] The ith row determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is defined for all* $i = 1, \ldots, n$ *by putting*

$$\mathrm{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} \left(a_{i\, i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots . a_{i_{k_1+l_1} i}\right) \ldots \left(a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}\right),$$

$$\sigma = \left(i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}\right) \left(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}\right) \ldots \left(i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}\right),$$

with conditions $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ *and* $i_{k_t} < i_{k_t+s}$ *for* $t = 2, \ldots, r$ *and* $s = 1, \ldots, l_t$.

Definition 2.2. *[14] The* j*th column determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is defined for all* $j = 1, \ldots, n$ *by putting*

$$\mathrm{cdet}_j\, \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} \left(a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} i_{k_r}}\right) \ldots . \left(a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}\right),$$

$$\tau = \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right)\ \left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right),$$

with conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ *and* $j_{k_t} < j_{k_t+s}$ *for* $t = 2, \ldots, r$ *and* $s = 1, \ldots, l_t$.

Suppose $\mathbf{A}^{ij}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting both the ith row and the jth column. Let $\mathbf{a}_{.j}$ be the jth column and $\mathbf{a}_{i.}$ be the ith row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its jth column with the column $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$. We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$.

Proposition 2.1. *[14] If* $b \in \mathbb{H}$, *then*

$$\mathrm{rdet}_i \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i \mathbf{A},$$

$$\mathrm{cdet}_i\, \mathbf{A}_{.i}(\mathbf{a}_{.i} \cdot b) = \mathrm{cdet}_i\, \mathbf{A} \cdot b,$$

for all $i = 1, \ldots, n$.

Proposition 2.2. *[14] If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in I_n$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = 1, \ldots, n$, *then*

$$\begin{array}{l}\mathrm{rdet}_i\, \mathbf{A} = \mathrm{rdet}_i\, \mathbf{A}_{t.}\,(\mathbf{b}) + \mathrm{rdet}_i\, \mathbf{A}_{t.}\,(\mathbf{c})\,,\\ \mathrm{cdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}_{t.}\,(\mathbf{b}) + \mathrm{cdet}_i\, \mathbf{A}_{t.}\,(\mathbf{c})\,,\end{array}$$

where $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$ *and for all* $i = 1, \ldots, n$.

Proposition 2.3. *[14] If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in J_n$ *such that* $a_{i\,t} = b_i + c_i$ *for all* $i = 1, \ldots, n$, *then*

$$\begin{array}{l}\mathrm{rdet}_j\, \mathbf{A} = \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{b}) + \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{c})\,,\\ \mathrm{cdet}_j\, \mathbf{A} = \mathrm{cdet}_j\, \mathbf{A}_{\,.\,t}\,(\mathbf{b}) + \mathrm{cdet}_j\mathbf{A}_{\,.\,t}\,(\mathbf{c})\,,\end{array}$$

where $\mathbf{b} = (b_1, \ldots, b_n)^T$, $\mathbf{c} = (c_1, \ldots, c_n)^T$ *and for all* $j = 1, \ldots, n$.

Remark 2.1. *Let* $\mathrm{rdet}_i\, \mathbf{A} = \sum\limits_{j=1}^{n} a_{ij} \cdot R_{ij}$ *and* $\mathrm{cdet}_j\, \mathbf{A} = \sum\limits_{i=1}^{n} L_{ij} \cdot a_{ij}$ *for all* $i, j = 1, \ldots, n$, *where by* R_{ij} *and* L_{ij} *denote the right and left* (ij)*th cofactors of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, *respectively. It means that* $\mathrm{rdet}_i\, \mathbf{A}$ *can be expand by right cofactors along the* i*th row and* $\mathrm{cdet}_j\mathbf{A}$ *can be expand by left cofactors along the* j*th column, respectively, for all* $i, j = 1, \ldots, n$.

The main property of the usual determinant is that the determinant of a non-invertible matrix must be equal zero. But the row and column determinants don't satisfy it, in general. Therefore, these matrix functions can be consider as some pre-determinants. The following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1. *[14] If* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $\mathrm{rdet}_1\mathbf{A} = \cdots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\mathbf{A} = \cdots = \mathrm{cdet}_n\mathbf{A} \in \mathbb{R}$.

Due to Theorem 2.1, we can define the determinant of a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ by putting,

$$\det \mathbf{A} := \mathrm{rdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}, \tag{5}$$

for all $i = 1, \ldots, n$. By using its row and column determinants, the determinant of a quaternion Hermitian matrix has properties similar to the usual determinant. These properties are completely explored in [14] and can be summarized in the following theorems.

Theorem 2.2. *If the ith row of a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $\mathbf{a}_{i.} = c_1\mathbf{a}_{i_1.} + \ldots + c_k\mathbf{a}_{i_k.}$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{i, i_l\} \subset I_n$, then*

$$\operatorname{rdet}_i \mathbf{A}_{i.}\left(c_1\mathbf{a}_{i_1.} + \ldots + c_k\mathbf{a}_{i_k.}\right) = \operatorname{cdet}_i \mathbf{A}_{i.}\left(c_1\mathbf{a}_{i_1.} + \ldots + c_k\mathbf{a}_{i_k.}\right) = 0.$$

Theorem 2.3. *If the jth column of a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $\mathbf{a}_{.j} = \mathbf{a}_{.j_1}c_1 + \ldots + \mathbf{a}_{.j_k}c_k$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{j, j_l\} \subset J_n$, then*

$$\operatorname{cdet}_j \mathbf{A}_{.j}\left(\mathbf{a}_{.j_1}c_1 + \ldots + \mathbf{a}_{.j_k}c_k\right) = \operatorname{rdet}_j \mathbf{A}_{.j}\left(\mathbf{a}_{.j_1}c_1 + \ldots + \mathbf{a}_{.j_k}c_k\right) = 0.$$

The following theorem about determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 2.4. *[14] If a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is such that $\det \mathbf{A} \neq 0$, then there exist a unique right inverse matrix $(R\mathbf{A})^{-1}$ and a unique left inverse matrix $(L\mathbf{A})^{-1}$, and $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$, which possess the following determinantal representations:*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \tag{6}$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \tag{7}$$

where $\det \mathbf{A} = \sum\limits_{j=1}^{n} a_{ij} \cdot R_{ij} = \sum\limits_{i=1}^{n} L_{ij} \cdot a_{ij}$,

$$R_{ij} = \begin{cases} -\operatorname{rdet}_j \mathbf{A}^{ii}_{.j}\left(\mathbf{a}_{.i}\right), & i \neq j, \\ \operatorname{rdet}_k \mathbf{A}^{ii}, & i = j, \end{cases} \quad L_{ij} = \begin{cases} -\operatorname{cdet}_i \mathbf{A}^{jj}_{i.}\left(\mathbf{a}_{j.}\right), & i \neq j, \\ \operatorname{cdet}_k \mathbf{A}^{jj}, & i = j. \end{cases}$$

The submatrix $\mathbf{A}^{ii}_{.j}\left(\mathbf{a}_{.i}\right)$ is obtained from $\mathbf{A}$ by replacing the jth column with the ith column and then deleting both the ith row and column, $\mathbf{A}^{jj}_{i.}\left(\mathbf{a}_{j.}\right)$ is obtained by replacing the ith row with the jth row, and then by deleting both the jth row and column, respectively. $I_n = \{1, \ldots, n\}$, $k = \min\{I_n \setminus \{i\}\}$, for all $i, j = 1, \ldots, n$.

Theorem 2.5. *[14] If an arbitrary column of* $\mathbf{A} \in \mathbf{H}^{m\times n}$ *is a right linear combination of its other columns, or an arbitrary row of* $\mathbf{A}^*$ *is a left linear combination of its others, then* $\det \mathbf{A}^*\mathbf{A} = 0$.

Theorem 2.6. *[14] The right-linearly independence of columns of* $\mathbf{A} \in \mathbf{H}^{m\times n}$ *or the left-linearly independence of rows of* $\mathbf{A}^*$ *is the necessary and sufficient condition for* $\det \mathbf{A}^*\mathbf{A} \neq 0$.

Theorem 2.7. *If* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, then* $\det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

Definition 2.3. *For* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, the double determinant of* $\mathbf{A}$ *is defined by putting,* $\mathrm{ddet}\,\mathbf{A} := \det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

Theorem 2.8. *The necessary and sufficient condition of invertibility of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is* $\mathrm{ddet}\mathbf{A} \neq 0$*. Then there exists* $\mathbf{A}^{-1} = (L\mathbf{A})^{-1} = (R\mathbf{A})^{-1}$*, where*

$$(L\mathbf{A})^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\,\mathbf{A}^* = \frac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \dots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \dots & \mathbb{L}_{n2} \\ \dots & \dots & \dots & \dots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \dots & \mathbb{L}_{nn} \end{pmatrix}$$

$$(R\mathbf{A})^{-1} = \mathbf{A}^*\,(\mathbf{A}\mathbf{A}^*)^{-1} = \frac{1}{\mathrm{ddet}\mathbf{A}^*}\begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \dots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \dots & \mathbb{R}_{n2} \\ \dots & \dots & \dots & \dots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \dots & \mathbb{R}_{nn} \end{pmatrix}$$

and

$$\mathbb{L}_{ij} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\,(\mathbf{a}^*_{.i})\,,\ \ \mathbb{R}_{\,ij} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\,\left(\mathbf{a}^*_{j.}\right),$$

for all $i, j = 1, \ldots, n$.

Moreover, the following criterion of invertibility of a quaternion matrix can be obtained.

Theorem 2.9. *If* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, then the following statements are equivalent.*

i) $\mathbf{A}$ *is invertible, i.e.* $\mathbf{A} \in GL\,(n, \mathbb{H})$;

ii) rows of $\mathbf{A}$ *are left-linearly independent;*

iii) columns of $\mathbf{A}$ *are right-linearly independent;*

iv) $\mathrm{ddet}\,\mathbf{A} \neq 0$.

2.1. Some Provisions of Quaternion Matrices and Quaternion Vector Spaces

Due to real-scalar multiplying on the right, quaternion column-vectors form a right vector $\mathbb{R}$-space, and, by real-scalar multiplying on the left, quaternion row-vectors form a left vector $\mathbb{R}$-space. Moreover, we define right and left quaternion vector spaces, denoted by $\mathcal{H}_r$ and $\mathcal{H}_l$, respectively, with corresponding $\mathbb{H}$-valued inner products $\langle\cdot,\cdot\rangle$ which satisfy, for every $\alpha,\beta\in\mathbb{H}$, and $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}_r(\mathcal{H}_l)$, the relations:

$$\langle\mathbf{x},\mathbf{y}\rangle=\overline{\langle\mathbf{y},\mathbf{x}\rangle};$$

$$\langle\mathbf{x},\mathbf{x}\rangle\geq 0\in\mathbb{R}\ \text{ and } \|\mathbf{x}\|^2:=\langle\mathbf{x},\mathbf{x}\rangle=0\ \Leftrightarrow\ \mathbf{x}=\mathbf{0};$$

$$\begin{array}{ll}\langle\mathbf{x}\alpha+\mathbf{y}\beta,\mathbf{z}\rangle=\langle\mathbf{x},\mathbf{z}\rangle\alpha+\langle\mathbf{y},\mathbf{z}\rangle\beta & \text{when } \mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}_r\\ \langle\alpha\mathbf{x}+\beta\mathbf{y},\mathbf{z}\rangle=\alpha\langle\mathbf{x},\mathbf{z}\rangle+\beta\langle\mathbf{y},\mathbf{z}\rangle & \text{when } \mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}_l;\end{array}$$

$$\begin{array}{ll}\langle\mathbf{x},\mathbf{y}\alpha+\mathbf{z}\beta\rangle=\overline{\alpha}\langle\mathbf{x},\mathbf{y}\rangle+\overline{\beta}\langle\mathbf{x},\mathbf{z}\rangle & \text{when } \mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}_r\\ \langle\mathbf{x},\alpha\mathbf{y}+\beta\mathbf{z}\rangle=\langle\mathbf{x},\mathbf{y}\rangle\overline{\alpha}+\langle\mathbf{y},\mathbf{z}\rangle\overline{\beta} & \text{when } \mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}_l.\end{array}$$

It can be achieved by putting $\langle\mathbf{x},\mathbf{y}\rangle_r=\overline{y}_1x_1+\cdots+\overline{y}_nx_n$ for $\mathbf{x}=(x_i)_{i=1}^n,\mathbf{y}=(y_i)_{i=1}^n\in\mathcal{H}_r$, and $\langle\mathbf{x},\mathbf{y}\rangle_l=x_1\overline{y}_1+\cdots+x_n\overline{y}_n$ for $\mathbf{x},\mathbf{y}\in\mathcal{H}_l$.

The right vector spaces $\mathcal{H}_r$ possess the Gram-Schmidt process which takes a nonorthogonal set of linearly independent vectors $S=\{\mathbf{v}_1,...,\mathbf{v}_k\}$ for $k\leq n$ and constructs an orthogonal (or orthonormal) basis $S^{'}=\{\mathbf{u}_1,...,\mathbf{u}_k\}$ that spans the same k-dimensional subspace of $\mathcal{H}_r$ as S. To $\mathcal{H}_r$, the following projection operator is defined by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}):=\mathbf{u}\frac{\langle\mathbf{u},\mathbf{v}\rangle_r}{\langle\mathbf{u},\mathbf{u}\rangle_r},$$

which orthogonally projects the vector $\mathbf{v}$ onto the line spanned by the vector $\mathbf{u}$. Then, the GramSchmidt process works as follows:

$$\begin{array}{ll}\mathbf{u}_1=\mathbf{v}_1, & \mathbf{e}_1=\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|},\\ \mathbf{u}_k=\mathbf{v}_k-\sum_{j=1}^{k-1}\mathrm{proj}_{\mathbf{u_j}}(\mathbf{v_k}), & \mathbf{e}_k=\frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.\end{array}$$

The sequence $\mathbf{u}_1,...,\mathbf{u}_k$ is the required system of orthogonal vectors, and the normalized vectors $\mathbf{e}_1,...,\mathbf{e}_k$ form an orthonormal set.

The GramSchmidt process for the left vector spaces $\mathcal{H}_l$ can be realize by the same algorithm but with the projection operator

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) := \frac{\langle \mathbf{u}, \mathbf{v} \rangle_l}{\langle \mathbf{u}, \mathbf{u} \rangle_l}\mathbf{u}.$$

Definition 2.4. *Suppose* $\mathbf{U} \in \mathrm{M}\,(n, \mathbb{H})$ *and* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$*, then the matrix* $\mathbf{U}$ *is called unitary.*

Clear, that columns of $\mathbf{U}$ form a system of normalized vectors in $\mathcal{H}_r$, rows of $\mathbf{U}^*$ is a system of normalized vectors in $\mathcal{H}_l$.

We shall also need the following facts about eigenvalues of quaternion matrices.

Due to the noncommutativity of quaternions, there are two types of eigenvalues. A quaternion λ is said to be a left eigenvalue of $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ if

$$\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}, \tag{8}$$

and a right eigenvalue if

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda \tag{9}$$

for some nonzero quaternion column-vector $\mathbf{x}$. Then, the set $\{\lambda \in \mathbb{H} | \mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{0} \in \mathbb{H}^n\}$ is called the left spectrum of $\mathbf{A}$, denoted by $\sigma_l(\mathbf{A})$. The right spectrum is similarly defined by putting, $\sigma_r(\mathbf{A}) := \{\lambda \in \mathbb{H} | \mathbf{A}\mathbf{x} = \mathbf{x}\lambda, \mathbf{x} \neq \mathbf{0} \in \mathbb{H}^n\}$.

The theory on the left eigenvalues of quaternion matrices has been investigated in particular in [29–31]. The theory on the right eigenvalues of quaternion matrices is more developed [32–37]. We consider this is a natural consequence of the fact that quaternion column vectors form a right vector space for which left eigenvalues from (8) seem to be ”exotic” because of their multiplying from the left. Left eigenvalues may appear natural in the equation

$$\mathbf{x}\mathbf{A} = \lambda\mathbf{x}. \tag{10}$$

Since $\mathbf{x}\mathbf{A} = \lambda\mathbf{x}$ if and only if $\mathbf{A}^*\mathbf{x}^* = \mathbf{x}^*\overline{\lambda}$, then the theory of such ”natural” left eigenvalues from (10) be identical to the theory of right eigenvalues from (9). Similarly, the theory of right eigenvalues from $\mathbf{x}\mathbf{A} = \mathbf{x}\lambda$ is identical to the theory of left eigenvalues from (8).

Now, we present the some known results from the theory of right eigenvalues (9) that will be applied hereinafter. Due to the above, henceforth, we will avoid the ”right” specification.

In particular, it's well known that if λ is a nonreal eigenvalue of $\mathbf{A}$, so is any element in the equivalence class containing $[\lambda]$, i.e. $[\lambda] = \{x | x = u^{-1}\lambda u,\ u \in \mathbb{H},\ \|u\| = 1\}$.

Theorem 2.10. *[32] Any quaternion matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ *has exactly* n *eigenvalues which are complex numbers with nonnegative imaginary parts.*

Proposition 2.4. *[34] Suppose that* $\lambda_1, \ldots, \lambda_n$ *are distinct eigenvalues for* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, no two of which are conjugate, and let* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *be corresponding eigenvectors. Then* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are (right) linearly independent.*

Moreover, similarly to the complex case, the following theorem can be proved.

Theorem 2.11. *A matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is diagonalizable if and only if* $\mathbf{A}$ *has a set of* n *right-linearly independent eigenvectors. Furthermore, if* λ_i*,* $\mathbf{v}_1$*, for* $i = 1, \ldots, n$*, are eigenpairs of* $\mathbf{A}$ *, then*

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where $\mathbf{P} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$*,* $\mathbf{D} = diag\,[\lambda_1, \ldots, \lambda_n]$*.*

Proof. ($\Rightarrow$) Since matrix $\mathbf{A}$ is diagonalizable, there exist an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{11}$$

Since $\mathbf{D} \in \mathrm{M}(n, \mathbb{H})$ is diagonal, it has a right-linearly independent set of n right eigenvectors, given by the column vectors of the identity matrix, i.e.,

$$\mathbf{D}\mathbf{e}_i = \mathbf{e}_i d_{ii},\ \ \mathbf{D} = diag\,[d_{11}, \ldots, d_{nn}]\,,\ \ \mathbf{I} = [\mathbf{e}_1, \ldots, \mathbf{e}_n]\,.$$

So, the pair d_{ii}, $\mathbf{e}_i$ is an eigenvalue-eigenvector pair of $\mathbf{D}$, for all $i = 1, \ldots, n$. Due to $\mathbf{e}_i d_{ii} = d_{ii}\mathbf{e}_i$ and using (11), we have

$$\mathbf{e}_i d_{ii} = d_{ii}\mathbf{e}_i = \mathbf{D}\mathbf{e}_i = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{e}_i. \tag{12}$$

By multiplying the extreme members of (12) by $\mathbf{P}$ on the left, we obtain

$$\mathbf{A}(\mathbf{P}\mathbf{e}_i) = (\mathbf{P}\mathbf{e}_i)d_{ii}.$$

It means that the vectors $\mathbf{v}_i = \mathbf{P}\mathbf{e}_i$ are right eigenvectors of $\mathbf{A}$ with eigenvalue d_{ii} for all $i = 1, \ldots, n$. Since the matrix $\mathbf{P} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is invertible, then, by Theorem 2.9, the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is right-linearly independent.

($\Leftarrow$) Let λ_i, $\mathbf{v}_1$, be eigenvalue-eigenvector pairs of $\mathbf{A}$, for $i = 1, \ldots, n$. Consider the matrix $\mathbf{P} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Computing the product, we obtain

$$\mathbf{AP} = \mathbf{A}\left[\mathbf{v}_1, \ldots, \mathbf{v}_n\right] = \left[\mathbf{A}\mathbf{v}_1, \ldots, \mathbf{A}\mathbf{v}_n\right] = \left[\mathbf{v}_1\lambda_1, \ldots, \mathbf{v}_n\lambda_n\right].$$

Since the eigenvector set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is right-linearly independent, then, by Theorem 2.9, $\mathbf{P}$ is invertible. There exists $\mathbf{P}^{-1}$, and

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\left[\mathbf{v}_1\lambda_1, \ldots, \mathbf{v}_n\lambda_n\right] = \left[\mathbf{P}^{-1}\mathbf{v}_1\lambda_1, \ldots, \mathbf{P}^{-1}\mathbf{v}_n\lambda_n\right].$$

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, then $\mathbf{P}^{-1}\mathbf{v}_i = \mathbf{e}_i$ for all $i = 1, \ldots, n$. So,

$$\mathbf{P}^{-1}\mathbf{AP} = \left[\mathbf{e}_1\lambda_1, \ldots, \mathbf{e}_n\lambda_n\right] = diag\left[\lambda_1, \ldots, \lambda_n\right]$$

Denoting $\mathbf{D} = diag\left[\lambda_1, \ldots, \lambda_n\right]$, we conclude that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, or equivalently, $\mathbf{A} = \mathbf{PDP}^{-1}$. $\square$

Corollary 2.1. *[34]If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *has* n *non-conjugate eigenvalues, then it can be diagonalized in the sense that there is a* $\mathbf{P} \in GL_n(\mathbb{H})$ *for which* $\mathbf{PAP}^{-1}$ *is diagonal.*

Those eigenvalues $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$, where $k_t \geq 0$ and $h_t, k_t \in \mathbb{R}$ for all $t = 1, \ldots, n$, are said to be the standard eigenvalues of $\mathbf{A}$.

Theorem 2.12. *[32] Let* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$. *Then there exists a unitary matrix* $\mathbf{U}$ *such that* $\mathbf{U}^*\mathbf{AU}$ *is an upper triangular matrix with diagonal entries* $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ *which are the standard eigenvalues of* $\mathbf{A}$.

Corollary 2.2. *[36] Let* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *with the standard eigenvalues* $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$. *Then* $\sigma_r = [h_1 + k_1\mathbf{i}] \cup \cdots \cup [h_n + k_n\mathbf{i}]$.

Corollary 2.3. *[36]* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *is normal if and only if there exists an unitary matrix* $\mathbf{U} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *such that*

$$\mathbf{U}^*\mathbf{AU} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\},$$

where $\lambda_i = h_i + k_i\mathbf{i} \in \mathbb{C}$ *is standard eigenvalues for all* $i = 1, \ldots, n$.

Corollary 2.4. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be given. Then,* $\mathbf{A}$ *is Hermitian if and only if there are a unitary matrix* $\mathbf{U} \in \mathrm{M}(n, \mathbb{H})$ *and a real diagonal matrix* $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *such that* $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$*, where* $\lambda_1, ..., \lambda_n$ *are right eigenvalues of* $\mathbf{A}$*.*

The right (9) and left (8) eigenvalues are in general unrelated [38], but it is not for Hermitian matrices. Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda\mathbf{I} - \mathbf{A}$ is Hermitian.

Definition 2.5. *If* $t \in \mathbb{R}$*, then for a Hermitian matrix* $\mathbf{A}$ *the polynomial* $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$ *is said to be the characteristic polynomial of* $\mathbf{A}$*.*

Lemma 2.1. *[14] If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $p_{\mathbf{A}}(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^n d_n$*, where* d_k *is the sum of principle minors of* $\mathbf{A}$ *of order* k*,* $1 \leq k < n$*, and* $d_n = \det \mathbf{A}$*.*

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well.

Definition 2.6. *Let* $\mathbf{A} \in \mathbb{H}^{n \times n}$ *be a Hermitian matrix and* $\pi(\mathbf{A}) = \pi$ *be the number of positive eigenvalues of* $\mathbf{A}$*,* $\nu(\mathbf{A}) = \nu$ *be the number of negative eigenvalues of* $\mathbf{A}$*, and* $\delta(\mathbf{A}) = \delta$ *be the number of zero eigenvalues of* $\mathbf{A}$*. Then the ordered triple* $\omega = (\pi, \nu, \delta)$ *will be called the inertia of* $\mathbf{A}$*. We shall write* $\omega = In\,\mathbf{A}$*.*

We have [9, 10] the following theorem about the singular value decomposition (SVD) of quaternion matrices and their Moore-Penrose inverses.

Theorem 2.13. *(SVD) Let* $\mathbf{A} \in \mathbb{H}_r^{m \times n}$*. There exist unitary quaternion matrices* $\mathbf{V} \in \mathbb{H}^{m \times m}$ *and* $\mathbf{W} \in \mathbb{H}^{n \times n}$ *such that* $\mathbf{A} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*$*, where* $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{H}_r^{m \times n}$*, and* $\mathbf{D} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$*, and* σ_i^2 *is the nonzero eigenvalues of* $\mathbf{A}^*\mathbf{A}$ *for all* $i = 1, ..., r$*. Then* $\mathbf{A}^\dagger = \mathbf{W}\mathbf{\Sigma}^\dagger\mathbf{V}^*$*, where* $\mathbf{\Sigma}^\dagger = \begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{H}_r^{n \times m}$*.*

Due to Theorem 2.13, the limit and determinantal representations of the Moore-Penrose inverse over the quaternion skew field have been obtained as follows.

Lemma 2.2. *[14] If* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{A}^{\dagger}$ *is its Moore-Penrose inverse, then* $\mathbf{A}^{\dagger} = \lim\limits_{\alpha\to 0} \mathbf{A}^{*}\left(\mathbf{A}\mathbf{A}^{*} + \alpha\mathbf{I}\right)^{-1} = \lim\limits_{\alpha\to 0}\left(\mathbf{A}^{*}\mathbf{A} + \alpha\mathbf{I}\right)^{-1}\mathbf{A}^{*}$*, where* $\alpha \in \mathbb{R}_{\dagger}$.

Theorem 2.14. *[14] If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, then the Moore-Penrose inverse* $\mathbf{A}^{\dagger} = \left(a_{ij}^{\dagger}\right) \in \mathbb{H}^{n\times m}$ *possess the following determinantal representations:*

$$a_{ij}^{\dagger} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{*}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{*}\right)\right){}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{*}\mathbf{A}\right){}_{\beta}^{\beta}\right|}, \tag{13}$$

or

$$a_{ij}^{\dagger} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^{*}\right)_{j\,.}\left(\mathbf{a}_{i.}^{*}\right)\right){}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{A}\mathbf{A}^{*}\right){}_{\alpha}^{\alpha}\right|}. \tag{14}$$

Definition 2.7. *A Hermitian matrix* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is called positive (semi)definite if* $\mathbf{x}^{*}\mathbf{A}\mathbf{x} > 0 (\geq 0)$ *for any nonzero vector* $\mathbf{x} \in \mathbb{H}^{n}$.

For quaternion positive definite matrices, the following properties from the set of complex matrices can be obviously expanded. Their proofs are similar to the proofs in the complex case, then we list their without complete profs but with some comments.

Proposition 2.5. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *be a positive definite matrix. Then following properties are equivalent.*

1. *All its eigenvalues are positive.*
2. *Its leading principal minors are all positive.*
3. *The associated sesquilinear form is an inner product.*
4. *It is the Gram matrix of linearly independent vectors.*
5. *It has a unique Cholesky decomposition.*

Proof.

1. By Corollary 2.4 there exists an unitary matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}^*\mathbf{D}\mathbf{U}$, where right eigenvalues $\lambda_1, ..., \lambda_n$ are real. Then $\mathbf{x}^*\mathbf{A}\mathbf{x} = (\mathbf{U}\mathbf{x})^*\,\mathbf{D}\mathbf{U}\mathbf{x}$. By the one-to-one change of variable $\mathbf{y} = \mathbf{U}\mathbf{x}$, $\mathbf{y}^*\mathbf{A}\mathbf{y} > 0$, for any nonzero vector $\mathbf{y} \in \mathbb{H}^n$ when $\mathbf{D}$ is positive definite, It means each element of the main diagonal of $\mathbf{D}$ – that is, every eigenvalue of $\mathbf{A}$ – is positive.

2. Since all leading principal submatrices of a Hermitian matrix are Hermitian, then we can define leading principal minors as determinants of Hermitian submatrices in terms of (5).

3. Let the sesquilinear form by $\mathbf{A}$ be defined the function $\langle\cdot,\cdot\rangle_{r,A}$ from $\mathbb{H}^n \times \mathbb{H}^n$ to $\mathbb{H}$ such that $\langle\mathbf{x},\mathbf{y}\rangle_{r,A} := \mathbf{y}^*\mathbf{A}\mathbf{x}$ for all $\mathbf{x},\mathbf{y} \in \mathbb{H}^n$. By (2.1), the form is an inner product on $\mathbb{H}$ if and only if $\langle\mathbf{x},\mathbf{x}\rangle_{r,A}$ is real and positive for all nonzero $\mathbf{x}$; that is if and only if $\mathbf{A}$ is positive definite.

4. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a set of n linearly independent vectors of $\mathcal{H}_r^n$ with an inner product $\langle\cdot,\cdot\rangle_r$. It can be verified that the Gram matrix $\mathbf{A}$ of those entries, defined by $a_{ij} = \langle x_i, x_j\rangle_r$, is always positive definite.

5. There exists a unique lower triangular matrix $\mathbf{L}$, with real and strictly positive diagonal elements, such that $\mathbf{A} = \mathbf{L}\mathbf{L}^*$. This factorization is called the Cholesky decomposition of $\mathbf{A}$.

Every positive definite matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ has a unique square root defined by $\mathbf{A}^{\frac{1}{2}}$. It means, if $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$ then $\mathbf{A}^{\frac{1}{2}} = \mathbf{U}\mathbf{D}^{\frac{1}{2}}\mathbf{U}^*$.

Lemma 2.3. *[14] Let* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, then* $\mathbf{A}^*\mathbf{A}$ *and* $\mathbf{A}\mathbf{A}^*$ *are both positive (semi)definite, and* r *nonzero eigenvalues of* $\mathbf{A}^*\mathbf{A}$ *and* $\mathbf{A}\mathbf{A}^*$ *coincide.*

Proof. The proof of the second part immediately follows from the singular value decomposition of $\mathbf{A} \in \mathbb{H}_r^{m\times n}$.□

Definition 2.8. *A square matrix* $\mathbf{Q} \in \mathbb{H}^{m\times m}$ *is called* $\mathbf{H}$*-weighted unitary (unitary with weight* $\mathbf{H}$ *) if* $\mathbf{Q}^*\mathbf{H}\mathbf{Q} = \mathbf{I}_m$*, where* $\mathbf{I}_m$ *is the identity matrix.*

The following well-known two facts (see, e.g. [39]) on positive definite and Hermitian matrices and their product obviously can be extended to quaternion matrices.

Lemma 2.4. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *be positive definite and* $\mathbf{B} \in \mathbb{H}^{n\times n}$ *be Hermitian matrices, respectively. Then* $\mathbf{AB}$ *is a diagonalizable matrix, it's all eigenvalues are real, and* $In\mathbf{AB} = In\mathbf{A}$.

Lemma 2.5. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *be positive definite and* $\mathbf{B} \in \mathbb{H}^{n\times n}$ *be Hermitian matrices, respectively. Then there exists nonsingular* $\mathbf{C} \in \mathbb{H}^{n\times n}$ *such that* $\mathbf{C}^*\mathbf{AC} = \mathbf{I}_n$, *and* $\mathbf{C}^*\mathbf{BC} = \mathbf{\Lambda}$, *where* $\mathbf{\Lambda}$ *is a diagonal matrix.*

3. Weighted Singular Value Decomposition and Representations of the Weighted Moore-Penrose Inverse of Quaternion Matrices

3.1. Representations of the Weighted Moore-Penrose Inverse of Quaternion Matrices by WSVD

Denote $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$. Now, we prove the following theorem about the weighted singular value decomposition (WSVD) of quaternion matrices. A similar theorem for real matrices was independently proved by Loan [11] using the Cholesky decomposition, and by Galba [12], where WSVD of $\mathbf{A} \in \mathbb{R}^{m\times n}$ with positive definite weights $\mathbf{B}$ and $\mathbf{C}$ has been described as $\mathbf{A} = \mathbf{UDV}^T\mathbf{C}$. Our method of proving different from analogous for real matrices in [11], and has more similar manner to [12].

Theorem 3.1. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, *and* $\mathbf{M}$ *and* $\mathbf{N}$ *be positive definite matrices of order* m *and* n, *respectively. Then there exist* $\mathbf{U} \in \mathbb{H}^{m\times m}$, $\mathbf{V} \in \mathbb{H}^{n\times n}$ *satisfying* $\mathbf{U}^*\mathbf{MU} = \mathbf{I}_m$ *and* $\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$ *such that*

$$\mathbf{A} = \mathbf{UDV}^*, \tag{15}$$

where $\mathbf{D} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\mathbf{\Sigma} = diag(\sigma_1, \sigma_2, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ *and* σ_i^2 *is the nonzero eigenvalues of* $\mathbf{A}^\sharp\mathbf{A}$ *or* $\mathbf{AA}^\sharp$, *which coincide.*

Proof. First, consider $\mathbf{A}^\sharp\mathbf{A} = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{MA}$. Since $\mathbf{A}^*\mathbf{MA} = (\mathbf{M}^{\frac{1}{2}}\mathbf{A})^*\mathbf{AM}^{\frac{1}{2}}$, then, by Lemma 2.3, $\mathbf{A}^*\mathbf{MA}$ is Hermitian positive semidefinite, and by Lemma 2.4 all eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$ are nonnegative. Denote them by σ_i^2, where $\sigma_1 \geq ... \geq \sigma_n \geq 0$.

Denote $\mathbf{L} = \mathbf{A}^{\sharp}\mathbf{A}$. Since $\mathbf{L}\mathbf{N}^{-1} = \mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}\mathbf{A}\mathbf{N}^{-1}$ is Hermitian and there exists a nonsingular $\mathbf{V} \in \mathbb{H}^{n\times n}$ such that $\mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$, then by Lemma 2.5,

$$\mathbf{V}^{*}\mathbf{L}\mathbf{N}^{-1}\mathbf{V} = \mathbf{\Lambda}, \tag{16}$$

where $\mathbf{V}$ is unitary with weight $\mathbf{N}^{-1}$, and $\mathbf{\Lambda}$ is a diagonal matrix.

It follows from $\mathbf{L} = \mathbf{N}^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{*} = \left(\mathbf{V}^{*}\right)^{-1}\mathbf{\Lambda}\mathbf{V}^{*}$ that $\mathbf{\Lambda} \equiv \mathbf{\Sigma}_1^2$, where $\mathbf{\Sigma}_1^2$ is diagonal with eigenvalues of $\mathbf{A}^{\sharp}\mathbf{A}$ on the principal diagonal, $\sigma_{ii}^2 = \sigma_i^2$ for all $i = 1, ..., n$. Since $\operatorname{rank}\mathbf{L} = \operatorname{rank}\mathbf{A} = r$, then the number of nonzero diagonal elements of $\mathbf{\Sigma}_1^2$ is equal r. Also, we note that

$$\mathbf{V}^{*}\mathbf{L}\mathbf{N}^{-1}\mathbf{V} = \mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}\mathbf{A}\mathbf{N}^{-1}\mathbf{V} = \\ \mathbf{V}^{-1}\mathbf{A}^{*}\left(\mathbf{U}^{*}\right)^{-1}\mathbf{U}^{-1}\mathbf{A}\left(\mathbf{V}^{*}\right)^{-1} = \mathbf{\Sigma}_1^2. \tag{17}$$

Consider the following matrix,

$$\mathbf{P} = \mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-1}\mathbf{V} \in \mathbb{H}^{m\times n}. \tag{18}$$

By virtue of (16),

$$\mathbf{P}^{*}\mathbf{P} = \left(\mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}^{\frac{1}{2}}\right)\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-1}\mathbf{V} = \mathbf{\Sigma}_1^2. \tag{19}$$

Let us introduce the following $m \times n$ matrix $\mathbf{D} \in \mathbb{H}^{m\times n}$,

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{20}$$

where $\mathbf{\Sigma} \in \mathbb{H}^{r\times r}$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ on the principal diagonal. Then,

$$\mathbf{P} = \mathbf{M}^{\frac{1}{2}}\mathbf{U}\mathbf{D}. \tag{21}$$

By (18) and (21), we have $\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-1}\mathbf{V} = \mathbf{M}^{\frac{1}{2}}\mathbf{U}\mathbf{D}$. Due to the equality $\left(\mathbf{N}^{-1}\mathbf{V}\right)^{-1} = \mathbf{V}^{*}$, it follows (15).

Now we shall prove (15), where σ_i^2 is the nonzero eigenvalues of $\mathbf{A}\mathbf{A}^{\sharp} = \mathbf{A}\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}$. Since $\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^{*}$ and $\mathbf{M}$ are respectively Hermitian positive semidefinite and definite, then by by Lemma 2.4 all eigenvalues of $\mathbf{A}\mathbf{A}^{\sharp}$ are nonnegative. Primarily, denote them by τ_i^2, where $\tau_1 \geq ... \geq \tau_m \geq 0$, and denote $\mathbf{Q} = \mathbf{A}\mathbf{A}^{\sharp}$. Since $\mathbf{M}\mathbf{Q} = \mathbf{M}\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}$ is Hermitian and there exists a nonsingular $\mathbf{U} \in \mathbb{H}^{m\times m}$ such that $\mathbf{U}^{*}\mathbf{M}\mathbf{U} = \mathbf{I}_m$, then by Lemma 2.5,

$$\mathbf{U}^{*}\mathbf{M}\mathbf{Q}\mathbf{U} = \mathbf{\Omega}, \tag{22}$$

where $\mathbf{U}$ is unitary with weight $\mathbf{M}$, and $\mathbf{\Omega}$ is a diagonal matrix.

It follows from $\mathbf{Q} = \mathbf{U\Omega U}^*\mathbf{M} = \mathbf{U\Omega U}^{-1}$ that $\mathbf{\Omega} \equiv \mathbf{\Sigma}_2^2$, where $\mathbf{\Sigma}_2^2$ is diagonal with eigenvalues of $\mathbf{AA}^\sharp$ on the principal diagonal, $\tau_{ii}^2 = \tau_i^2$ for all $i = 1, ..., m$. Since $\operatorname{rank} \mathbf{Q} = \operatorname{rank} \mathbf{A} = r$, then the number of nonzero diagonal elements of $\mathbf{\Sigma}_2^2$ is equal r. Also, we have

$$\mathbf{U}^*\mathbf{MQU} = \mathbf{U}^*\mathbf{MAN}^{-1}\mathbf{A}^*\mathbf{MU} = \\ \mathbf{U}^{-1}\mathbf{A}\left(\mathbf{V}^*\right)^{-1}\mathbf{V}^{-1}\mathbf{A}^*\left(\mathbf{U}^*\right)^{-1} = \mathbf{\Sigma}_2^2. \quad (23)$$

Comparing (17) and (23), and due to Lemma 2.3, we have that r nonzero eigenvalues of $\mathbf{AA}^\sharp$ coincide with r nonzero eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$, i.e. $\sigma_i^2 = \tau_i^2$ for all $i = 1, ..., r$.

Consider the following matrix,

$$\mathbf{S} = \mathbf{U}^*\mathbf{MAN}^{-\frac{1}{2}} \in \mathbb{H}^{m\times n}. \quad (24)$$

By virtue of (22),

$$\mathbf{SS}^* = \mathbf{U}^*\mathbf{MAN}^{-\frac{1}{2}}\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{MU}\right) = \mathbf{\Sigma}_2^2. \quad (25)$$

Consider again the matrix $\mathbf{D} \in \mathbb{H}^{m\times n}$ from (20). Then,

$$\mathbf{S} = \mathbf{DV}^*\mathbf{N}^{-\frac{1}{2}}. \quad (26)$$

By (24) and (26), we have $\mathbf{U}^*\mathbf{MAN}^{-\frac{1}{2}} = \mathbf{DV}^*\mathbf{N}^{-\frac{1}{2}}$. From this, due to $(\mathbf{U}^*\mathbf{M})^{-1} = \mathbf{U}$, we again obtain (15).□

Now, we prove the following theorem about a representation of $\mathbf{A}_{M,N}^\dagger$ by WSVD of $\mathbf{A} \in \mathbb{H}_r^{m\times n}$ with weights $\mathbf{M}$ and $\mathbf{N}$.

Theorem 3.2. *Let $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, $\mathbf{M}$ and $\mathbf{N}$ be positive definite matrices of order m and n, respectively. There exist $\mathbf{U} \in \mathbb{H}^{m\times m}$, $\mathbf{V} \in \mathbb{H}^{n\times n}$ satisfying $\mathbf{U}^*\mathbf{MU} = \mathbf{I}_m$ and $\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$ such that $\mathbf{A} = \mathbf{UDV}^*$, where $\mathbf{D} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Then the weighted Moore-Penrose inverse $\mathbf{A}_{M,N}^\dagger$ can be represented*

$$\mathbf{A}_{M,N}^\dagger = \mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*\mathbf{M}, \quad (27)$$

where $\mathbf{\Sigma} = diag(\sigma_1, \sigma_2, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ and σ_i^2 is the nonzero eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$ or $\mathbf{AA}^\sharp$, which coincide.

Proof. To prove the theorem it is enough to show that $\mathbf{X} = \mathbf{A}^{\dagger}_{M,N}$ expressed by (27) satisfies the equations (1), (2), (3N), and (4M).

$$1)\ \mathbf{AXA} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*} = \mathbf{A},$$

$$2)\ \mathbf{XAX} = \mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}\times$$
$$\mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{M} = \mathbf{X},$$

$$(3M)\ (\mathbf{MAX})^{*} = \left(\mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{M}\right)^{*} =$$
$$\mathbf{MU}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{m\times m}\mathbf{U}^{*}\mathbf{M} = \mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}\mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{M} =$$
$$\mathbf{MAX},$$

$$(4N)\ (\mathbf{NXA})^{*} = \left(\mathbf{NN}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}\right)^{*} =$$
$$\mathbf{NN}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{n\times n}\mathbf{V}^{*} = \mathbf{NN}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^{*}\mathbf{MU}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*} =$$
$$\mathbf{NXA}.$$

□

3.2. Limit Representations of the Weighted Moore-Penrose Inverse Over the Quaternion Skew Field

Due to [3] the following limit representation can be extended to $\mathbb{H}$. We give the proof of the following lemma that different from ([3], Corollary 3.4.) and based on WSVD.

Lemma 3.1. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r}$*, and* $\mathbf{M}$ *and* $\mathbf{N}$ *be positive definite matrices of order* m *and* n*, respectively. Then*

$$\mathbf{A}^{\dagger}_{M,N} = \lim_{\lambda\to 0}(\lambda\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A})^{-1}\mathbf{A}^{\sharp}. \tag{28}$$

where $\mathbf{A}^{\sharp} = \mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}$*,* $\lambda \in \mathbb{R}_{+}$ *and* $\mathbb{R}_{+}$ *is the set of all positive real numbers.*

Proof. By Theorems 3.1 and 3.2, respectively, we have

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*,\ \mathbf{A}^{\dagger}_{M,N} = \mathbf{N}^{-1}\mathbf{V}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*\mathbf{M},$$

where $\mathbf{\Sigma} = diag(\sigma_1, \sigma_2, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ and $\sigma_i^2 \in \mathbb{R}$ is the nonzero eigenvalues of $\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\mathbf{A}$. Consider the matrix

$$\mathbf{D} := \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{D} = (\sigma_{ij}) \in \mathbb{H}^{m\times n}_r$ is such that $\sigma_{11} \geq \sigma_{22} \geq \ldots \geq \sigma_{rr} > \sigma_{r+1\,r+1} = \ldots = \sigma_{qq} = 0$, $q = \min\{n, m\}$. Then

$$\mathbf{D}^* = \begin{pmatrix} \mathbf{\Sigma}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ \mathbf{D}^{\dagger} = \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$, $\mathbf{A}^{\sharp} = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M}$, $\mathbf{A}^{\dagger}_{M,N} = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^{\dagger}\mathbf{U}^*\mathbf{M}$. Since $\mathbf{N}^{-1}\mathbf{V} = (\mathbf{V}^*)^{-1}$, then we have

$$\lambda\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A} = \lambda\mathbf{I} + \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M}\mathbf{U}\mathbf{D}\mathbf{V}^* = \lambda\mathbf{I} + (\mathbf{V}^*)^{-1}\mathbf{D}^*\mathbf{D}\mathbf{V}^* = (\mathbf{V}^*)^{-1}(\lambda\mathbf{I} + \mathbf{D}^2)\mathbf{V}^*.$$

Further,

$$(\lambda\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A})^{-1}\mathbf{A}^{\sharp} = (\mathbf{V}^*)^{-1}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M} = \mathbf{N}^{-1}\mathbf{V}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D}^*\mathbf{U}^*\mathbf{M}.$$

Consider the matrix

$$\left(\lambda\mathbf{I} + \mathbf{D}^2\right)^{-1}\mathbf{D} = \begin{pmatrix} \frac{\sigma_1}{\sigma_1^2+\lambda} & \ldots & 0 & & \\ \ldots & \ldots & \ldots & & \mathbf{0} \\ 0 & \ldots & \frac{\sigma_r}{\sigma_r^2+\lambda} & & \vdots \\ & \vdots & & \ddots & \\ & \mathbf{0} & & & \mathbf{0} \end{pmatrix}.$$

It is obviously that $\lim\limits_{\lambda\to 0}\left(\lambda\mathbf{I} + \mathbf{D}^2\right)^{-1}\mathbf{D} = \mathbf{D}^{\dagger}$. Then,

$$\lim_{\lambda\to 0}\mathbf{N}^{-1}\mathbf{V}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D}^*\mathbf{U}^*\mathbf{M} = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^{\dagger}\mathbf{U}^*\mathbf{M} = \mathbf{A}^{\dagger}_{M,N}.$$

The lemma is proofed.□

The following lemma gives another limit representation of $\mathbf{A}^{\dagger}_{M,N}$.

Lemma 3.2. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r}$*, and* $\mathbf{M}$ *and* $\mathbf{N}$ *be positive definite matrices of order* m *and* n*, respectively. Then*

$$\mathbf{A}^{\dagger}_{M,N} = \lim_{\lambda\to 0} \mathbf{A}^{\sharp}(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^{\sharp})^{-1},$$

where $\mathbf{A}^{\sharp} = \mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}$*,* $\lambda \in \mathbb{R}_{+}$.

Proof. The proof is similar to the proof of Lemma 3.1 by using the fact from Theorem 3.1 that the nonzero eigenvalues of $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\sharp}$ coincide.□

It is evidently the following corollary.

Corollary 3.1. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, then the following statements are true.*

i) If $\mathrm{rank}\,\mathbf{A} = \mathrm{n}$*, then* $\mathbf{A}^{\dagger}_{M,N} = \left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{-1}\mathbf{A}^{\sharp}$.

ii) If $\mathrm{rank}\,\mathbf{A} = \mathrm{m}$*, then* $\mathbf{A}^{\dagger}_{M,N} = \mathbf{A}^{\sharp}\left(\mathbf{A}\mathbf{A}^{\sharp}\right)^{-1}$.

iii) If $\mathrm{rank}\,\mathbf{A} = \mathrm{n} = \mathrm{m}$*, then* $\mathbf{A}^{\dagger}_{M,N} = \mathbf{A}^{-1}$.

4. Determinantal Representations of the Weighted Moore-Penrose Inverse Over the Quaternion Skew Field

Even though the eigenvalues of $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\sharp}$ are real and nonnegative, they are not Hermitian in general. Therefor, we consider two cases, when $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\sharp}$ both or one of them are Hermitian, and when they are non-Hermitian.

4.1. The Case of Hermitian $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\sharp}$

Let $(\mathbf{A}^{\sharp}\mathbf{A}) \in \mathbb{H}^{n\times n}$ be Hermitian. It means that $(\mathbf{A}^{\sharp}\mathbf{A})^{*} = (\mathbf{A}^{\sharp}\mathbf{A})$. Since $\mathbf{N}^{-1}$ and $\mathbf{M}$ are Hermitian, then

$$(\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}\mathbf{A})^{*} = \mathbf{A}^{*}\mathbf{M}\mathbf{A}\mathbf{N}^{-1} = \mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}\mathbf{A}.$$

So, to the matrix $(\mathbf{A}^{\sharp}\mathbf{A})$ be Hermitian the matrices $\mathbf{N}^{-1}$ and $(\mathbf{A}^{*}\mathbf{M}\mathbf{A})$ should be commutative. Similarly, to $(\mathbf{A}\mathbf{A}^{\sharp})$ be Hermitian the matrices $\mathbf{M}$ and $(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^{*})$ should be commutative.

Denote by $\mathbf{a}^{\sharp}_{.j}$ and $\mathbf{a}^{\sharp}_{i.}$ the jth column and the ith row of $\mathbf{A}^{\sharp}$ respectively.

Lemma 4.1. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, *then* $\operatorname{rank}\,\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right) \le r.$

Proof. Let's lead elementary transformations of the matrix $\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)$ right-multiplying it by elementary unimodular matrices $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$, $k \ne j$. The matrix $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$ has $-a_{j\,k}$ in the (i,k) entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. Right-multiplying a matrix by $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$, where $k \ne j$, means adding to k-th column its i-th column right-multiplying on $-a_{jk}$. Then we get

$$\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,j}\right)\cdot\prod_{k\ne i}\mathbf{P}_{i\,k}\left(-a_{j\,k}\right) = \begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\ne j} a^{\sharp}_{1k}a_{k1} & \ldots & a^{\sharp}_{1j} & \ldots & \sum\limits_{k\ne j} a^{\sharp}_{1k}a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\ne j} a^{\sharp}_{nk}a_{k1} & \ldots & a^{\sharp}_{nj} & \ldots & \sum\limits_{k\ne j} a^{\sharp}_{nk}a_{kn} \end{array}\right) \\ \qquad\qquad\quad ith \end{array}.$$

The obtained matrix has the following factorization.

$$\begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\ne j} a^{\sharp}_{1k}a_{k1} & \ldots & a^{\sharp}_{1j} & \ldots & \sum\limits_{k\ne j} a^{\sharp}_{1k}a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\ne j} a^{\sharp}_{nk}a_{k1} & \ldots & a^{\sharp}_{nj} & \ldots & \sum\limits_{k\ne j} a^{\sharp}_{nk}a_{kn} \end{array}\right) \\ \qquad\qquad ith \end{array} =$$

$$= \left(\begin{array}{cccc} a^{\sharp}_{11} & a^{\sharp}_{12} & \ldots & a^{\sharp}_{1m} \\ a^{\sharp}_{21} & a^{\sharp}_{22} & \ldots & a^{\sharp}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a^{\sharp}_{n1} & a^{\sharp}_{n2} & \ldots & a^{\sharp}_{nm} \end{array}\right) \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \ldots & 0 & \ldots & a_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & 0 & \ldots & a_{mn} \end{array}\right) \\ \qquad\quad ith \end{array} jth.$$

Denote by $\tilde{\mathbf{A}} := \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \ldots & 0 & \ldots & a_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & 0 & \ldots & a_{mn} \end{array}\right) \\ \qquad\quad ith \end{array} jth.$ The matrix $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by replacing all entries of the jth row and of the ith column with zeroes except that the (j,i)th entry equals 1. Elementary transformations of a matrix do not change its rank and a rank of a matrix product does not exceed ranks of factors. It follows that $\operatorname{rank}\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right) \le$

$\min\left\{\operatorname{rank}\mathbf{A}^{\sharp}, \operatorname{rank}\tilde{\mathbf{A}}\right\}$. It is obviously that $\operatorname{rank}\tilde{\mathbf{A}} \geq \operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}^{\sharp}$. This completes the proof. □

The following lemma can be proved similarly.

Lemma 4.2. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, *then* $\operatorname{rank}\left(\mathbf{A}\mathbf{A}^{\sharp}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{\cdot j}\right) \leq r$.

Analogues of the characteristic polynomial are considered in the following lemmas.

Lemma 4.3. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $t \in \mathbb{R}$, *and* $(\mathbf{A}^{\sharp}\mathbf{A})$ *is Hermitian, then*

$$\operatorname{cdet}_i\left(t\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{\cdot j}\right) = c_1^{(ij)}t^{n-1} + c_2^{(ij)}t^{n-2} + \cdots + c_n^{(ij)}, \tag{29}$$

where $c_n^{(ij)} = \operatorname{cdet}_i\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{\cdot\, j}\right)$ *and*

$$c_k^{(ij)} = \sum_{\beta\in J_{k,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{\cdot\, j}\right)\right)\,{}^{\beta}_{\beta}$$

for all $k = 1,\ldots,n-1$, $i = 1,\ldots,n$, *and* $j = 1,\ldots,m$.

Proof. Denote by $\mathbf{b}_{.i}$ the ith column of the Hermitian matrix $\mathbf{A}^{\sharp}\mathbf{A} =: (b_{ij})_{n\times n}$. Consider the Hermitian matrix $\left(t\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A}\right)_{.i}(\mathbf{b}_{.i}) \in \mathbb{H}^{n\times n}$. It differs from $\left(t\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A}\right)$ in an entry b_{ii}. Taking into account Lemma 2.1, we obtain

$$\det\left(t\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{b}_{.i}) = d_1t^{n-1} + d_2t^{n-2} + \ldots + d_n, \tag{30}$$

where $d_k = \sum\limits_{\beta\in J_{k,\,n}\{i\}} \det\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}$ is the sum of all principal minors of order k that contain the ith column for all $k = 1,\ldots,n-1$ and $d_n = \det\left(\mathbf{A}^{\sharp}\mathbf{A}\right)$. Therefore, we have

$$\mathbf{b}_{.\,i} = \begin{pmatrix} \sum\limits_l a^{\sharp}_{1l}a_{li} \\ \sum\limits_l a^{\sharp}_{2l}a_{li} \\ \vdots \\ \sum\limits_l a^{\sharp}_{nl}a_{li} \end{pmatrix} = \sum_l \mathbf{a}^{\sharp}_{.\,l}a_{li},$$

where $\mathbf{a}^{\sharp}_{.\,l}$ is the lth column-vector of $\mathbf{A}^{\sharp}$ for all $l = 1, \ldots, m$. Taking into account Theorem 2.1, Remark 2.1 and Proposition 2.1, on the one hand we obtain

$$\begin{gathered}\det\left(t\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{b}_{.\,i}) = \mathrm{cdet}_i\left(t\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{b}_{.\,i}) = \\ = \sum_{l}\mathrm{cdet}_i\left(t\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,l}\left(\mathbf{a}^{\sharp}_{.\,l}a_{l\,i}\right) = \sum_{l}\mathrm{cdet}_i\left(t\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.l}\right)\cdot a_{li}\end{gathered} \tag{31}$$

On the other hand having changed the order of summation, we get for all $k = 1, \ldots, n-1$

$$\begin{gathered}d_k = \sum_{\beta\in J_{k,\,n}\{i\}}\det\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta} = \sum_{\beta\in J_{k,\,n}\{i\}}\mathrm{cdet}_i\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta} = \\ \sum_{\beta\in J_{k,\,n}\{i\}}\sum_{l}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,l}a_{l\,i}\right)\right)_{\beta}^{\beta} = \\ \sum_{l}\sum_{\beta\in J_{k,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,l}\right)\right)_{\beta}^{\beta}\cdot a_{l\,i}.\end{gathered} \tag{32}$$

By substituting (31) and (32) in (30), and equating factors at $a_{l\,i}$ when $l = j$, we obtain the equality (29). □

The following lemma can be proved similarly.

Lemma 4.4. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $t \in \mathbb{R}$, *and* $\mathbf{A}\mathbf{A}^{\sharp}$ *is Hermitian, then*

$$\mathrm{rdet}_j(t\mathbf{I}+\mathbf{A}\mathbf{A}^{\sharp})_{j\,.}(\mathbf{a}^{\sharp}_{i.}) = r_1^{(ij)}t^{n-1} + r_2^{(ij)}t^{n-2} + \cdots + r_n^{(ij)},$$

where $r_n^{(ij)} = \mathrm{rdet}_j(\mathbf{A}\mathbf{A}^{\sharp})_{j\,.}(\mathbf{a}^{\sharp}_{i.})$ *and*

$$r_k^{(ij)} = \sum_{\alpha\in I_{r,m}\{j\}}\mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^{\sharp})_{j\,.}(\mathbf{a}^{\sharp}_{i.})\right)_{\alpha}^{\alpha}$$

for all $k = 1, \ldots, n-1$, $i = 1, \ldots, n$, *and* $j = 1, \ldots, m$.

The following theorem introduces the determinantal representations of the weighted Moore-Penrose by analogs of the classical adjoint matrix.

Denote the (ij)th entry of $\mathbf{A}^{\dagger}_{M,N}$ by $a^{\ddagger}_{ij}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Theorem 4.1. *Let* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$. *If* $\mathbf{A}^\sharp\mathbf{A}$ *or* $\mathbf{A}\mathbf{A}^\sharp$ *are Hermitian, then the weighted Moore-Penrose inverse* $\mathbf{A}^\dagger_{M,N} = \left(a^\ddagger_{ij}\right) \in \mathbb{H}^{n\times m}$ *possess the following determinantal representations, respectively,*

$$a^\ddagger_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{\cdot j}\right)\right){}_\beta^\beta}{\sum\limits_{\beta\in J_{r,\ n}} \left|\left(\mathbf{A}^\sharp\mathbf{A}\right){}_\beta^\beta\right|}, \tag{33}$$

or

$$a^\ddagger_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^\sharp)_{j\,.}(\mathbf{a}^\sharp_{i.})\right){}_\alpha^\alpha}{\sum\limits_{\alpha\in I_{r,\ m}} |(\mathbf{A}\mathbf{A}^\sharp){}_\alpha^\alpha|}. \tag{34}$$

Proof. At first we prove (33). By Lemma 3.1, $\mathbf{A}^\dagger = \lim\limits_{\lambda\to 0}\left(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}\right)^{-1}\mathbf{A}^\sharp$. Let $\mathbf{A}^\sharp\mathbf{A}$ is Hermitian. Then the matrix $\left(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}\right) \in \mathbb{H}^{n\times n}$ is a full-rank Hermitian matrix for $\forall\lambda \in \mathbb{R}_+$. Taking into account Theorem 2.4 it has an inverse, which we represent as a left inverse matrix

$$\left(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}\right)^{-1} = \frac{1}{\det\left(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}\right)}\begin{pmatrix} L_{11} & L_{21} & \ldots & L_{n1} \\ L_{12} & L_{22} & \ldots & L_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ L_{1n} & L_{2n} & \ldots & L_{nn} \end{pmatrix},$$

where L_{ij} is a left (ij)th cofactor of $\alpha\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}$. Then we have

$$\begin{gathered}\left(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A}\right)^{-1}\mathbf{A}^\sharp = \\ = \frac{1}{\det\left(\lambda\mathbf{I}+\mathbf{A}^\sharp\mathbf{A}\right)}\begin{pmatrix} \sum\limits_{k=1}^n L_{k1}a^\sharp_{k1} & \sum\limits_{k=1}^n L_{k1}a^\sharp_{k2} & \ldots & \sum\limits_{k=1}^n L_{k1}a^\sharp_{km} \\ \sum\limits_{k=1}^n L_{k2}a^\sharp_{k1} & \sum\limits_{k=1}^n L_{k2}a^\sharp_{k2} & \ldots & \sum\limits_{k=1}^n L_{k2}a^\sharp_{km} \\ \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k=1}^n L_{kn}a^\sharp_{k1} & \sum\limits_{k=1}^n L_{kn}a^\sharp_{k2} & \ldots & \sum\limits_{k=1}^n L_{kn}a^\sharp_{km} \end{pmatrix}.\end{gathered}$$

Using the definition of a left cofactor, we obtain

$$\mathbf{A}_{M,N}^{\dagger} = \lim_{\alpha \to 0} \begin{pmatrix} \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.1}\left(\mathbf{a}_{.1}^{\sharp}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)} & \ldots & \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.1}\left(\mathbf{a}_{.m}^{\sharp}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)} \\ \ldots & \ldots & \ldots \\ \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.n}\left(\mathbf{a}_{.1}^{\sharp}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)} & \ldots & \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)_{.n}\left(\mathbf{a}_{.m}^{\sharp}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right)} \end{pmatrix}. \tag{35}$$

By Lemma 2.1, we have $\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_n$, where $d_k = \sum\limits_{\beta \in J_{k,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta}\right|$ is a sum of principal minors of $\mathbf{A}^{\sharp}\mathbf{A}$ of order k for all $k = 1, \ldots, n-1$ and $d_n = \det \mathbf{A}^{\sharp}\mathbf{A}$. Since $\operatorname{rank} \mathbf{A}^{\sharp}\mathbf{A} = \operatorname{rank} \mathbf{A} = r$ and $d_n = d_{n-1} = \cdots = d_{r+1} = 0$, it follows that

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{\sharp}\mathbf{A}\right) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_r\lambda^{n-r}.$$

Using (29), we have

$$\mathrm{cdet}_i\left(\lambda\mathbf{I}+\mathbf{A}^{*}\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.j}^{\sharp}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where

$$c_k^{(ij)} = \sum_{\beta \in J_{k,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right)\right)_{\beta}^{\beta}$$

for all $k = 1, \ldots, n-1$ and $c_n^{(ij)} = \mathrm{cdet}_i\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.j}^{\sharp}\right)$.

Now we prove that $c_k^{(ij)} = 0$, when $k \geq r+1$ for all $i = 1, \ldots, n$, and $j = 1, \ldots, m$. By Lemma 4.1 $\operatorname{rank}\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right) \leq r$, then the matrix $\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right)$ has no more r right-linearly independent columns.

Consider $\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right)\right)_{\beta}^{\beta}$, when $\beta \in J_{k,n}\{i\}$. It is a principal submatrix of $\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}_{.j}^{\sharp}\right)$ of order $k \geq r+1$. Deleting both its ith row and column, we obtain a principal submatrix of order $k-1$ of $\mathbf{A}^{\sharp}\mathbf{A}$. We denote it by $\mathbf{M}$. The following cases are possible.

Let $k = r+1$ and $\det \mathbf{M} \neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta}$ keeps their right-linear independence. Hence, they are basis in a matrix $\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta}$, and the ith column is the right linear combination of its basic columns. From this by Theorem 2.5, we get $\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta} = 0$, when $\beta \in J_{k,n}\{i\}$ and $k \geq r+1$.

If $k = r+1$ and $\det \mathbf{M} = 0$, than p, ($p < k$), columns are basis in $\mathbf{M}$ and in $\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta}$. Then by Theorem 2.5, we obtain $\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta} = 0$ as well.

If $k > r+1$, then by Theorem 2.6 it follows that $\det \mathbf{M} = 0$ and p, ($p < k-1$), columns are basis in the both matrices $\mathbf{M}$ and $\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta}$. Therefore, by Theorem 2.5, we obtain $\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta} = 0$.

Thus in all cases, we have $\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta} = 0$, when $\beta \in J_{k,n}\{i\}$ and $r+1 \leq k < n$, and for all $i = 1, \dots, n$ and $j = 1, \dots, m$,

$$c_k^{(ij)} = \sum_{\beta \in J_{k,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right)^{\beta}_{\beta} = 0,$$

$$c_n^{(ij)} = \mathrm{cdet}_i\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right) = 0.$$

Hence, $\mathrm{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,j}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \cdots + c_r^{(ij)}\lambda^{n-r}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. By substituting these values in the matrix from (35), we obtain

$$\mathbf{A}^{\dagger}_{M,N} = \lim_{\lambda \to 0} \begin{pmatrix} \frac{c_1^{(11)}\lambda^{n-1}+\ldots+c_r^{(11)}\lambda^{n-r}}{\alpha^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \cdots & \frac{c_1^{(1m)}\lambda^{n-1}+\ldots+c_r^{(1m)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \\ \cdots & \cdots & \cdots \\ \frac{c_1^{(n1)}\lambda^{n-1}+\ldots+c_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \cdots & \frac{c_1^{(nm)}\lambda^{n-1}+\ldots+c_r^{(nm)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \end{pmatrix} = \begin{pmatrix} \frac{c_r^{(11)}}{d_r} & \cdots & \frac{c_r^{(1m)}}{d_r} \\ \cdots & \cdots & \cdots \\ \frac{c_r^{(n1)}}{d_r} & \cdots & \frac{c_r^{(nm)}}{d_r} \end{pmatrix}.$$

Here

$$c_r^{(ij)} = \sum_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^\sharp \mathbf{A} \right)_{.\,i} \left(\mathbf{a}^\sharp_{.j} \right) \right)^{\beta}_{\beta}$$

and $d_r = \sum\limits_{\beta \in J_{r,\ n}} \left| \left(\mathbf{A}^\sharp \mathbf{A} \right)^{\beta}_{\beta} \right|$. Thus, we have obtained the determinantal representation of $\mathbf{A}^\dagger_{M,N}$ by (33).

The determinantal representation of $\mathbf{A}^\dagger_{M,N}$ by (34) can be proved similarly. □

Corollary 4.1. *If* $\operatorname{rank}\mathbf{A} = n < m$*, and* $(\mathbf{A}^\sharp \mathbf{A})$ *is Hermitian, then we get the following representation of* $\mathbf{A}^\dagger_{M,N} = \left(a^\ddagger_{ij} \right) \in \mathbb{H}^{n\times m}$*,*

$$a^\ddagger_{ij} = \frac{\operatorname{cdet}_i (\mathbf{A}^\sharp \mathbf{A})_{.\,i} \left(\mathbf{a}^\sharp_{.\,j} \right)}{\det(\mathbf{A}^\sharp \mathbf{A})}, \tag{36}$$

or the determinantal representation (33) can be applicable as well.

Proof. By Corollary 3.1, $\mathbf{A}^\dagger_{M,N} = \left(\mathbf{A}^\sharp \mathbf{A} \right)^{-1} \mathbf{A}^\sharp$. Considering Hermitian $\left(\mathbf{A}^\sharp \mathbf{A} \right)^{-1}$ as a left inverse by (7), we obtain

$$a^\ddagger_{ij} = \frac{\sum_k L_{ki} a^\sharp_{kj}}{\det(\mathbf{A}^\sharp \mathbf{A})},$$

where L_{ki} is the (ki)th cofactor of $\det(\mathbf{A}^\sharp \mathbf{A}) = \operatorname{cdet}_i(\mathbf{A}^\sharp \mathbf{A})$. By its definition, $\sum_k L_{ki} a^\sharp_{kj} = \operatorname{cdet}_i (\mathbf{A}^\sharp \mathbf{A})_{.i} \left(\mathbf{a}^\sharp_{.j} \right)$. From this, (36) follows immediately. □

Corollary 4.2. *If* $\operatorname{rank}\mathbf{A} = m$*, and* $(\mathbf{A}\mathbf{A}^\sharp)$ *is Hermitian, then*

$$a^\ddagger_{ij} = \frac{\operatorname{rdet}_j (\mathbf{A}\mathbf{A}^\sharp)_{j.} \left(\mathbf{a}^\sharp_{i.} \right)}{\det(\mathbf{A}\mathbf{A}^\sharp)}. \tag{37}$$

or the determinantal representation (34) can be applicable as well.

Proof. By Corollary 3.1, $\mathbf{A}^\dagger_{M,N} = \mathbf{A}^\sharp \left(\mathbf{A}\mathbf{A}^\sharp \right)^{-1}$. Considering $\left(\mathbf{A}\mathbf{A}^\sharp \right)^{-1}$ as a right inverse by (6), we get the following representation

$$a^\ddagger_{ij} = \frac{\sum_k a^\sharp_{ik} R_{jk}}{\det(\mathbf{A}^\sharp \mathbf{A})},$$

where R_{jk} is the (jk)th cofactor of $\det(\mathbf{A}\mathbf{A}^{\sharp}) = \mathrm{rdet}_j(\mathbf{A}\mathbf{A}^{\sharp})$. By its definition, $\sum_k a^{\sharp}_{ik} R_{jk} = \mathrm{rdet}_j(\mathbf{A}\mathbf{A}^{\sharp})_{j.}(\mathbf{a}^{\sharp}_{i.})$. From this, (37) follows immediately. □

We also can obtain determinantal representations of the projection matrices $\mathbf{A}^{\dagger}_{M,N}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\dagger}_{M,N}$ in the following corollaries.

Corollary 4.3. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r}$ *and* $\mathbf{A}^{\sharp}\mathbf{A}$ *is Hermitian, then the projection matrix* $\mathbf{A}^{\dagger}_{M,N}\mathbf{A} =: \mathbf{P} = (p_{ij})_{n\times n}$ *possess the following determinantal representation,*

(i) if $r < \min\{m, n\}$ *or* $r = m < n$*,*

$$p_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{d}_{.j})\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|}, \tag{38}$$

where $\mathbf{d}_{.j}$ *is the* j*th column of* $\mathbf{A}^{\sharp}\mathbf{A} \in \mathbb{H}^{n\times n}$ *and for all* $i, j = 1, \ldots, n$*;*

(ii) if $r = n$*,*

$$p_{ij} = \frac{\mathrm{cdet}_i(\mathbf{A}^{\sharp}\mathbf{A})_{.\,i}(\mathbf{d}_{.j})}{\det(\mathbf{A}^{\sharp}\mathbf{A})}.$$

Proof.

(i) If $\mathbf{A} \in \mathbb{H}^{m\times n}_{r}$, $\mathbf{A}^{\sharp}\mathbf{A}$ is Hermitian, and $r < \min\{m, n\}$ or $r = m < n$, we can represent $\mathbf{A}^{\dagger}$ by (33). Right-multiplying it by $\mathbf{A}$ gives the following presentation of an entry p_{ij} of $\mathbf{A}^{\dagger}_{M,N}\mathbf{A} =: \mathbf{P} = (p_{ij})_{n\times n}$,

$$p_{ij} = \sum_k \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,k}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^{\sharp}\mathbf{A})^{\beta}_{\beta}\right|} \cdot a_{kj} =$$

$$= \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \sum\limits_{k} \mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.k}\right)\right)^{\beta}_{\beta} \cdot a_{kj}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^{\sharp}\mathbf{A})^{\beta}_{\beta}\right|}.$$

Since $\sum\limits_{k} \mathbf{a}^{\sharp}_{.k} a_{kj} = \mathbf{d}_{.j}$, where $\mathbf{d}_{.j}$ denote the jth column of $\mathbf{A}^{\sharp}\mathbf{A} \in \mathbb{H}^{n\times n}$, then it follows (38).

(ii) If $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, $\mathbf{A}^\sharp\mathbf{A}$ is Hermitian, and $r = n$, we can represent $\mathbf{A}^\dagger$ by (36). Right-multiplying it by $\mathbf{A}$ gives the following presentation of an entry p_{ij} of $\mathbf{A}_{M,N}^\dagger\mathbf{A} =: \mathbf{P} = (p_{ij})_{n\times n}$,

$$p_{ij} = \sum_{k=1}^{n} \frac{\operatorname{cdet}_i(\mathbf{A}^\sharp\mathbf{A})_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right)}{\det(\mathbf{A}^\sharp\mathbf{A})} \cdot a_{kj} = \frac{\operatorname{cdet}_i(\mathbf{A}^\sharp\mathbf{A})_{.\,i}\left(\mathbf{d}_{.j}\right)}{\det(\mathbf{A}^\sharp\mathbf{A})}.$$

□

The following corollary can be proved by analogy.

Corollary 4.4. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$ *and* $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m\times m}$ *is Hermitian, then the projection matrix* $\mathbf{A}\mathbf{A}_{M,N}^\dagger =: \mathbf{Q} = (q_{ij})_{m\times m}$ *possess the following determinantal representation,*

(i) if $r < \min\{m, n\}$ *or* $r = n < m$*,*

$$q_{ij} = \frac{\sum\limits_{\alpha\in I_{r,\,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}\,\mathbf{A}^\sharp)_{j.}\,(\mathbf{g}_{i.}\,)\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^\sharp)\,{}^{\alpha}_{\alpha}\right|},$$

where $\mathbf{g}_{i.}$ *is the ith row of* $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m\times m}$ *and for all* $i, j = 1, \ldots, m$*.*

(ii) if $r = m$*,*

$$q_{ij} = \frac{\operatorname{rdet}_j(\mathbf{A}\mathbf{A}^\sharp)_{j.}\,(\mathbf{g}_{i.})}{\det(\mathbf{A}\mathbf{A}^\sharp)}.$$

4.2. The Case of Non-Hermitian $\mathbf{A}^\sharp\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\sharp$

In this subsection we derive determinantal representations of the weighted Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ when $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m\times m}$ and $(\mathbf{A}^\sharp\mathbf{A}) \in \mathbb{H}^{n\times n}$ are non-Hermitian.

First, let $(\mathbf{A}^\sharp\mathbf{A}) \in \mathbb{H}^{n\times n}$ be non-Hermitian and $\operatorname{rank}(\mathbf{A}^\sharp\mathbf{A}) < n$. Consider $(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A})^{-1}\mathbf{A}^\sharp$. We have,

$$\begin{aligned}(\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A})^{-1}\mathbf{A}^\sharp = (\lambda\mathbf{I} + \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\mathbf{A})^{-1}\mathbf{A}^\sharp = (\mathbf{N}^{-1}(\lambda\mathbf{N} + \mathbf{A}^*\mathbf{M}\mathbf{A}))^{-1}\mathbf{A}^\sharp = \\ (\lambda\mathbf{N} + \mathbf{A}^*\mathbf{M}\mathbf{A})^{-1}\mathbf{A}^*\mathbf{M} = \mathbf{N}^{-\frac{1}{2}}(\lambda + \mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}})^{-1}\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M} = \\ \mathbf{N}^{-\frac{1}{2}}\left(\lambda + \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^*\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{-1}\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}^{\frac{1}{2}}\right)\mathbf{M}^{\frac{1}{2}} \quad (39)\end{aligned}$$

Since by Lemma 2.2

$$\lim_{\lambda\to 0}\left(\lambda+\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{-1}\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}^{\frac{1}{2}}\right)=\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{\dagger}, \tag{40}$$

then combining (40) and (39), we obtain the well-known representation of the weighted Moore-Penrose inverse(see, e.g., [40]),

$$\mathbf{A}^{\dagger}_{M,N}=\mathbf{N}^{-\frac{1}{2}}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{\dagger}\mathbf{M}^{\frac{1}{2}}. \tag{41}$$

Denote $\mathbf{M}^{\frac{1}{2}}=\left(m_{ij}^{(\frac{1}{2})}\right)$, $\mathbf{N}^{-\frac{1}{2}}=\left(n_{ij}^{(-\frac{1}{2})}\right)$, and $\widetilde{\mathbf{A}}:=\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}=(\widetilde{a}_{ij})\in\mathbb{H}^{m\times n}$, then $\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}^{\frac{1}{2}}=\widetilde{\mathbf{A}}^{*}=\left(\widetilde{a}^{*}_{ij}\right)$, $\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{\dagger}=\widetilde{\mathbf{A}}^{\dagger}=\left(\widetilde{a}^{\dagger}_{ij}\right)$. By determinantal representing (13) for $\widetilde{\mathbf{A}}^{\dagger}$, we obtain

$$\widetilde{a}^{\dagger}_{ij}=\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\operatorname{cdet}_i\left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)_{.\,i}\left(\mathbf{m}^{\frac{1}{2}}\mathbf{a}\mathbf{n}^{-\frac{1}{2}}\right)^{*}_{.j}\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)^{\beta}_{\beta}\right|}=$$

$$\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\operatorname{cdet}_i\left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)_{.\,i}\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}^{\frac{1}{2}}\right)_{.j}\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)^{\beta}_{\beta}\right|}$$

where $\left(\mathbf{m}^{\frac{1}{2}}\mathbf{a}\mathbf{n}^{-\frac{1}{2}}\right)^{*}_{.j}$ denote the j-th column of $\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}$, $\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}^{\frac{1}{2}}\right)_{.j}$ denote the j-th column of $\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}^{\frac{1}{2}}\right)$ for all $j=1,\ldots,m$. Since $\sum_l\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}^{\frac{1}{2}}\right)_{.\,l}m^{\frac{1}{2}}_{lj}=\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}\right)_{.j}$, then for the weighted Moore-

Penrose inverse $\mathbf{A}^{\dagger}_{M,N} = \left(a^{\ddagger}_{ij}\right) \in \mathbb{H}^{n\times m}$, we have

$$a^{\ddagger}_{ij} = \sum_{k}^{n}\sum_{l}^{m} n_{ik}^{-\frac{1}{2}}\widetilde{a}^{\dagger}_{kl}m_{lj}^{\frac{1}{2}} =$$

$$\frac{\sum\limits_{k} n_{ik}^{-\frac{1}{2}} \cdot \sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_k\left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)_{.\,k}\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}\right)_{.j}\right){\,}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\;n}}\left|\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right){\,}^{\beta}_{\beta}\right|} =$$

$$\frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^{*}\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widehat{a}_{ij}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\;n}}\left|\left(\widetilde{\mathbf{A}}^{*}\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|},$$

where $\widehat{\mathbf{a}}_{.j} := \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}\right)_{.j}$ denote jth column of $\widehat{\mathbf{A}} = (\widehat{a}_{ij}) := \left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}\right)$ for all $i = 1,\ldots,n$, $j = 1,\ldots,m$.

If $\operatorname{rank}(\mathbf{A}^{\sharp}\mathbf{A}) = n$, then by Corollary 3.1, $\mathbf{A}^{\dagger}_{M,N} = (\mathbf{A}^{\sharp}\mathbf{A})^{-1}\mathbf{A}^{\sharp} =: \left(a^{\ddagger}_{ij}\right)$. So, $\mathbf{A}^{\dagger}_{M,N} = \left(\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M}\mathbf{A}\right)^{-1}\mathbf{N}^{-1}\mathbf{A}^{*}\mathbf{M} = \left(\mathbf{A}^{*}\mathbf{M}\mathbf{A}\right)^{-1}\mathbf{A}^{*}\mathbf{M}$. Since $\mathbf{A}^{*}\mathbf{M}\mathbf{A}$ is Hermitian, then we can use the determinantal representation of a Hermitian inverse matrix (7). Denote $\mathbf{A}^{*}\mathbf{M} =: \hat{\mathbf{A}} = (\hat{a})_{ij} \in \mathbb{H}^{n\times m}$. So, we have

$$a^{\ddagger}_{ij} = \frac{\sum_{k=1}^{n} L_{ki}\hat{a}_{kj}}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})} = \frac{\operatorname{cdet}_i(\mathbf{A}^{*}\mathbf{M}\mathbf{A})_{.i}\left(\hat{\mathbf{a}}_{.j}\right)}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})}.$$

where $\hat{\mathbf{a}}_{.j}$ is the jth column of $\mathbf{A}^{*}\mathbf{M}$ for all $j = 1,\ldots,m$.

Now, let $(\mathbf{A}\mathbf{A}^{\sharp}) \in \mathbb{H}^{m\times m}$ be non-Hermitian and $\operatorname{rank}(\mathbf{A}\mathbf{A}^{\sharp}) < m$. By determinantal representing (14) for $\widetilde{\mathbf{A}}^{\dagger}$, we similarly obtain

$$\widetilde{a}^{\dagger}_{ij} = \frac{\sum\limits_{\alpha\in I_{r,\,m}\{j\}} \operatorname{rdet}_j\left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\right)^{*}_{j.}\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}^{\frac{1}{2}}\right)_{i.}\right){\,}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\;m}}\left|\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^{*}\right){\,}^{\alpha}_{\alpha}\right|},$$

where $\left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^{*}\mathbf{m}^{\frac{1}{2}}\right)_{i.}$ denote the ith row of $\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}^{\frac{1}{2}}\right)$ for all $i = \overline{1,n}$.

Since $\sum\limits_k n_{ik}^{-\frac{1}{2}} \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^*\mathbf{m}^{\frac{1}{2}}\right)_{k.} = \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^*\mathbf{m}^{\frac{1}{2}}\right)_{i.}$, then we get

$$a_{ij}^{\ddagger} = \sum_k^n \sum_l^m n_{ik}^{-\frac{1}{2}} \widetilde{a}_{kl}^{\dagger} m_{lj}^{\frac{1}{2}} =$$

$$\sum_l \frac{\sum\limits_{\alpha \in I_{r,\,m}\{l\}} \operatorname{rdet}_l \left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right) \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^* \right)_{l.} \left(\mathbf{n}^{-1}\mathbf{a}^*\mathbf{m}^{\frac{1}{2}}\right)_{i.} \right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right) \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^* \right) {}_{\alpha}^{\alpha} \right|} \cdot m_{lj}^{\frac{1}{2}} =$$

$$\frac{\sum\limits_l \sum\limits_{\alpha \in I_{r,\,m}\{l\}} \operatorname{rdet}_l \left(\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^*\right)_{l.} (\widehat{\mathbf{a}}_{i.}) \right)_{\alpha}^{\alpha} \cdot m_{lj}^{\frac{1}{2}}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^*\right)_{\alpha}^{\alpha} \right|},$$

where, in this case, $\widehat{\mathbf{a}}_{i.} := \left(\mathbf{n}^{-1}\mathbf{a}^*\mathbf{m}^{\frac{1}{2}}\right)_{i.}$ denote ith row of $\widehat{\mathbf{A}} = (\widehat{a}_{ij}) := \left(\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}^{\frac{1}{2}}\right)$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

If $\operatorname{rank}(\mathbf{A}\mathbf{A}^{\sharp}) = m$, then by Corollary 3.1, $\mathbf{A}_{M,N}^{\dagger} = \mathbf{A}^{\sharp}(\mathbf{A}\mathbf{A}^{\sharp})^{-1}$. So, $\mathbf{A}_{M,N}^{\dagger} = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\left(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\right)^{-1} = \mathbf{N}^{-1}\mathbf{A}^*\left(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*\right)^{-1}$. Since $\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*$ is Hermitian and full-rank, then we can use the determinantal representation of a Hermitian inverse matrix (6). Denote $\mathbf{N}^{-1}\mathbf{A}^* =: (\widehat{a})_{ij} \in \mathbb{H}^{n \times m}$. So, we have

$$a_{ij}^{\ddagger} = \frac{\sum_{k=1}^n \check{a}_{ik} R_{jk}}{\det(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)} = \frac{\operatorname{rdet}_j(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)_{j.}(\widehat{\mathbf{a}}_{i.})}{\det(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)},$$

where $\widehat{\mathbf{a}}_{i.}$ is the ith row of $\mathbf{N}^{-1}\mathbf{A}^*$ for all $i = 1, \ldots, n$.

Thus, we have proved the following theorem.

Theorem 4.2. *Let* $\mathbf{A} \in \mathbb{H}_r^{m \times n}$.

(i) If $\mathbf{A}^{\sharp}\mathbf{A}$ *is non-Hermitian, then the weighted Moore-Penrose inverse* $\mathbf{A}_{M,N}^{\dagger} = \left(a_{ij}^{\ddagger}\right) \in \mathbb{H}^{n \times m}$ *possess the determinantal representations*

(a) if $r < n$

$$a_{ij}^{\ddagger} = \frac{\sum\limits_k n_{ik}^{(-\frac{1}{2})} \sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_k \left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k} (\widehat{\mathbf{a}}_{.j}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta} \right|}, \tag{42}$$

where $\widehat{\mathbf{a}}_{.j}$ *is the* j*th column of* $\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}$*;*

(b) if $r = n$

$$a_{ij}^{\ddagger} = \frac{\operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.i}(\widehat{\mathbf{a}}_{.j})}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})}, \tag{43}$$

where $\widehat{\mathbf{a}}_{.j}$ *is the* j*th column of* $\mathbf{A}^*\mathbf{M}$ *for all* $j = 1, \dots, m$*.*

(ii) If $\mathbf{A}\mathbf{A}^{\sharp}$ *is non-Hermitian, then* $\mathbf{A}_{M,N}^{\dagger} = \left(a_{ij}^{\ddagger}\right)$ *possess the determinantal representation*

(a) if $r < m$*,*

$$a_{ij}^{\ddagger} = \frac{\sum\limits_{l} \sum\limits_{\alpha \in I_{r,\,m}\{l\}} \operatorname{rdet}_l \left(\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^* \right)_{l.} (\widehat{\mathbf{a}}_{i.}) \right)_{\alpha}^{\alpha} \cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^* \right)_{\alpha}^{\alpha} \right|}, \tag{44}$$

where $\widehat{\mathbf{a}}_{i.}$ *is the* i*th row of* $\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}^{\frac{1}{2}}$*;*

(b) if $r = m$*,*

$$a_{ij}^{\ddagger} = \frac{\operatorname{rdet}_j(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)_{j.}(\widehat{\mathbf{a}}_{i.})}{\det(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)}. \tag{45}$$

where $\widehat{\mathbf{a}}_{i.}$ *is the* i*th row of* $\mathbf{N}^{-1}\mathbf{A}^*$ *for all* $i = 1, \dots, n$*.*

Remark 4.1. *To give determinantal representations of* $\mathbf{A}_{M,N}^{\dagger}$ *over the complex field it is enough in Theorem 4.1 substitute all row-column determinants by usual determinants.*

5. Cramer's Rule for Systems of Quaternion Linear Equations with Restrictions

Consider a right system of linear equations over the quaternion skew field,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{46}$$

where $\mathbf{A} \in \mathbb{H}^{m\times n}$ is a coefficient matrix, $\mathbf{b} \in \mathbb{H}^{m\times 1}$ is a column of constants, and $\mathbf{x} \in \mathbb{H}^{n\times 1}$ is a unknown column. Due to [16], we have the following theorem that characterizes the weighted Moore-Penrose solution of (46).

Theorem 5.1. *[16] The right system of linear equations (46) with restriction*

$$\mathbf{x} \in \mathcal{R}_r(\mathbf{A}^\sharp) \tag{47}$$

has the unique solution $\tilde{\mathbf{x}} = \mathbf{A}^\dagger_{M,N}\mathbf{b}$.

The following theorems give analogs of Cramer's rule for solutions of the system (46) with the restriction (47).

Theorem 5.2. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{A}^\sharp\mathbf{A} \in \mathbb{H}^{n\times n}$ *be Hermitian.*

(i) If $\operatorname{rank}\mathbf{A} = r \leq m < n$*, then the solution* $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T \in \mathbb{H}^{n\times 1}$ *of (46) possess the following determinantal representations*

$$\tilde{x}_i = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}(\mathbf{f})\right)^\beta_\beta}{\sum\limits_{\beta\in J_{r,\ n}} \left|(\mathbf{A}^\sharp\mathbf{A})\,^\beta_\beta\right|}, \tag{48}$$

where $\mathbf{f} = \mathbf{A}^\sharp\,\mathbf{b} \in \mathbb{H}^{n\times 1}$.

(ii) If $\operatorname{rank}\mathbf{A} = n$*, then*

$$\tilde{x}_i = \frac{\operatorname{cdet}_i\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.i}(\mathbf{f})}{\det \mathbf{A}^\sharp\mathbf{A}}. \tag{49}$$

Proof. (i) If $\operatorname{rank}\mathbf{A} = r \leq m < n$, then by Theorem 4.1 we can represent $\mathbf{A}^\dagger_{M,N}$ by (33). By component-wise of $\tilde{\mathbf{x}} = \mathbf{A}^\dagger_{M,N}\mathbf{b}$, we have

$$\tilde{x}_i = \sum_{j=1}^{m} \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.\,j}\right)\right)^\beta_\beta}{\sum\limits_{\beta\in J_{r,\ n}} \left|(\mathbf{A}^\sharp\mathbf{A})^\beta_\beta\right|} \cdot b_j =$$

$$= \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \sum\limits_{j} \operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^\sharp_{.j}\right)\right)^\beta_\beta \cdot b_j}{\sum\limits_{\beta\in J_{r,\ n}} \left|(\mathbf{A}^\sharp\mathbf{A})^\beta_\beta\right|} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}(\mathbf{f})\right)\,^\beta_\beta}{\sum\limits_{\beta\in J_{r,\ n}} \left|(\mathbf{A}^\sharp\mathbf{A})^\beta_\beta\right|},$$

where $\mathbf{f} = \mathbf{A}^\sharp\mathbf{b}$ and for all $i = 1, \ldots, n$.

(ii) If $\operatorname{rank}\mathbf{A} = n$, then $\mathbf{A}^\dagger_{M,N}$ can be represented by (36). Representing $\mathbf{A}^\dagger\mathbf{b}$ by component-wise directly gives (49). □

Theorem 5.3. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{A}^\sharp\mathbf{A} \in \mathbb{H}^{n\times n}$ *be non-Hermitian.*

(i) If $\operatorname{rank}\mathbf{A} = r \leq m < n$*, then the solution* $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T \in \mathbb{H}^{n\times 1}$ *of (46) possess the following determinantal representation*

$$\tilde{x}_i = \frac{\sum\limits_k n_{ik}^{(-\frac{1}{2})} \sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_k \left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}(\mathbf{f})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|},$$

where $\mathbf{f} = \left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^\sharp\mathbf{M}\right)\mathbf{b} \in \mathbb{H}^{n\times 1}$ *and* $n_{ik}^{(-\frac{1}{2})}$ *is the* (ik)*th entry of* $\mathbf{N}^{-\frac{1}{2}}$ *for all* $i, k = 1, \ldots, n$*.*

(ii) If $\operatorname{rank}\mathbf{A} = n$*, then*

$$\tilde{x}_i = \frac{\operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.i}(\mathbf{f})}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})},$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{M}\mathbf{b} \in \mathbb{H}^{n\times 1}$ *for all* $j = 1, \ldots, m$*.*

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}_{M,N}^\dagger$ by (42) in the (i) point and by (43) in the (ii) point, respectively. □

Consider a left system of linear equations over the quaternion skew field,

$$\mathbf{x}\mathbf{A} = \mathbf{b} \tag{50}$$

where $\mathbf{A} \in \mathbb{H}^{m\times n}$ is a coefficient matrix, $\mathbf{b} \in \mathbb{H}^{1\times n}$ is a row of constants, and $\mathbf{x} \in \mathbb{H}^{1\times m}$ is a unknown row. The following theorem characterizes the weighted Moore-Penrose solution of (50).

Theorem 5.4. *The left system of linear equations (50) with restriction* $\mathbf{x} \in \mathcal{R}_l(\mathbf{A}^\sharp)$ *has the unique solution* $\tilde{\mathbf{x}} = \mathbf{b}\mathbf{A}_{M,N}^\dagger$*.*

Theorem 5.5. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{A}\mathbf{A}^\sharp \in \mathbb{H}^{m\times m}$ *be Hermitian.*

(i) If $\operatorname{rank}\mathbf{A} = r \leq n < m$*, then the restricted solution* $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ *of (50) possess the following determinantal representation*

$$\tilde{x}_j = \frac{\sum\limits_{\alpha\in I_{r,\,m}\{j\}} \operatorname{rdet}_j \left(\left(\mathbf{A}\mathbf{A}^\sharp\right)_{j\,.}(\mathbf{g})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{A}\mathbf{A}^\sharp\right)_{\alpha}^{\alpha}\right|},$$

where $\mathbf{g} = \mathbf{b}\,\mathbf{A}^\sharp \in \mathbb{H}^{1\times m}$*.*

(ii) *If* $\operatorname{rank}\mathbf{A} = m$*, then*

$$\tilde{x}_j = \frac{\operatorname{rdet}_j\left(\mathbf{A}\mathbf{A}^\sharp\right)_{j.}(\mathbf{g})}{\det \mathbf{A}\mathbf{A}^\sharp}.$$

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}^\dagger_{M,N}$ by (34) in the (i) point, and (37) in the (ii) point, respectively. □

Theorem 5.6. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{A}\mathbf{A}^\sharp \in \mathbb{H}^{m\times m}$ *be non-Hermitian.*

(i) *If* $\operatorname{rank}\mathbf{A} = k \leq n < m$*, then the solution* $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ *of (50) possess the following determinantal representation*

$$\tilde{x}_j = \frac{\sum\limits_{l}\sum\limits_{\alpha\in I_{r,\,m}\{l\}} \operatorname{rdet}_l\left(\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^*\right)_{l.}(\mathbf{g})\right)^{\alpha}_{\alpha} m_{lj}^{\frac{1}{2}}}{\sum\limits_{\alpha\in I_{r,\,m}}\left|\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^*\right)^{\alpha}_{\alpha}\right|},$$

where $\mathbf{g} = \mathbf{b}\left(\mathbf{N}^{-1}\mathbf{A}^\sharp\mathbf{M}^{\frac{1}{2}}\right) \in \mathbb{H}^{1\times m}$ *and* $m_{lj}^{(\frac{1}{2})}$ *is the* (lj)*th entry of* $\mathbf{M}^{\frac{1}{2}}$ *for all* $l, j = 1, \ldots, m$*.*

(ii) *If* $\operatorname{rank}\mathbf{A} = m$*, then*

$$\tilde{x}_j = \frac{\operatorname{rdet}_j(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)_{j.}(\mathbf{g})}{\det(\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*)}.$$

where $\mathbf{g} = \mathbf{b}\mathbf{N}^{-1}\mathbf{A}^* \in \mathbb{H}^{1\times m}$.

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}^\dagger_{M,N}$ by (44) in the (i) point, and (45) in the (ii) point, respectively. □

6. Cramer's Rule for Two-sided Restricted Quaternion Matrix Equation

Definition 6.1. *For an arbitrary matrix over the quaternion skew field,* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, we denote by*

- $\mathcal{R}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^{m\times 1} : \mathbf{y} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{H}^{n\times 1}\}$, *the column right space of* $\mathbf{A}$,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{x} \in \mathbb{H}^{n\times 1} :\ \mathbf{A}\mathbf{x} = 0\}$, *the right null space of* $\mathbf{A}$,
- $\mathcal{R}_l(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^{1\times n} :\ \mathbf{y} = \mathbf{x}\mathbf{A},\ \mathbf{x} \in \mathbb{H}^{1\times m}\}$, *the row left space of* $\mathbf{A}$,
- $\mathcal{N}_l(\mathbf{A}) = \{\mathbf{x} \in \mathbb{H}^{1\times m} :\ \mathbf{x}\mathbf{A} = 0\}$, *the left null space of* $\mathbf{A}$.

It is easy to see, if $\mathbf{A} \in \mathbb{H}^{n\times n}_n$, then $\mathcal{R}_r \oplus \mathcal{N}_r = \mathbb{H}^{n\times 1}$, and $\mathcal{R}_l \oplus \mathcal{N}_l = \mathbb{H}^{1\times n}$. Suppose that $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{B} \in \mathbb{H}^{p\times q}$. Denote

$$\begin{aligned}
\mathcal{R}_r(\mathbf{A},\mathbf{B}) &:= \mathcal{N}_r(\mathbf{Y}) = \{\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} : \mathbf{X}^{n\times q}\},\\
\mathcal{N}_r(\mathbf{A},\mathbf{B}) &:= \mathcal{R}_r(\mathbf{X}) = \{\mathbf{X}^{n\times p} : \mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{0}\},\\
\mathcal{R}_l(\mathbf{A},\mathbf{A} :) &= \mathcal{R}_l(\mathbf{Y}) = \{\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} : \mathbf{X}^{n\times q}\},\\
\mathcal{N}_l(\mathbf{A},\mathbf{B}) &:= \mathcal{N}_l(\mathbf{X}) = \{\mathbf{X}^{n\times p} : \mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{0}\}.
\end{aligned}$$

Lemma 6.1. *[28] Suppose that* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$, *and* $\mathbf{Q}$ *are Hermitian positive definite matrices of order* m, n, p, *and* q, *respectively. Denote* $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$ *and* $\mathbf{B}^\sharp = \mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}$. *If* $\mathbf{D} \subset \mathcal{R}_r\left(\mathbf{A}\mathbf{A}^\sharp, \mathbf{B}^\sharp\mathbf{B}\right)$ *and* $\mathbf{D} \subset \mathcal{R}_l\left(\mathbf{A}^\sharp\mathbf{A}, \mathbf{B}\mathbf{B}^\sharp\right)$,

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{51}$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathbf{N}^{-1}\mathcal{R}_r(\mathbf{A}^*),\ \mathcal{N}_r(\mathbf{X}) \supset \mathbf{P}^{-1}\mathcal{N}_r(\mathbf{B}^*), \tag{52}$$

$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{A}^*)\mathbf{M},\ \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l(\mathbf{B}^*)\mathbf{Q} \tag{53}$$

then the unique solution of (51) with the restrictions (52)-(53) is

$$\mathbf{X} = \mathbf{A}^\dagger_{M,N}\mathbf{D}\mathbf{B}^\dagger_{P,Q}. \tag{54}$$

In this chapter, we get determinantal representations of (54) that are intrinsically analogs of the classical Cramer's rule. We will consider several cases depending on whether the matrices $\mathbf{A}^\sharp\mathbf{A}$ and $\mathbf{B}\mathbf{B}^\sharp$ are Hermitian or not.

6.1. The Case of Both Hermitian Matrices $\mathbf{A}^\sharp\mathbf{A}$ and $\mathbf{B}\mathbf{B}^\sharp$

Denote $\widetilde{\mathbf{D}} = \mathbf{A}^\sharp\mathbf{D}\mathbf{B}^\sharp$.

Theorem 6.1. *Let* $\mathbf{A}^\sharp\mathbf{A}$ *and* $\mathbf{B}\mathbf{B}^\sharp$ *be Hermitian. Then the solution (54) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^\sharp \mathbf{A} \right)_{.\,i} \left(\mathbf{d}^{\mathbf{B}}_{.\,j} \right) \right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{r_1,n}} \left| \left(\mathbf{A}^\sharp \mathbf{A} \right)^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} \left| \left(\mathbf{B}\mathbf{B}^\sharp \right)^{\alpha}_{\alpha} \right|}, \tag{55}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \operatorname{rdet}_j \left(\left(\mathbf{B}\mathbf{B}^\sharp \right)_{j\,.} \left(\mathbf{d}^{\mathbf{A}}_{i\,.} \right) \right)^{\alpha}_{\alpha}}{\sum\limits_{\beta \in J_{r_1,n}} \left| \left(\mathbf{A}^\sharp \mathbf{A} \right)^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} \left| \left(\mathbf{B}\mathbf{B}^\sharp \right)^{\alpha}_{\alpha} \right|}, \tag{56}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha \in I_{r_2,p}\{j\}} \operatorname{rdet}_j \left(\left(\mathbf{B}\mathbf{B}^\sharp \right)_{j.} \left(\tilde{\mathbf{d}}_{k.} \right) \right)^{\alpha}_{\alpha} \right) \in \mathbb{H}^{n \times 1} \tag{57}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta \in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^\sharp \mathbf{A} \right)_{.i} \left(\tilde{\mathbf{d}}_{.l} \right) \right)^{\beta}_{\beta} \right) \in \mathbb{H}^{1 \times p} \tag{58}$$

are the column-vector and the row-vector, respectively. $\tilde{\mathbf{d}}_{k.}$ *and* $\tilde{\mathbf{d}}_{.l}$ *are the* k*th row and the* l*th column of* $\widetilde{\mathbf{D}}$ *for all* $k = 1, ..., n$*,* $l = 1, ..., p$*.*

(ii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\operatorname{cdet}_i (\mathbf{A}^\sharp \mathbf{A})_{.\,i} \left(\mathbf{d}^{\mathbf{B}}_{.j} \right)}{\det(\mathbf{A}^\sharp \mathbf{A}) \cdot \det(\mathbf{B}\mathbf{B}^\sharp)}, \tag{59}$$

or

$$x_{i\,j} = \frac{\operatorname{rdet}_j (\mathbf{B}\mathbf{B}^\sharp)_{j.} \left(\mathbf{d}^{\mathbf{A}}_{i\,.} \right)}{\det(\mathbf{A}^\sharp \mathbf{A}) \cdot \det(\mathbf{B}\mathbf{B}^\sharp)}, \tag{60}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.j} := \left(\operatorname{rdet}_j (\mathbf{B}\mathbf{B}^\sharp)_{j.} \left(\tilde{\mathbf{d}}_{k\,.} \right) \right) \in \mathbb{H}^{n \times 1}, \tag{61}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} := \left(\operatorname{cdet}_i (\mathbf{A}^\sharp \mathbf{A})_{.\,i} \left(\tilde{\mathbf{d}}_{.l} \right) \right) \in \mathbb{H}^{1 \times p}. \tag{62}$$

(iii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right)\right)}{\det(\mathbf{A}^{\sharp}\mathbf{A}) \cdot \sum\limits_{\alpha \in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{\sharp})^{\alpha}_{\alpha}\right|}, \tag{63}$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j\,.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)\right){}^{\alpha}_{\alpha}}{\det(\mathbf{A}^{\sharp}\mathbf{A}) \cdot \sum\limits_{\alpha \in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{\sharp})^{\alpha}_{\alpha}\right|}, \tag{64}$$

where $\mathbf{d}^{\mathbf{B}}_{.\,j}$ *is (57) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (62).*

(iv) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\operatorname{rdet}_j(\mathbf{B}\mathbf{B}^{\sharp})_{j.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)}{\sum\limits_{\beta \in J_{r_1,n}} \left|(\mathbf{A}^{\sharp}\mathbf{A})^{\beta}_{\beta}\right| \cdot \det(\mathbf{B}\mathbf{B}^{\sharp})}, \tag{65}$$

or

$$x_{i\,j} = \frac{\sum\limits_{\beta \in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{r_1,n}} \left|(\mathbf{A}^{\sharp}\mathbf{A})^{\beta}_{\beta}\right| \cdot \det(\mathbf{B}\mathbf{B}^{\sharp})}, \tag{66}$$

where $\mathbf{d}^{\mathbf{B}}_{.\,j}$ *is (61) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (58).*

Proof. (i) If $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$ and $r_1 < n$, $r_2 < p$, then, by Theorem 4.1, the weighted Moore-Penrose inverses $\mathbf{A}^{\dagger} = \left(a^{\ddagger}_{ij}\right) \in \mathbb{H}^{n\times m}$ and $\mathbf{B}^{\dagger} = \left(b^{\ddagger}_{ij}\right) \in \mathbb{H}^{q\times p}$ possess the following determinantal representations, respectively,

$$a^{\ddagger}_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.j}\right)\right){}^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right){}^{\beta}_{\beta}\right|}, \tag{67}$$

$$b^{\ddagger}_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \operatorname{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{b}^{\sharp}_{i.})\right){}^{\alpha}_{\alpha}}{\sum\limits_{\alpha \in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{\sharp})\,{}^{\alpha}_{\alpha}\right|}. \tag{68}$$

By Lemma 6.1, $\mathbf{X} = \mathbf{A}^{\dagger}_{M,N}\mathbf{D}\mathbf{B}^{\dagger}_{P,Q}$ and entries of $\mathbf{X} = (x_{ij})$ are

$$x_{ij} = \sum_{s=1}^{q}\left(\sum_{k=1}^{m} a^{\ddagger}_{ik} d_{ks}\right) b^{\ddagger}_{sj}. \tag{69}$$

for all $i = 1, ..., n, j = 1, ..., p$.

Denote by $\hat{\mathbf{d}}_{.s}$ the sth column of $\mathbf{A}^{\sharp}\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times q}$ for all $s = 1, ..., q$. It follows from $\sum\limits_{k} \mathbf{a}^{\sharp}_{.\,k} d_{ks} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{k=1}^{m} a^{\ddagger}_{ik} d_{ks} = \sum_{k=1}^{m} \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.k}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|} \cdot d_{ks} =$$

$$\frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \sum\limits_{k=1}^{m} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^{\sharp}_{.k}\right)\right)^{\beta}_{\beta} \cdot d_{ks}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|}. \tag{70}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are the unit row-vector and the unit column-vector, respectively, such that all their components are 0, except the sth components, which are 1. Substituting (70) and (68) in (69), we obtain

$$x_{ij} = \sum_{s=1}^{q} \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|} \frac{\sum\limits_{\alpha\in I_{r_2,p}\{j\}} \operatorname{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{b}^{\sharp}_{s.})\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{\sharp})^{\alpha}_{\alpha}\right|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{l=1}^{n} \mathbf{e}_{.\,l}\hat{d}_{ls},\ \mathbf{b}^{\sharp}_{s.} = \sum_{t=1}^{p} b^{\sharp}_{st}\mathbf{e}_{t.},\ \sum_{s=1}^{q} \hat{d}_{ls}b^{\sharp}_{st} = \widetilde{d}_{lt},$$

then we have

$$x_{ij} =$$

$$\frac{\sum\limits_{s=1}^{q}\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n} \sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\right)^{\beta}_{\beta} \hat{d}_{ls}b^{\sharp}_{st} \sum\limits_{\alpha\in I_{r_2,p}\{j\}} \operatorname{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{e}_{t.})\right)^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|(\mathbf{B}\mathbf{B}^{\sharp})^{\alpha}_{\alpha}\right|} =$$

$$\frac{\sum\limits_{t=1}^{p}\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\right)_{\beta}^{\beta}\widetilde{d}_{lt}\sum\limits_{\alpha\in I_{r_2,p}\{j\}}\mathrm{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|(\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha}\right|}. \tag{71}$$

Denote by

$$d_{it}^{\mathbf{A}}:=$$

$$\sum_{\beta\in J_{r_1,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\widetilde{\mathbf{d}}_{.t}\right)\right)_{\beta}^{\beta}=\sum_{l=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.l})\right)_{\beta}^{\beta}\widetilde{d}_{lt}$$

the tth component of a row-vector $\mathbf{d}_{i\,.}^{\mathbf{A}}=(d_{i1}^{\mathbf{A}},...,d_{ip}^{\mathbf{A}})$ for all $t=1,...,p$. Substituting it in (71), we have

$$x_{ij}=\frac{\sum\limits_{t=1}^{p}d_{it}^{\mathbf{A}}\sum\limits_{\alpha\in I_{r_2,p}\{j\}}\mathrm{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|(\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{p}d_{it}^{\mathbf{A}}\mathbf{e}_{t.}=\mathbf{d}_{i\,.}^{\mathbf{A}}$, then it follows (56).

If we denote by

$$d_{lj}^{\mathbf{B}}:=\sum_{t=1}^{p}\widetilde{d}_{lt}\sum_{\alpha\in I_{r_2,p}\{j\}}\mathrm{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}=\sum_{\alpha\in I_{r_2,p}\{j\}}\mathrm{rdet}_j\left((\mathbf{B}\mathbf{B}^{\sharp})_{j\,.}(\widetilde{\mathbf{d}}_{l.})\right)_{\alpha}^{\alpha}$$

the lth component of a column-vector $\mathbf{d}_{.\,j}^{\mathbf{B}}=(d_{1j}^{\mathbf{B}},...,d_{jn}^{\mathbf{B}})^{T}$ for all $l=1,...,n$ and substitute it in (71), we obtain

$$x_{ij}=\frac{\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{e}_{.\,l})\right)_{\beta}^{\beta}d_{lj}^{\mathbf{B}}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|(\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{l=1}^{n}\mathbf{e}_{.l}d_{lj}^{\mathbf{B}}=\mathbf{d}_{.\,j}^{\mathbf{B}}$, then it follows (55).

(ii) If $\operatorname{rank}\mathbf{A} = n$ and $\operatorname{rank}\mathbf{B} = p$, then by Theorem 4.1 the weighted Moore-Penrose inverses $\mathbf{A}^{\dagger}_{M,N} = \left(a^{\ddagger}_{ij}\right) \in \mathbb{H}^{n\times m}$ and $\mathbf{B}^{\dagger}_{P,Q} = \left(b^{\ddagger}_{ij}\right) \in \mathbb{H}^{q\times p}$ possess the following determinantal representations, respectively,

$$a^{\ddagger}_{ij} = \frac{\operatorname{cdet}_i(\mathbf{A}^{\sharp}\mathbf{A})_{.\,i}\left(\mathbf{a}^{\sharp}_{.\,j}\right)}{\det(\mathbf{A}^{\sharp}\mathbf{A})} \tag{72}$$

$$b^{\ddagger}_{ij} = \frac{\operatorname{rdet}_j(\mathbf{B}\mathbf{B}^{\sharp})_{j.}\left(\mathbf{b}^{\sharp}_{i.}\right)}{\det(\mathbf{B}\mathbf{B}^{\sharp})}. \tag{73}$$

By their substituting in (69) and pondering ahead as in the previous case, we obtain (59) and (60).

(iii) If $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$ and $r_1 = n$, $r_2 < p$, then, for the weighted Moore-Penrose inverses $\mathbf{A}^{\dagger}_{M,N}$ and $\mathbf{B}^{\dagger}_{P,Q}$, the determinantal representations (72) and (67) are more applicable to use, respectively. By their substituting in (69) and pondering ahead as in the previous case, we finally obtain (63) and (64) as well.

(iv) In this case for $\mathbf{A}^{\dagger}_{M,N}$ and $\mathbf{B}^{\dagger}_{P,Q}$, we use the determinantal representations (72) and (68), respectively. □

Corollary 6.1. *Suppose that* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{D} \in \mathbb{H}^{m\times p}$, $\mathbf{M}$, $\mathbf{N}$ *are Hermitian positive definite matrices of order* m *and* n, *respectively,* $\mathbf{A}^{\sharp}\mathbf{A}$ *is Hermitian. Denote* $\widehat{\mathbf{D}} = \mathbf{A}^{\sharp}\mathbf{D}$. *If* $\mathbf{D} \subset \mathcal{R}_r(\mathbf{A}\mathbf{A}^{\sharp})$ *and* $\mathbf{D} \subset \mathcal{R}_l(\mathbf{A}^{\sharp}\mathbf{A})$,

$$\mathbf{A}\mathbf{X} = \mathbf{D}, \tag{74}$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathbf{N}^{-1}\mathcal{R}_r(\mathbf{A}^*),\ \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{A}^*)\mathbf{M}, \tag{75}$$

then the unique solution of (74) with the restrictions (75) is

$$\mathbf{X} = \mathbf{A}^{\dagger}_{M,N}\mathbf{D}$$

which possess the following determinantal representations.

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$, *then*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\widehat{\mathbf{d}}_{.\,j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,n}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|},$$

where $\widehat{\mathbf{d}}_{.j}$ *are the* j*th column of* $\widehat{\mathbf{D}}$ *for all* $i = 1, ..., n$, $j = 1, ..., p$.

(ii) If $\operatorname{rank}\mathbf{A} = n$*, then*

$$x_{i\,j} = \frac{\mathrm{cdet}_i(\mathbf{A}^\sharp\mathbf{A})_{.\,i}\left(\widehat{\mathbf{d}}_{.j}\right)}{\det(\mathbf{A}^\sharp\mathbf{A})},$$

Proof. The proof follows evidently from Theorem 6.1 when $\mathbf{B}$ be removed, and unit matrices insert instead $\mathbf{P}$, $\mathbf{Q}$.

Corollary 6.2. *Suppose that* $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$*,* $\mathbf{D} \in \mathbb{H}^{n\times q}$*,* $\mathbf{P}$*, and* $\mathbf{Q}$ *are Hermitian positive definite matrices of order* p *and* q*, respectively,* $\mathbf{B}\mathbf{B}^\sharp$ *is Hermitian. Denote* $\check{\mathbf{D}} = \mathbf{D}\mathbf{B}^\sharp$*. If* $\mathbf{D} \subset \mathcal{R}_r(\mathbf{B}^\sharp\mathbf{B})$ *and* $\mathbf{D} \subset \mathcal{R}_l(\mathbf{B}\mathbf{B}^\sharp)$*,*

$$\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{76}$$

$$\mathcal{N}_r(\mathbf{X}) \supset \mathbf{P}^{-1}\mathcal{N}_r(\mathbf{B}^*),\, \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l(\mathbf{B}^*)\mathbf{Q}, \tag{77}$$

then the unique solution of (76) with the restrictions (77) is

$$\mathbf{X} = \mathbf{D}\mathbf{B}^\dagger_{P,Q}$$

which possess the following determinantal representations.

(i) If $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,q}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{B}\mathbf{B}^\sharp\right)_{j\,.}\left(\check{\mathbf{d}}_{\,i\,.}\right)\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r_2,q}} \left|\left(\mathbf{B}\mathbf{B}^\sharp\right)^{\alpha}_{\alpha}\right|},$$

where $\check{\mathbf{d}}_{i.}$ *are the* i*th row of* $\check{\mathbf{D}}$ *for all* $i = 1, ..., n$*,* $j = 1, ..., p$*.*

(ii) If $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\mathrm{rdet}_j\left(\mathbf{B}\mathbf{B}^\sharp\right)_{j.}\left(\check{\mathbf{d}}_{i.}\right)}{\det\left(\mathbf{B}\mathbf{B}^\sharp\right)}.$$

Proof. The proof follows evidently from Theorem 6.1 when $\mathbf{A}$ be removed and unit matrices insert instead $\mathbf{M}$, $\mathbf{N}$.

6.2. The Case of Both Non-Hermitian Matrices $\mathbf{A}^\sharp\mathbf{A}$ and $\mathbf{B}\mathbf{B}^\sharp$

Theorem 6.2. *Let* $\mathbf{A}^\sharp\mathbf{A}$ *and* $\mathbf{B}\mathbf{B}^\sharp$ *be both non-Hermitian. Then the solution (54) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})} \sum\limits_{\beta\in J_{r_1,\,n}\{k\}} \mathrm{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\mathbf{d}^{\mathbf{B}}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|}, \tag{78}$$

or

$$x_{ij} = \frac{\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}} \mathrm{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\mathbf{d}^{\mathbf{A}}_{i.}\right)\right)^{\alpha}_{\alpha}\cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|}, \tag{79}$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{l}\sum_{\alpha\in I_{r_2,\,p}\{l\}} \mathrm{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\widetilde{\mathbf{d}}_{t.}\right)\right)^{\alpha}_{\alpha}\cdot m_{lj}^{(\frac{1}{2})}\right) \in \mathbb{H}^{n\times 1} \tag{80}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{k} n_{ik}^{(-\frac{1}{2})} \sum_{\beta\in J_{r_1,\,n}\{k\}} \mathrm{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widetilde{\mathbf{d}}_{.f}\right)\right)^{\beta}_{\beta}\right) \in \mathbb{H}^{1\times p} \tag{81}$$

are the column-vector and the row-vector, respectively. $\tilde{\mathbf{d}}_{t.}$ *and* $\tilde{\mathbf{d}}_{.f}$ *are the tth row and the fth column of* $\widetilde{\mathbf{D}} := \mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}^{\frac{1}{2}} = (\widetilde{d}_{ij}) \in \mathbb{H}^{n\times p}$ *for all* $t = 1, ..., n$*,* $f = 1, ..., p$*.*

(ii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\mathrm{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.j}\right)}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})\cdot\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)}, \tag{82}$$

or

$$x_{i\,j} = \frac{\mathrm{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})\cdot\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)}, \tag{83}$$

where

$$\mathbf{d}^{\mathbf{B}}_{\cdot j} := \left(\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}\left(\widetilde{\mathbf{d}}_{t\,.}\right)\right) \in \mathbb{H}^{n\times 1}, \tag{84}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} := \left(\operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.\,i}\left(\widetilde{\mathbf{d}}_{.f}\right)\right) \in \mathbb{H}^{1\times p}, \tag{85}$$

$\widetilde{\mathbf{d}}_{t\,.}$ *and* $\widetilde{\mathbf{d}}_{.f}$ *are the* t*th row and the* f*th column of* $\widetilde{\mathbf{D}} := \mathbf{A}^*\mathbf{M}\mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^* \in \mathbb{H}^{n\times p}$, *respectively.*

(iii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$, *then*

$$x_{ij} = \frac{\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{M}\mathbf{A})_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{\cdot j}\right)\right)}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})\cdot \sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|}, \tag{86}$$

or

$$x_{ij} = \frac{\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\mathbf{d}^{\mathbf{A}}_{i.}\right)\right)^{\alpha}_{\alpha}\cdot m^{(\frac{1}{2})}_{lj}}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})\cdot \sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|}, \tag{87}$$

where $\mathbf{d}^{\mathbf{B}}_{\cdot j}$ *is (80) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (85).*

(iv) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = p$, *then*

$$x_{i\,j} = \frac{\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|\cdot\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)}, \tag{88}$$

or

$$x_{i\,j} = \frac{\sum\limits_{k} n^{(-\frac{1}{2})}_{ik}\sum\limits_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\mathbf{d}^{\mathbf{B}}_{\cdot j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|\cdot\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)}, \tag{89}$$

where $\mathbf{d}^{\mathbf{B}}_{\cdot j}$ *is (84) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (81).*

Proof. (i) If $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$ are both non-Hermitian, and $r_1 < n$, $r_2 < p$, then, by Theorem 4.2, the weighted Moore-Penrose inverses $\mathbf{A}^{\dagger} =$

$\left(a_{ij}^{\ddagger}\right)\in\mathbb{H}^{n\times m}$ and $\mathbf{B}^{\dagger}=\left(b_{ij}^{\ddagger}\right)\in\mathbb{H}^{q\times p}$ posses the following determinantal representations, respectively,

$$a_{ij}^{\ddagger}=\frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widehat{\mathbf{a}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|},\tag{90}$$

where $\widehat{\mathbf{a}}_{.j}$ is the jth column of $\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}$;

$$b_{ij}^{\ddagger}=\frac{\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\widehat{\mathbf{b}}_{i.}\right)\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\alpha\in I_{2,\,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{\alpha}^{\alpha}\right|},\tag{91}$$

where $\widehat{\mathbf{b}}_{i.}$ is the ith row of $\mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}^{\frac{1}{2}}$. By Lemma 6.1, $\mathbf{X}=\mathbf{A}_{M,N}^{\dagger}\mathbf{D}\mathbf{B}_{P,Q}^{\dagger}$ and entries of $\mathbf{X}=(x_{ij})$ are

$$x_{ij}=\sum_{s=1}^{q}\left(\sum_{t=1}^{m}a_{it}^{\ddagger}d_{ts}\right)b_{sj}^{\ddagger}.\tag{92}$$

for all $i=1,...,n, j=1,...,p$.

Denote by $\widehat{\mathbf{d}}_{.s}$ the sth column of $\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{D}=:\widehat{\mathbf{D}}=(\widehat{d_{ij}})\in\mathbb{H}^{n\times q}$ for all $s=1,...,q$. It follows from $\sum\limits_{t}\widehat{\mathbf{a}}_{.\,t}d_{ts}=\widehat{\mathbf{d}}_{.\,s}$ that

$$\sum_{t=1}^{m}a_{it}^{\ddagger}d_{ts}=\sum_{t=1}^{m}\frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widehat{\mathbf{a}}_{.t}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|}\cdot d_{ts}=$$
$$\frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widehat{\mathbf{d}}_{.s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|}.\tag{93}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are the unit row-vector and the unit column-vector, respectively, such that all their components are 0, except the sth components, which are 1. Substituting (93) and (91) in (92), we obtain

$$x_{ij}=\sum_{s=1}^{q}\frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widehat{\mathbf{d}}_{.s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|}\times$$
$$\frac{\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\widehat{\mathbf{b}}_{s.}\right)\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\alpha\in I_{2,\,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{\alpha}^{\alpha}\right|}.$$

Since

$$\widehat{\mathbf{d}}_{.\,s}=\sum_{l=1}^{n}\mathbf{e}_{.\,l}\widehat{d}_{ls},\ \widehat{\mathbf{b}}_{s.}=\sum_{t=1}^{p}\widehat{b}_{st}\mathbf{e}_{t.},\ \sum_{s=1}^{q}\widehat{d}_{ls}\widehat{b}_{st}=\widetilde{d}_{lt},$$

then we have

$$x_{ij}=$$
$$\frac{\sum\limits_{t=1}^{p}\sum\limits_{f=1}^{n}\sum\limits_{k}n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\mathbf{e}_{.\,f}\right)\right)_{\beta}^{\beta}\tilde{d}_{ft}\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\mathbf{e}_{t.}\right)\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{\alpha}^{\alpha}\right|}. \tag{94}$$

Denote by

$$d_{it}^{\mathbf{A}}:=\sum_{k}n_{ik}^{(-\frac{1}{2})}\sum_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widetilde{\mathbf{d}}_{.t}\right)\right)_{\beta}^{\beta}=$$
$$\sum_{f=1}^{n}\sum_{k}n_{ik}^{(-\frac{1}{2})}\sum_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\mathbf{e}_{.f}\right)\right)_{\beta}^{\beta}\widetilde{d}_{ft}$$

the tth component of the row-vector $\mathbf{d}_{i.}^{\mathbf{A}}=(d_{i1}^{\mathbf{A}},...,d_{ip}^{\mathbf{A}})$ for all $t=1,...,p$. Substituting it in (94), we have

$$x_{ij}=\frac{\sum\limits_{t=1}^{p}d_{it}^{\mathbf{A}}\sum\limits_{l}\sum\limits_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\mathbf{e}_{t.}\right)\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{p} d_{it}^{\mathbf{A}}\mathbf{e}_{t.} = \mathbf{d}_{i\,.}^{\mathbf{A}}$, then it follows (79).

If we denote by

$$\sum_{t=1}^{p}\widetilde{d}_{ft}\sum_{l}\sum_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})} = \sum_{l}\sum_{\alpha\in I_{r_2,\,p}\{l\}}\operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}(\widetilde{\mathbf{d}}_{f.})\right)_{\alpha}^{\alpha}\cdot m_{lj}^{(\frac{1}{2})} =: d_{fj}^{\mathbf{B}}$$

the fth component of the column-vector $\mathbf{d}_{\cdot j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}},...,d_{jn}^{\mathbf{B}})^T$ for all $f = 1,...,n$ and substitute it in (94), then

$$x_{ij} = \frac{\sum\limits_{f=1}^{n}\sum\limits_{k} n_{ik}^{(-\frac{1}{2})}\sum\limits_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}(\mathbf{e}_{.f})\right)_{\beta}^{\beta} d_{fj}^{\mathbf{B}}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|(\mathbf{A}^\sharp\mathbf{A})\,{}_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}|(\mathbf{B}\mathbf{B}^\sharp)\,{}_{\alpha}^{\alpha}|}.$$

Since $\sum\limits_{f=1}^{n}\mathbf{e}_{.f}d_{fj}^{\mathbf{B}} = \mathbf{d}_{.\,j}^{\mathbf{B}}$, then it follows (78).

(ii) If $\operatorname{rank}\mathbf{A} = n$ and $\operatorname{rank}\mathbf{B} = p$, then by Theorem 4.2 the weighted Moore-Penrose inverses $\mathbf{A}_{M,N}^{\dagger} = \left(a_{ij}^{\ddagger}\right) \in \mathbb{H}^{n\times m}$ and $\mathbf{B}_{P,Q}^{\dagger} = \left(b_{ij}^{\ddagger}\right) \in \mathbb{H}^{q\times p}$ possess the following determinantal representations, respectively,

$$a_{ij}^{\ddagger} = \frac{\operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.i}\,(\widehat{\mathbf{a}}_{.j})}{\det(\mathbf{A}^*\mathbf{M}\mathbf{A})}, \tag{95}$$

$$b_{ij}^{\ddagger} = \frac{\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}(\widehat{\mathbf{b}}_{i.})}{\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)}. \tag{96}$$

where $\widehat{\mathbf{a}}_{.j}$ is the jth column of $\mathbf{A}^*\mathbf{M}$ for all $j = 1,\ldots,m$, and $\widehat{\mathbf{b}}_{i.}$ is the ith row of $\mathbf{Q}^{-1}\mathbf{B}^*$ for all $i = 1,\ldots,n$.

By their substituting in (92), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{p}\sum\limits_{f=1}^{n}\operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.i}\,(\mathbf{e}_{.f})\,\widetilde{d}_{ft}\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}(\mathbf{e}_{t.})}{\det\,(\mathbf{A}^*\mathbf{M}\mathbf{A})\det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)},$$

where $\widetilde{d}_{ft}$ is the (ft)th entry of $\widetilde{\mathbf{D}} := \mathbf{A}^*\mathbf{M}\mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*$ in this case. Denote by

$$d_{it}^{\mathbf{A}} := \operatorname{cdet}_i(\mathbf{A}^*\mathbf{M}\mathbf{A})_{.i}\,(\widetilde{\mathbf{d}}_{.t})$$

the tth component of the row-vector $\mathbf{d}^{\mathbf{A}}_{i.} = (d^{\mathbf{A}}_{i1}, ..., d^{\mathbf{A}}_{ip})$ for all $t = 1, ..., p$. Substituting it in (94), it follows (82).

Similarly, we can obtain (83).

(iii) If $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{p\times q}_{r_2}$ and $r_1 = n$, $r_2 < p$, then, for the weighted Moore-Penrose inverses $\mathbf{A}^{\dagger}_{M,N}$ and $\mathbf{B}^{\dagger}_{P,Q}$, the determinantal representations (95) and (90) are more applicable to use, respectively. By their substituting in (92) and pondering ahead as in the previous case, we finally obtain (86) and (87) as well.

(iv) In this case for $\mathbf{A}^{\dagger}_{M,N}$ and $\mathbf{B}^{\dagger}_{P,Q}$, we use the determinantal representations (90) and (96), respectively. □

Corollary 6.3. *Suppose that* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{D} \in \mathbb{H}^{m\times p}$, $\mathbf{M}$, $\mathbf{N}$ *are Hermitian positive definite matrices of order* m *and* n*, respectively, and* $\mathbf{A}^{\sharp}\mathbf{A}$ *is non-Hermitian. If* $\mathbf{D} \subset \mathcal{R}_r(\mathbf{A}\mathbf{A}^{\sharp})$ *and* $\mathbf{D} \subset \mathcal{R}_l(\mathbf{A}^{\sharp}\mathbf{A})$*, then the unique solution* $\mathbf{X} = \mathbf{A}^{\dagger}_{M,N}\mathbf{D}$ *of the equation* $\mathbf{A}\mathbf{X} = \mathbf{D}$ *with the restrictions (75) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$*, then*

$$x_{ij} = \frac{\sum\limits_k n_{ik}^{(-\frac{1}{2})} \sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widetilde{\mathbf{d}}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}\right)^{\beta}_{\beta}\right|},$$

where $\widetilde{\mathbf{d}}_{.j}$ *are the* j*th column of* $\widetilde{\mathbf{D}} = \mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{D}$ *for all* $i = 1, ..., n$, $j = 1, ..., p$.

(ii) If $\operatorname{rank}\mathbf{A} = n$*, then*

$$x_{i\,j} = \frac{\operatorname{cdet}_i\left(\mathbf{A}^*\mathbf{M}\mathbf{A}\right)_{.\,i}\left(\widetilde{\mathbf{d}}_{.j}\right)}{\det\left(\mathbf{A}^*\mathbf{M}\mathbf{A}\right)},$$

where $\widetilde{\mathbf{d}}_{.j}$ *are the* j*th column of* $\widetilde{\mathbf{D}} = \mathbf{A}^*\mathbf{M}\mathbf{D}$.

Proof. The proof follows evidently from Theorem 6.2 when $\mathbf{B}$ be removed and unit matrices insert instead $\mathbf{P}$, $\mathbf{Q}$.

Corollary 6.4. *Suppose that* $\mathbf{B} \in \mathbb{H}_{r_2}^{p\times q}$*,* $\mathbf{D} \in \mathbb{H}^{n\times q}$*,* $\mathbf{P}$*, and* $\mathbf{Q}$ *are Hermitian positive definite matrices of order* p *and* q*, respectively, and* $\mathbf{B}\mathbf{B}^{\sharp}$ *is non-Hermitian. If* $\mathbf{D} \subset \mathcal{R}_r(\mathbf{B}^{\sharp}\mathbf{B})$ *and* $\mathbf{D} \subset \mathcal{R}_l(\mathbf{B}\mathbf{B}^{\sharp})$*, then the unique solution* $\mathbf{X} = \mathbf{D}\mathbf{B}^{\dagger}_{P,Q}$ *of the equation* $\mathbf{X}\mathbf{B} = \mathbf{D}$ *with the restrictions (77) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,q}\{j\}} \operatorname{rdet}_j\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{j\,.}\left(\widetilde{\mathbf{d}}_{\,i\,.}\right)\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r_2,q}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|},$$

where $\widetilde{\mathbf{d}}_{i.}$ *are the ith row of* $\widetilde{\mathbf{D}} = \mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}^{\frac{1}{2}}$ *for all* $i = 1,...,n$*,* $j = 1,...,p$*.*

(ii) If $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\operatorname{rdet}_j\left(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*\right)_{j.}\left(\widetilde{\mathbf{d}}_{i.}\right)}{\det\left(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*\right)}, \tag{97}$$

where $\widetilde{\mathbf{d}}_{i.}$ *are the ith row of* $\widetilde{\mathbf{D}} = \mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*$*.*

Proof. The proof follows evidently from Theorem 6.2 when $\mathbf{A}$ be removed and unit matrices insert instead $\mathbf{M}$, $\mathbf{N}$.

6.3. Mixed Cases

In this subsection we consider mixed cases when only one from the pair $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{B}\mathbf{B}^{\sharp}$ is non-Hermitian. We give this theorems without proofs, since their proofs are similar to the proof of Theorems 6.1 and 6.2.

Theorem 6.3. *Let* $\mathbf{A}^{\sharp}\mathbf{A}$ *be Hermitian and* $\mathbf{B}\mathbf{B}^{\sharp}$ *be non-Hermitian. Then the solution (54) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{.\,j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)^{\beta}_{\beta}\right|\sum\limits_{\alpha\in I_{r_2,p}}\left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|},$$

or

$$x_{ij} = \frac{\sum\limits_{l} \sum\limits_{\alpha\in I_{r_2,\,p}\{l\}} \operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\mathbf{d}^{\mathbf{A}}_{i.}\right)\right)^{\alpha}_{\alpha} \cdot m^{(\frac{1}{2})}_{lj}}{\sum\limits_{\beta\in J_{r_1,n}} \left|\left(\mathbf{A}^\sharp\mathbf{A}\right)^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,p}} \left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|},$$

where

$$\mathbf{d}^{\mathbf{B}}_{.j} = \left(\sum_{l} \sum_{\alpha\in I_{r_2,\,p}\{l\}} \operatorname{rdet}_l\left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)_{l.}\left(\widetilde{\mathbf{d}}_{t.}\right)\right)^{\alpha}_{\alpha} \cdot m^{(\frac{1}{2})}_{lj}\right) \in \mathbb{H}^{n\times 1} \quad (98)$$

$$\mathbf{d}^{\mathbf{A}}_{i.} = \left(\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.i}\left(\widetilde{\mathbf{d}}_{.f}\right)\right)^{\beta}_{\beta}\right) \in \mathbb{H}^{1\times p} \quad (99)$$

are the column-vector and the row-vector, respectively. $\tilde{\mathbf{d}}_{t.}$ *and* $\widetilde{\mathbf{d}}_{.f}$ *are the tth row and the fth column of* $\widetilde{\mathbf{D}} := \mathbf{A}^\sharp\mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}^{\frac{1}{2}}$ *for all* $t = 1, ..., n$, $f = 1, ..., p$.

(ii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = p$, *then*

$$x_{ij} = \frac{\operatorname{cdet}_i(\mathbf{A}^\sharp\mathbf{A})_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.j}\right)}{\det(\mathbf{A}^\sharp\mathbf{A}) \cdot \det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)},$$

or

$$x_{ij} = \frac{\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)}{\det(\mathbf{A}^\sharp\mathbf{A}) \cdot \det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)},$$

where

$$\mathbf{d}^{\mathbf{B}}_{.j} := \left(\operatorname{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.}\left(\tilde{\mathbf{d}}_{t\,.}\right)\right) \in \mathbb{H}^{n\times 1}, \quad (100)$$

$$\mathbf{d}^{\mathbf{A}}_{i.} := \left(\operatorname{cdet}_i(\mathbf{A}^\sharp\mathbf{A})_{.\,i}\left(\tilde{\mathbf{d}}_{.f}\right)\right) \in \mathbb{H}^{1\times p}, \quad (101)$$

$\tilde{\mathbf{d}}_{t\,.}$, $\tilde{\mathbf{d}}_{.f}$ *are the tth row and fth column of* $\tilde{\mathbf{D}} = \mathbf{A}^\sharp\mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^*$.

(iii) If $\operatorname{rank}\mathbf{A} = n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$, *then*

$$x_{ij} = \frac{\operatorname{cdet}_i\left(\left(\mathbf{A}^\sharp\mathbf{A}\right)_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{.j}\right)\right)}{\det(\mathbf{A}^\sharp\mathbf{A}) \cdot \sum\limits_{\alpha\in I_{r_2,p}} \left|\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^*\right)^{\alpha}_{\alpha}\right|},$$

or

$$x_{ij} = \frac{\sum\limits_{l} \sum\limits_{\alpha \in I_{r_2,\, p}\{l\}} \mathrm{rdet}_l \left(\left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^* \right)_{l.} (\mathbf{d}_{i.}^{\mathbf{A}}) \right)_{\alpha}^{\alpha} \cdot m_{lj}^{(\frac{1}{2})}}{\det(\mathbf{A}^{\sharp}\mathbf{A}) \cdot \sum\limits_{\alpha \in I_{r_2,p}} \left| \left(\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^* \right)_{\alpha}^{\alpha} \right|},$$

where $\mathbf{d}_{.\,j}^{\mathbf{B}}$ *is (98) and* $\mathbf{d}_{i\,.}^{\mathbf{A}}$ *is (101).*

(iv) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{i\,j} = \frac{\mathrm{rdet}_j(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)_{j.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}}\right)}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^{\sharp}\mathbf{A})_{\beta}^{\beta} \right| \cdot \det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)},$$

or

$$x_{i\,j} = \frac{\sum\limits_{\beta \in J_{r_1,\, n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i} \left(\mathbf{d}_{\cdot j}^{\mathbf{B}}\right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_1,n}} \left| (\mathbf{A}^{\sharp}\mathbf{A})_{\beta}^{\beta} \right| \cdot \det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^*)},$$

where $\mathbf{d}_{.\,j}^{\mathbf{B}}$ *is (100) and* $\mathbf{d}_{i\,.}^{\mathbf{A}}$ *is (99).*

Theorem 6.4. *Let* $\mathbf{A}^{\sharp}\mathbf{A}$ *be non-Hermitian, and* $\mathbf{B}\mathbf{B}^{\sharp}$ *be Hermitian. Denote* $\widetilde{\mathbf{D}} := \widetilde{\mathbf{A}}^*\mathbf{D}\mathbf{B}^{\sharp}$*. Then the solution (54) possess the following determinantal representations.*

(i) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = r_2 < p$*, then*

$$x_{ij} = \frac{\sum\limits_{k} n_{ik}^{(-\frac{1}{2})} \sum\limits_{\beta \in J_{r_1,\, n}\{k\}} \mathrm{cdet}_k \left(\left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}} \right)_{.\,k} \left(\mathbf{d}_{.j}^{\mathbf{B}}\right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| \left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}} \right)_{\beta}^{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} \left| (\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha} \right|},$$

or

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{r_2,p}\{j\}} \mathrm{rdet}_j \left(\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j\,.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}}\right) \right)_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| \left(\widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}} \right)_{\beta}^{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} \left| (\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha} \right|},$$

where

$$\mathbf{d}^{\mathbf{B}}_{.\,j}=\left(\sum_{\alpha\in I_{r_2,p}\{j\}}\operatorname{rdet}_j\left(\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j\,.}\left(\widetilde{\mathbf{d}}_{\,t\,.}\right)\right)_{\alpha}^{\alpha}\right)\in\mathbb{H}^{n\times 1}\tag{102}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.}=\left(\sum_{k}n_{ik}^{(-\frac{1}{2})}\sum_{\beta\in J_{r_1,\,n}\{k\}}\operatorname{cdet}_k\left(\left(\widetilde{\mathbf{A}}^{*}\widetilde{\mathbf{A}}\right)_{.\,k}\left(\widetilde{\mathbf{d}}_{.f}\right)\right)_{\beta}^{\beta}\right)\in\mathbb{H}^{1\times p}\tag{103}$$

are the column-vector and the row-vector, respectively. $\tilde{\mathbf{d}}_{t.}$ *and* $\tilde{\mathbf{d}}_{.f}$ *are the* t*th row and the* f*th column of* $\widetilde{\mathbf{D}}:=\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^{*}\mathbf{M}\mathbf{D}\mathbf{B}^{\sharp}$ *for all* $t=1,...,n$, $f=1,...,p$.

(ii) If $\operatorname{rank}\mathbf{A}=n$ *and* $\operatorname{rank}\mathbf{B}=p$*, then*

$$x_{i\,j}=\frac{\operatorname{cdet}_i(\mathbf{A}^{*}\mathbf{M}\mathbf{A})_{.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.j}\right)}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})\cdot\det\left(\mathbf{B}\mathbf{B}^{\sharp}\right)},$$

or

$$x_{i\,j}=\frac{\operatorname{rdet}_j\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j.}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})\cdot\det\left(\mathbf{B}\mathbf{B}^{\sharp}\right)},$$

where

$$\mathbf{d}^{\mathbf{B}}_{.j}:=\left(\operatorname{rdet}_j\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j.}\left(\tilde{\mathbf{d}}_{t\,.}\right)\right)\in\mathbb{H}^{n\times 1},\tag{104}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.}:=\left(\operatorname{cdet}_i(\mathbf{A}^{*}\mathbf{M}\mathbf{A})_{.\,i}\left(\tilde{\mathbf{d}}_{.f}\right)\right)\in\mathbb{H}^{1\times p},\tag{105}$$

$\tilde{\mathbf{d}}_{t\,.}$, $\tilde{\mathbf{d}}_{.f}$ *are the* t*th row and* f*th column of* $\tilde{\mathbf{D}}=\mathbf{A}^{*}\mathbf{M}\mathbf{D}\mathbf{B}^{\sharp}$.

(iii) If $\operatorname{rank}\mathbf{A}=n$ *and* $\operatorname{rank}\mathbf{B}=r_2<p$*, then*

$$x_{ij}=\frac{\operatorname{cdet}_i\left((\mathbf{A}^{*}\mathbf{M}\mathbf{A})_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{.j}\right)\right)}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})\cdot\sum\limits_{\alpha\in I_{r_2,p}}\left|(\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha}\right|},$$

or

$$x_{ij}=\frac{\sum\limits_{\alpha\in I_{r_2,p}\{j\}}\operatorname{rdet}_j\left(\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j\,.}\left(\mathbf{d}^{\,\mathbf{A}}_{i\,.}\right)\right)_{\alpha}^{\alpha}}{\det(\mathbf{A}^{*}\mathbf{M}\mathbf{A})\cdot\sum\limits_{\alpha\in I_{r_2,p}}\left|(\mathbf{B}\mathbf{B}^{\sharp})_{\alpha}^{\alpha}\right|},$$

where $\mathbf{d}^{\mathbf{B}}_{.\,j}$ *is (102) and* $\mathbf{d}^{\mathbf{A}}_{i\,.}$ *is (105).*

(iv) If $\operatorname{rank}\mathbf{A} = r_1 < n$ *and* $\operatorname{rank}\mathbf{B} = p$*, then*

$$x_{ij} = \frac{\operatorname{rdet}_j \left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j.} \left(\mathbf{d}_{i\,.}^{\mathbf{A}}\right)}{\sum\limits_{\beta \in J_{r_1,n}} \left|\left(\widetilde{\mathbf{A}}^{*}\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right| \cdot \det\left(\mathbf{B}\mathbf{B}^{\sharp}\right)},$$

or

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{d}_{.\,j}^{\mathbf{B}}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_1,n}} \left|\left(\widetilde{\mathbf{A}}^{*}\widetilde{\mathbf{A}}\right)_{\beta}^{\beta}\right| \cdot \det(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^{*})},$$

where $\mathbf{d}_{.\,j}^{\mathbf{B}}$ *is (104) and* $\mathbf{d}_{i\,.}^{\mathbf{A}}$ *is (103).*

7. Examples

In this section, we give examples to illustrate our results.

1. Let us consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & i & j \\ -k & i & 1 \\ k & j & -i \\ j & -1 & i \end{pmatrix}, \tag{106}$$

$$\mathbf{N}^{-1} = \begin{pmatrix} 23 & 16 - 2i - 2j + 10k & -16 + 10i - 2j - 2k \\ 16 + 2i + 2j - 10k & 29 & -19 - i - 13j - k \\ -16 - 10i + 2j + 2k & -19 + i + 13j + k & 29 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} 2 & k & i & 0 \\ -k & 2 & 0 & j \\ -i & 0 & 2 & k \\ 0 & -j & -k & 2 \end{pmatrix}. \tag{107}$$

By direct calculation we get that leading principal minors of $\mathbf{M}$ and $\mathbf{N}^{-1}$ are all positive. So, $\mathbf{M}$ and $\mathbf{N}^{-1}$ are positive definite matrices. Similarly, by direct calculation of leading principal minors of $\mathbf{A}^{*}\mathbf{A}$, we obtain $\operatorname{rank}\mathbf{A}^{*}\mathbf{A} = \operatorname{rank}\mathbf{A} = 2$. Further,

$$\mathbf{A}^{\sharp} = \mathbf{M}\mathbf{A}^{*}\mathbf{N}^{-1} =$$

$$\begin{pmatrix} 51 - 12i + 25j - 24k & -43 - 18i + 39k & -18 + 26i - 30j - 38k & 19 - i - 50j - 42k \\ -32i + 17j - 37k & -24 - 50i + 26j + 24k & -5 - 24i - 56j + k & -38 - 25i - 18j - 67k \\ 5 - 6i - 50j + 11k & 44 + 23i - 12j + 7k & 30 + 38i + 5j + 37k & 18 - 44i + 6j + 54k \end{pmatrix}.$$

Since,

$$\mathbf{A}^{\sharp}\mathbf{A} =$$

$$\begin{pmatrix} 178 & 41+47i+47j+43k & -41+43i+47j+47k \\ 41-47i-47j-43k & 176 & -40-46i-42j-46k \\ -41-43i-47j-47k & -40+46i+42j+46k & 176 \end{pmatrix}$$

are Hermitian, then we shall be obtain $\mathbf{A}^{\dagger}_{M,N} = \left(\tilde{a}^{\dagger}_{ij}\right) \in \mathbb{H}^{3\times 4}$ due to Theorem 4.1 by Eq. (33).

We have, $\sum\limits_{\beta\in J_{2,\ 3}} \left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right){}^{\beta}_{\beta}\right| = 23380+23380+23380 = 70140$, and

$$\sum_{\beta\in J_{2,\,3}\{1\}} \operatorname{cdet}_1\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,1}\left(\mathbf{a}^{\sharp}_{.1}\right)\right){}^{\beta}_{\beta} = 6680+1670i+3340j-5010k+$$

$$6680-5010i+3340j-1640k = 13360-3340i+6680j-6680k.$$

Then,

$$\tilde{a}^{\dagger}_{11} = \frac{8-2i+4j-4k}{42}.$$

Similarly, we obtain

$$\tilde{a}^{\dagger}_{12} = \tfrac{-7-3i+6k}{42},\quad \tilde{a}^{\dagger}_{13} = \tfrac{-3+4i-5j-6k}{42},\quad \tilde{a}^{\dagger}_{14} = \tfrac{3-8j-7k}{42},$$

$$\tilde{a}^{\dagger}_{21} = \tfrac{-5i+3j-6k}{42},\quad \tilde{a}^{\dagger}_{22} = \tfrac{-4-8i+2j+2k}{42},\quad \tilde{a}^{\dagger}_{23} = \tfrac{-1-4i-9j}{42}, \tilde{a}^{\dagger}_{24} = \tfrac{-6-4i-3j-11k}{42},$$

$$\tilde{a}^{\dagger}_{31} = \tfrac{-1-i-8j+2k}{42},\quad \tilde{a}^{\dagger}_{32} = \tfrac{7+4i-2j+k}{42},\quad \tilde{a}^{\dagger}_{33} = \tfrac{5+6i+j+6k}{42}, \tilde{a}^{\dagger}_{34} = \tfrac{3-7i+j+9k}{42}.$$

Finally, we obtain

$$\mathbf{A}^{\dagger}_{M,N} =$$

$$\frac{1}{42}\begin{pmatrix} 8-2i+4j-4k & -7-3i+6k & -3+4i-5j-6k & 3-8j-7k \\ -5i+3j-6k & -4-8i+2j+2k & -1-4i-9j & -6-4i-3j-11k \\ -1-i-8j+2k & 7+4i-2j+k & 5+6i+j+6k & 3-7i+j+9k \end{pmatrix}. \tag{108}$$

2. Consider the right system of linear equations,

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{109}$$

where the coefficient matrix $\mathbf{A}$ is (106) and the column $\mathbf{b} = (1\ 0\ i\ k)^T$. Using (108), by the matrix method we have for the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = \mathbf{A}^{\dagger}_{M,N}\mathbf{b}$ of (109) with weights $\mathbf{M}$ and $\mathbf{N}$ from (107),

$$\tilde{x}_1 = \frac{11 - 13i - 2j + 4k}{42}, \tilde{x}_2 = \frac{15 - 9i + 7j - 3k}{42}, \tilde{x}_3 = \frac{-16 + 5i + 5j + 4k}{42}. \tag{110}$$

Now, we shall find the weighted Moore-Penrose solution of (109) by Cramer's rule (48). Since

$$\mathbf{f} = \mathbf{A}^{\sharp}\mathbf{b} = \begin{pmatrix} 67 - 80i - 12j + 25k \\ 91 - 55i + 43j - 19k \\ -97 + 30i + 31j + 24k \end{pmatrix},$$

then we have

$$\tilde{x}_1 = \frac{\sum\limits_{\beta \in J_{2,\,3}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}(\mathbf{f}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\ n}} \left| \left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta} \right|} = \frac{18370 - 21710i - 3340j + 6680k}{70140} = \frac{11 - 13i - 2j + 4k}{42},$$

$$\tilde{x}_2 = \frac{25050 - 15030i + 11690j - 5010k}{70140} = \frac{15 - 9i + 7j - 3k}{42},$$
$$\tilde{x}_3 = \frac{-26720 + 8350i + 8350j + 6680k}{70140} = \frac{-16 + 5i + 5j + 4k}{42}.$$

As we expected, the weighted Moore-Penrose solutions by Cramer's rule and be the matrix method coincide.

3. Let us consider the restricted matrix equation

$$\mathbf{XB} = \mathbf{D},\ \ \mathcal{N}_r(\mathbf{X}) \supset \mathbf{P}^{-1}\mathcal{N}_r(\mathbf{B}^*), \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l(\mathbf{B}^*)\mathbf{Q}, \tag{111}$$

where

$$\mathbf{B} = \begin{pmatrix} k & -j & j \\ 0 & 1 & i \end{pmatrix},\ \mathbf{Q} = \begin{pmatrix} 2 & i & k \\ -i & 2 & -j \\ -k & j & 3 \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} 1 & j \\ -j & 2 \end{pmatrix}.$$

The inverse $\mathbf{Q}^{-1}$ can be obtain due to Theorem 2.4, then

$$\mathbf{Q}^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -4i & -3k \\ 4i & 5 & 3j \\ 3k & -3j & 3 \end{pmatrix}.$$

Since

$$\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^{*}\mathbf{P} = \begin{pmatrix} 4+i-j+2k & 2-4i+j+2k \\ 1+2i+\frac{1}{3}j-k & \frac{7}{3}+i+j+2k \end{pmatrix}$$

is not Hermitian, and

$$\det\left(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}\right) = \det\begin{pmatrix} 7 & 1-2i-3j+k \\ 1+2i+3j-k & \frac{8}{3} \end{pmatrix} = 11,$$

then we shall find the solution of (111) by (97). Therefore,

$$x_{11} = \frac{\operatorname{rdet}_1\left(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^{*}\right)_{1.}\left(\check{\mathbf{d}}_{i.}\right)}{\det\left(\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}^{*}\right)} = \frac{1}{11}\operatorname{rdet}_1\begin{pmatrix} 1+i+3j+2k & \frac{4}{3}+i-j+k \\ 1+2i+3j-k & \frac{8}{3} \end{pmatrix} = \\ \frac{1}{33}(-2+3i+6j+2k),$$

because $\widehat{\mathbf{d}}_{1.}$ is the first row of

$$\widehat{\mathbf{D}} = \mathbf{D}\mathbf{Q}^{-1}\mathbf{B}^{*} = \begin{pmatrix} 1+i+3j+2k & \frac{4}{3}+i-j+k \\ 1+3i+3j-k & \frac{8}{3}-\frac{1}{3}k \\ 1-i+2j-3k & 1-i+j+k \end{pmatrix}.$$

Similarly, we obtain

$$x_{21} = \frac{1}{11}\operatorname{rdet}_1\begin{pmatrix} 1+3i+3j-k & \frac{8}{3}-\frac{1}{3}k \\ 1+2i+3j-k & \frac{8}{3} \end{pmatrix} = \frac{1}{33}(1+5i+2j+k),$$

and

$$\begin{aligned} x_{31} &= \frac{1}{33}(11-17i+13j-3k), & x_{12} &= \frac{1}{11}(-2-3i-6j+3k), \\ x_{22} &= \frac{1}{33}(5-3i+3j+2k), & x_{32} &= \frac{1}{11}(-1+3i+j-12k). \end{aligned}$$

Note that we used Maple with the package CLIFFORD in the calculations.

Conclusion

In this chapter, we derive determinantal representations of the weighted Moore-Penrose by WSVD within the framework of the theory of noncommutative column-row determinants. By using the obtained analogs of the adjoint matrix, we get the Cramer rules for solutions of restricted left and right systems of quaternion linear equations. We give determinantal representations for solutions of quaternion restricted matrix equation $\mathbf{AXB} = \mathbf{D}$ in all cases with respect to weighted matrices.

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